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ASYMPTOTIC STABILITY OF SYSTEMS WITH IMPULSES BY THE DIRECT METHOD OF LYAPUNOV

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In the present paper the asymptotic and globally asymptotic stability of the zero solution of systems with impulses are investigated. For this purpose piecewise continuous auxiliary functions are used which are analogous to Lyapunov's functions. The theorem of Marachkov on the asymptotic stability of systems without impulses is generalised. The results obtained are formulated in four theorems.

1. INTRODUCTION

Systems of differential equations with impulses are used as mathematical models of various real processes and phenomens in physics, biology, control theory, etcetera, which are subject during their evolution to short-time perturbations in the form of impulses. That is why in recent years these systems have been an object of numerous investigations ([**1** - **4**], [**6** - **12**]).

In this paper the asymptotic stability of the zero solution of systems with impulses of the following form is studied:

$$\dot{x} = X(t,x),\ t \neq \tau_i(x);\ \triangle x/_{t=\tau_i(x)} = I_i(x),$$

where $x \in \mathbf{R}^n$, $X\colon I \times \mathbf{R}^n \to \mathbf{R}^n$, $I_i\colon \mathbf{R}^n \to \mathbf{R}^n$, $I = [0,\infty)$, $\triangle x/_{t=\tau_i(x)} = x(t+0) - x(t-0)$.

The investigations are carried out by means of piecewise continuous functions which are an analogue of Lyapunov's functions. The main result is a generalisation of the theorem of Marachkov (1940, [**5**]) on the asymptotic stability of systems without impulses.

2. PRELIMINARY NOTES AND DEFINITIONS

Let $\mathbf{R}^s$ be s-dimensional Euclidean space with norm $\|\cdot\|$. Let $I = [0,\infty)$, $\mathbf{R}^s_H = \{x \in \mathbf{R}^s\colon \|x\| < H\}$, $0 < H =$ constant.

Consider the following system of differential equations with impulses

$$(1)\qquad \begin{cases} \dot{x} = f(t,x) + g(t,y),\ \dot{y} = h(t,x,y),\ t \neq \tau_i(x,y) \\ \triangle x/_{t=\tau_i(x,y)} = A_i(x) + B_i(y),\ \triangle y/_{t=\tau_i(x,y)} = C_i(x,y), \end{cases}$$

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where $x \in \mathbf{R}^n$, $y \in \mathbf{R}^m$, $f\colon I \times \mathbf{R}^n_H \to \mathbf{R}^n$, $g\colon I \times \mathbf{R}^m_H \to \mathbf{R}^n$, $h\colon I \times \mathbf{R}^n_H \times \mathbf{R}^m_H \to \mathbf{R}^m$, $A_i\colon \mathbf{R}^n_H \to \mathbf{R}^n$, $B_i\colon \mathbf{R}^m_H \to \mathbf{R}^n$, $C_i\colon \mathbf{R}^n_H \times \mathbf{R}^m_H \to \mathbf{R}^m$, $\tau_i\colon \mathbf{R}^n_H \times \mathbf{R}^m_H \to \mathbf{R}$, $\triangle x/_{t=\tau_i(x,y)} = x(t+0) - x(t-0)$, $\triangle y/_{t=\tau_i(x,y)} = y(t+0) - y(t-0)$.

Let $t_0 \in I$, $x_0 \in \mathbf{R}^n_H$, $y_0 \in \mathbf{R}^m_H$. The solution of system (1) satisfying the initial conditions $x(t_0+0; t_0, x_0, y_0) = x_0$, $y(t_0+0; t_0, x_0, y_0) = y_0$ is denoted by $(x(t), y(t)) = (x(t; t_0, x_0, y(t)), y(t; t_0, x_0, y_0))$. The solutions $(x(t),\, y(t))$ of system (1) are piecewise continuous functions with points of discontinuity of the first type in which they are left continuous, that is at the moment t_i when the integral curve of the solution $(x(t),\, y(t))$ meets the hypersurface

$$\sigma_i = \{(t,x,y) \in I \times \mathbf{R}^n_H \times \mathbf{R}^m_H \colon t = \tau_i(x,y)\}$$

the following relations are satisfied

$$\begin{aligned} &x(t_i - 0) = x(t_i), x(t_i + 0) = x(t_i) + A_i(x(t_i)) + B_i(y(t_i)),\\ &y(t_i - 0) = y(t_i), y(t_i + 0) = y(t_i) + C_i(x(t_i),\, y(t_i)). \end{aligned}$$

Together with system (1) we consider the following system with impulses

$$\dot{x} = f(t,x),\; t \neq \tau_i(x,0);\; \triangle x/_{t=\tau_i(x,0)} = A_i(x). \tag{2}$$

We shall say that conditions (A) hold if the following conditions are satisfied:

A1. The functions $f(t,x)$, $g(t,y)$, $h(t,x,y)$ are continuous in their domains of definitions and $f(t,0) = 0$, $g(t,0) = 0$, $h(t,0,0) = 0$ for $t \in I$;

A2. There exists a constant $L > 0$ such that

$$\|h(t,x,y)\| \leqq L,\; (t,x,y) \in I \times \mathbf{R}^n_H \times \mathbf{R}^m_H;$$

A3. There exists a continuous function $P\colon I \to I$ such that $P(0) = 0$ and $\|g(t,y)\| \leqq P(\|y\|)$ for $(t,y) \in I \times \mathbf{R}^m_H$,

A4. The functions $A_i(x)$, $B_i(y)$, $C_i(x,y)$ are continuous in their domains of definition, $A_i(0) = 0$, $B_i(0)$, $C_i(0,0) = 0$;

A5. If $x \in \mathbf{R}^n_H$ and $y \in \mathbf{R}_H$, then $y + C_i(x,y) \in \mathbf{R}^m_H, i = 1,2,\ldots,$

A6. The functions $\tau_i(x,y)$ are continuous and for $(x,y) \in \mathbf{R}^n_H \times \mathbf{R}^m_H$ the following relations hold

$$0 < \tau_1(x,y) < \tau_2(x,y) < \ldots,\; \lim_{i\to\infty}(x,y) = \infty$$

uniformly in $\mathbf{R}^n_H \times \mathbf{R}^m_H$,

$$\inf_{\mathbf{R}^n_H \times \mathbf{R}^m_H} \tau_{i+1}(x,y) - \sup_{\mathbf{R}^n_H \times \mathbf{R}^m_H} \tau_i(x,y) \geqslant \theta > 0,\; i = 1,2,\ldots$$

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A7. For each point $(t_0, x_0, y_0) \in I \times R^n_H \times R^m_H$ C1 the solution $(x(t; t_0, x_0, y_0), y(t; t_0, x_0, y_0))$ of system (1) is unique and defined in (t_0, ∞).

A8. For each point $(t_0, x_0) \in I \times R^n_H$ the solution $x(t; t_0, x_0)$ of system (2) satisfying $x(t_0 + 0; t_0, x_0) = x_0$ is unique and exists for all (t_0, ∞).

We shall say that condition (B) is satisfied if the following condition holds:

B. The integral curve of each solution of system (1) meets each of the hypersurfaces $\{\sigma_i\}$ at most once.

Condition (B) means that for system (1) the phenomenon called "beating" is not observed. It is clear that in this case the integral curve of each solution of system (2) meets each of the hypersurfaces $s_i = \{(t, x) \in I \times R^n_H : t = \tau_i(x, 0)\}$ at most once, that is for system (2) the phenomenon of "beating" is not observed either. We shall point out that efficient sufficient conditions which guarantee the absence of the phenomenon "beating" are given in **[1]** and **[2]**.

We shall give definitions of some types of stability of the zero solution of system (1).

DEFINITION 1: The zero solution of system (1) is called:

a) *stable* if

$$(\forall \varepsilon > 0)(\forall t_0 \in I)(\exists \delta = \delta(t_0, \varepsilon) > 0)$$
$$(\forall (x_0, y_0) \in R^n_H \times R^m_H,\ \|x_0\| + \|y_0\| < \delta)(\forall t > t_0):$$
$$\|x(t; t_0, x_0, y_0)\| + \|y(t; t_0, x_0, x_0, y_0)\| < \varepsilon;$$

b) *uniformly stable* if the number δ from a) does not depend on $t_0 \in I$;

c) *attractive* if

$$(\forall t_0 \in I)(\exists \lambda = \lambda(t_0) > 0)(\forall \varepsilon > 0)$$
$$(\forall (x_0, y_0) \in R^n_H \times R^m_H,\ \|x_0\| + \|y_0\| < \lambda)(\exists \sigma = \sigma(t_0, x_0, y_0, \varepsilon) > 0)$$
$$(\forall t \geqq t_0 + \sigma):\ \|x(t; t_0, x_0, y_0)\| + \|y(t; t_0, x_0, y_0)\| < \varepsilon;$$

d) *equi-attractive* if the number σ from c) does not depend on $(x_0, y_0) \in R^n_H \times R^m_H$;

e) *asymptotically stable* if it is stable and attractive;

f) *equi-asymptotically stable* if it is stable and equi-attractive.

In the following considerations we shall use classes $\mathcal{V}_0$, and $\mathcal{W}_0$ of piecewise continuous functions which are analogous to Lyapunov's functions.

Let $\tau_0(x, y) = 0$ for $(x, y) \in R^n_H \times R^m_H$. Consider the sets

$$G_i = \{(t, x, y) \in I \times R^n_H \times R^m_H : \tau_{i-1}(x, y) < t < \tau_i(x, y)\}$$
$$\Omega_i = \{(t, x) \in I \times R^n_H : \tau_{i-1}(x, 0) < t < \tau_i(x, 0)\}.$$

DEFINITION 2.: We say that the function $V\colon I\times \mathbf{R}^n_H \times \mathbf{R}^m_H \to \mathbf{R}$ *belongs to class* $\mathcal{V}_0$ if the following conditions hold:

1. the function V is continuous in $\bigcup_1^\infty G_i$ and is locally Lipschitz with respect to x and y in each of the sets G_i;

2. $V(t,0,0)=0$ for $t\in I$;

3. For each $i=1,2,\dots$ and for any point $(t_0,x_0,y_0)\in\sigma_i$ the limits

$$V(t_0-0,x_0,y_0)=\lim_{\substack{(t,x,y)\to(t_0,x_0,y_0)\\(t,x,y)\in G_i}} V(t,x,y),$$

$$V(t_0+0,x_0,y_0)=\lim_{\substack{(t,x,y)\to(t_0,x_0,y_0)\\(t,x,y)\in G_{i+1}}} V(t,x,y)$$

exist and are finite, and the equality $V(t_0-0,x_0,y_0)=V(t_0,x_0,y_0)$ holds;

4. For any point $(t,x,y)\in\sigma_i$ the following inequality holds

$$V(t+0,\,x+A_i(x)+B_i(y),\,y+C_i(x,y))\leqq V(t,x,y). \tag{3}$$

DEFINITION 3: We say that the function $W\colon I\times\mathbf{R}^n_H\to\mathbf{R}$ *belongs to class* $\mathcal{W}_0$ if the following conditions hold:

1. The function W is continuous in $\bigcup\limits_1^\infty \Omega_i$ and is locally Lipschitz with respect to x in each of the sets Ω_i;

2. $W(t,0)=0$ for $t\in I$;

3. the following limits exist and are finite:

$$W(t_0-0,x_0)=\lim_{\substack{(t,x)\to(t_0,x_0)\\(t,x)\in\Omega_i}} W(t,x),$$

$$W(t_0+0,x_0)=\lim_{\substack{(t,x)\to(t_0,x_0)\\(t,x)\in\Omega_{i+1}}} W(t,x)$$

for each $i=1,2,\dots$ and any point $(t_0,x_0)\in s_i$, and the equality $W(t_0-0,x_0)=W(t_0,x_0)$ holds;

4. for any point $(t,x)\in s_i$ the following inequality holds:

$$W(t+0,\,x+A_i(x))\leqq W(t,x). \tag{4}$$

Let $V\in\mathcal{V}_0$. For $(t,x,y)\in\bigcup\limits_1^\infty G_i$ we set

$$\dot V_{(1)}(t,x,y)=\limsup_{s\to0+}\frac{1}{s}[V(t+s,\,x+sf(t,x)+sg(t,y),y+sh(t,x,y))-V(t,x,y)].$$

Note that if $x = x(t)$, $y = y(t)$ is any solution of system (1), then $\dot{V}_{(1)}(t,x,y) = D^+V(t,x,y)$, where $D^+V(t,x,y)$ is the upper right Dini derivative of the function $V(t,x(t),y(t))$.

Analogously one can define the function $\dot{W}_{(2)}(t,x)$ for an arbitrary function $W \in \mathcal{W}_0$ for $(t,x) \in \bigcup_1^\infty \Omega_i$.

We shall denote by $\mathcal{K}$ the class of all functions $a\colon I \to I$ which are continuous, strictly increasing and such that $a(0) = 0$.

3. Main results

We shall prove that the existence of functions of classes $\mathcal{V}_0$ and $\mathcal{W}_0$ with certain properties is a sufficient condition for the asymptotic and equi-asymptotic stability of the zero solution of system (1).

THEOREM 1. *Let the following conditions be fulfilled:*

1. *Conditions (A) and (B) hold;*
2. *There exist functions $V \in \mathcal{V}_0$ and $a, c \in \mathcal{K}$ such that*

$$a(\|x\| + \|y\|) \leqq V(t,x,y),\ (t,x,y) \in I \times \mathbf{R}^n_H \times \mathbf{R}^m_H \tag{5}$$

$$\dot{V}_{(1)}(t,x,y) \leqq -c(\|y\|) \quad \text{for} \quad (t,x,y) \in \bigcup_1^\infty G_i; \tag{6}$$

3. *there exist functions $W \in \mathcal{W}_0$ and $a_1, c_1 \in \mathcal{K}$ such that*

$$a_1(\|x\|) \leqq W(t,x),\ (t,x) \in I \times \mathbf{R}^n_H, \tag{7}$$

$$\dot{W}_{(2)}(t,x) \leqq -c_1(W(t,x)),\ (t,x) \in \bigcup_1^\infty \Omega_i, \tag{8}$$

$$|W(t,x_1) - W(t,x_2)| \leqq d\,\|x_1 - x_2\|,\ t \in I,\ x_1, x_2 \in \mathbf{R}^n_H, \tag{9}$$

$0 < d =$ *constant.*

Then the zero solution of system (1) is asymptotically stable.

PROOF: Let $0 < \varepsilon < H$ and $t_0 \in I$. Without loss of generality we can assume that $t_0 < \tau_1(x,y)$ for $(x,y) \in \mathbf{R}^n_H \times \mathbf{R}^m_H$. From the condition $V(t_0,0,0) = 0$ and Definition 2 if follows that there exists a number $\delta = \delta(t_0,\varepsilon) > 0$ such that if $\|x\| + \|y\| < \delta(t_0,\varepsilon)$, then $V(t_0+0,x,y) < a(\varepsilon)$.

Let $x_0 \in \mathbf{R}^n_H$, $y_0 \in \mathbf{R}^m_H$, $\|x_0\| + \|y_0\| < \delta(t_0,\varepsilon)$ and $x(t) = x(t;t_0,x_0,y_0)$, $y(t) = y(t;t_0,x_0,y_0)$ be a solution of (1). From (3) and (6) it follows that the function $V(t,x(t),y(t))$ is monotonic decreasing in (t_0,∞), whence in view of (5) we obtain

$$a(\|x(t)\| + \|y(t)\|) \leqq V(t,x(t),y(t)) \leqq V(t_0+0,x_0,y_0) < a(\varepsilon)$$

for $t \in (t_0, \infty)$. Hence the zero solution of system (1) is stable.

Then we can choose a number $\lambda = \lambda(t_0) > 0$ such that if $\|x_0\| + \|y_0\| < \lambda$, then $\|x(t)\| + \|y(t)\| < H$ for any $t > t_0$. We shall prove that in this case $\lim_{t\to\infty} y(t; t_0, x_0, y_0) = 0$.

If we suppose that this is not true, then for some $\varepsilon_0 > 0$ there exists a sequence $\{\xi_k\}$ tending to ∞ for $k \to \infty$ such that $\|y(\xi_k)\| \geqq \varepsilon_0$, $k = 1, 2, \ldots$. If $t_i (i = 1, 2, \ldots)$ are the moments when the integral curve of the solution $(x(t), y(t))$ meets the hypersurfaces σ_i, then for $t \neq t_i$ using A2 we obtain

$$\left| \frac{d}{dt} \|y(t)\| \right| \leqq \|\dot{y}(t)\| = \|h(t, x(t), y(t))\| \leqq L.$$

We shall prove that $\|y(t)\| \geqq \frac{\varepsilon_0}{2}$ for $t \in [\xi_R - \frac{\varepsilon_0}{2L} . \xi_R] = T_R$. In fact, let $0 \leqq \xi_R - t \leqq \frac{\varepsilon_0}{2L}$. Integrating the inequality $\frac{d}{dt} \|y(t)\| \leqq L$ from t to ξ_R, we obtain

$$\int_t^{\xi_R} \frac{d}{d\tau} \|y(\tau)\| \, d\tau \leqq L(\xi_R - t) \leqq \frac{\varepsilon_0}{2}$$

On the other hand, each of the intervals T_R, $R = 1, 2, \ldots$ contains a finite number of points $\{t_i\}$. Let, for instance, these be the points $t_s, t_{s+1}, \ldots, t_{s+p}$. Then, making use of A5, we obtain

$$\begin{aligned}
\int_t^{\xi_R} \frac{d}{d\tau} \|y(\tau)\| \, d\tau &= \int_t^{t_s} \frac{d}{d\tau} \|y(\tau)\| \, d\tau + \sum_{j=s+1}^{s+p} \int_{t_{j-1}}^{t_j} \frac{d}{d\tau} \|y(\tau)\| \, d\tau \\
&\quad + \int_{t_{s+p}}^{\xi_R} \frac{d}{d\tau} \|y(\tau)\| \, d\tau = \|y(t_s)\| - \|y(t+0)\| \\
&\quad + \sum_{j=s+1}^{s+p} [\|y(t_j)\| - \|y(t_{j-1}+0)\|] + \|y(\xi_R)\| - \|y(t_{s+p}+0)\| \\
&\geqq \|y(\xi_R)\| - \|y(t)\|.
\end{aligned}$$

Therefore,

$$\varepsilon_0 - \|y(t)\| \leqq \|y(\xi_R)\| - \|y(t)\| \leqq \int_t^{\xi_R} \frac{d}{d\tau} \|y(\tau)\| \, d\tau \leqq \frac{\varepsilon_0}{2},$$

whence we obtain that $\|y(t)\| \geqq \frac{\varepsilon_0}{2}$.

If we choose a suitable subsequence of the sequence $\{\xi_k\}$ (which we again denote by $\{\xi_k\}$), we can assume that the intervals T_k do not intersect each other and $t_0 < \xi_1 - \frac{\varepsilon_0}{2L}$.

Then from (6) we deduce that $\dot V_{(1)}(t, x(t), y(t)) \leqq -c(\frac{\varepsilon_0}{2})$ in the intervals T_k and $\dot V_{(1)}(t, x(t), y(t)) \leqq 0$ for the remaining values of t for which $(t, x(t), y(t)) \in \bigcup_1^\infty G_i$. Integrating and applying (3) we obtain

$$V(\xi_k, x(\xi_k), y(\xi_k)) \leqq V(t_0+0, x_0, y_0) - c\left(\frac{\varepsilon_0}{2}\right)\frac{\varepsilon_0}{L}k \to -\infty$$

for $k \to \infty$ which contradicts (5). Hence $\lim_{t\to\infty} y(t; t_0, x_0, y_0) = 0$.

Next we shall show that $w(t) = W(t, x(t)) \to 0$ for $t \to \infty$.

Applying (9) we obtain

$$\dot W_{(1)}(t, x) \leqq \dot W_{(2)}(t, x) + d\,\|g(t, y)\|$$

for $t \in I$, $x \in \mathbf{R}^n_H$, $y \in \mathbf{R}^m_H$, $t \neq \tau_i(x, y)$, $i = 1, 2, \ldots$ whence, by (8) and A3, we have

$$\dot W_{(1)}(t, x(t)) \leqq -c_1(W(t, x(t)) + dP(\|y(t)\|)) \tag{10}$$

for $t \neq \tau_i(x(t), y(t))$, $i = 1, 2, \ldots$.

We set $\lim_{t\to\infty} \sup w(t) = \alpha$, $\lim_{t\to\infty} \inf w(t) = \beta$. If we assume that $\alpha > \beta$, then for an arbitrarily small number $\mu > 0$ we can find sequences $q_n > p_n \to \infty$ for $n \to \infty$ such that $w(p_n) = \beta + \mu$, $w(q_n) = \alpha - \mu$ and $\beta + \mu < w(t) < \alpha - \mu$ for $p_n < t < q_n$.

Since the function P is continuous, $P(0) = 0$ and $\lim_{t\to\infty} y(t) = 0$, then there exists a positive integer ν such that for $n \geqq \nu$ and $t \geqq p_n$ the following inequality holds

$$P(\|y(t)\|) \leqq \frac{c_1(\beta+\mu)}{d}$$

Then from (10) we have

$$\dot W_{(1)}(t, x(t)) \leqq -c_1(\beta+\mu) + d\frac{c_1(\beta+\mu)}{d} = 0$$

for $n \geqq \nu$ and $t \in (p_n, q_n)$, $t \neq \tau_i(x(t), y(t))$, $i = 1, 2, \ldots$ which together with (4) yields $w(p_n) \geqq w(q_n)$.

Hence $\beta + \mu \geqq \alpha - \mu$ which contradicts the assumption that $\alpha > \beta$. This shows that the limit $\lim_{t\to\infty} W(t, x(t)) = \sigma \geqq 0$ exists.

If we assume now that $\sigma > 0$, then we can find a number $T > 0$ such that the following inequalities hold:

$$\frac{\sigma}{2} \leqq W(t, x(t)) \leqq \frac{3\sigma}{2},$$
$$P(\|y(t)\|) \leqq \frac{1}{2d}c_1\left(\frac{\sigma}{2}\right),$$

for all $t \geqq T$. Then, applying again (10), we obtain

$$\dot{W}_{(1)}(t, x(t)) \leqq -c_1\left(\frac{\sigma}{2}\right) + d \cdot \frac{1}{2d} c_1\left(\frac{\sigma}{2}\right) = -\frac{1}{2} c_1\left(\frac{\sigma}{2}\right) < 0$$

for $t \geqq T$, whence, in virtue of (4), by integration we get

$$W(t, x(t)) \leqq W(T, x(T)) - \frac{1}{2} c_1\left(\frac{\sigma}{2}\right)[t - T] \to -\infty \quad \text{for} \quad t \to \infty$$

which contradicts (7).

Hence $\lim_{t\to\infty} W(t, x(t; t_0, x_0, y_0)) = 0$, whence by (7) we obtain that $\lim_{t\to\infty} x(t; t_0, x_0, y_0) = 0$. Having in mind that $\lim_{t\to\infty} y(t; t_0, x_0, y_0) = 0$ we conclude that the zero solution of system (1) is attractive. This completes the proof of Theorem 1. ∎

THEOREM 2. *Let the conditions of Theorem 1 be satisfied and suppose there exists a function $C \in K$ such that*

(11) $$V(t + 0, x, y) \leqq C(\|x\| + \|y\|), \ (t, x, y) \in I \times \mathbf{R}^n_H \times \mathbf{R}^m_H$$

Then the zero solution of system (1) is uniformly stable and equi-asymptotically stable.

The proof of Theorem 2 is analogous to the proof of Theorem 1.

The method that was applied can be used for the investigation of the global stability of the zero solution of systems with impulses. We shall give only the formulations of the assertions since the proofs are analogous to the proofs of Theorem 1 and Theorem 2.

Next we consider a system of the form (1) for which

$$f: I \times \mathbf{R}^n \to \mathbf{R}^n, \ g: I \times \mathbf{R}^m \to \mathbf{R}^n, \ h: I \times \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^m, \ A_i: \mathbf{R}^n \to \mathbf{R}^n,$$
$$B_i: \mathbf{R}^m \to \mathbf{R}^n, \ C_i: \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^m, \ \tau_i: \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}.$$

DEFINITION 4: The zero solution of system (1) is called

a) *globally attractive* if
$(\forall \alpha > 0)(\forall \varepsilon > 0)(\forall t_0 \in I)(\forall (x_0, y_0) \in \mathbf{R}^n \times \mathbf{R}^m, \ \|x_0\| + \|y_0\| \leqq \alpha)$
$(\exists \sigma = \sigma(t_0, x_0, y_0, \alpha, \varepsilon) > 0)(\forall t \geqq t_0 + \sigma)$: $\|x(t\ t_0, x_0, y_0)\| + \|y(t; t_0, x_0, y_0)\| < \varepsilon$;

b) *globally equi-attractive* if the number σ from a) does not depend on $(x_0, y_0) \in \mathbf{R}^n \times \mathbf{R}^m$;

c) *globally asymptotically stable* if it is stable and globally attractive;

d) *globally equi-asymptotically stable* if it is stable and globally equi-attractive.

We shall say that conditions (C) are satisfied if the following conditions hold:

C1. the functions $f(t,x)$, $g(t,y)$ and $h(t,x,y)$ are continuous in their domains of definition and $f(t,0)=0$, $g(t,0)=0$, $h(t,0,0)=0$ for $t\in I$;

C2. for each $H>0$ there exists a constant $L(H)>0$ such that $\|h(t,x,y)\|\leqq L(H)$ for $t\in I$, $x\in\mathbf{R}^n_H$, $y\in\mathbf{R}^m_H$;

C3. there exists a continuous function $P\colon I\to I$ such that $P(0)=0$ and $\|g(t,y)\|\leqq P(\|y\|)$ for $t\in I$, $y\in\mathbf{R}^m$;

C4. the functions A_i, B_i, C_i are continuous in their definition domains, $A_i(0)=0$, $B_i(0)=0$, $C_i(0,0)=0$;

C5. the functions $\tau_i(x,y)$, $i=1,2,\dots$ are continuous in $\mathbf{R}^n\times\mathbf{R}^m$ and the relations

$$0<\tau_1(x,y)<\tau_2(x,y)<\dots,\ \lim_{i\to\infty}\tau_i(x,y)=\infty$$

hold uniformly in $\mathbf{R}^n\times\mathbf{R}^m$, and

$$\inf_{\mathbf{R}^n\times\mathbf{R}^m}\tau_{i+1}(x,y)-\sup_{\mathbf{R}^n\times\mathbf{R}^m}\tau_i(x,y)\geqslant\theta>0,\ i=1,2,\dots$$

C6. for any $H>0$ from $x\in\mathbf{R}^n_H$ and $y\in\mathbf{R}^m_H$ it follows that $y+C_i(x,y)\in\mathbf{R}^m_H, i=1,2,\dots$.

THEOREM 3. *Let the following conditions hold:*

1. conditions (B) and (C);

2. there exists a function $V\in\mathcal{V}_0$ defined in $I\times\mathbf{R}^n\times\mathbf{R}^m$ and functions $a,c\in\mathcal{K}$ such that

$$a(\|x\|+\|y\|)\leqq V(t,x,y),\ (t,x,y)\in I\times\mathbf{R}^n\times\mathbf{R}^m$$

$$\dot V_{(1)}(t,x,y)\leqq -c(\|y\|)\ \text{for}\ (t,x,y)\in\bigcup_1^\infty G_i$$

and $a(r)\to\infty$ for $r\to\infty$;

3. there exists a function $W\in\mathcal{W}_0$ defined in $I\times\mathbf{R}^n$ and functions $a_1,c_1\in\mathcal{K}$ such that

$$a_1(\|x\|)\leqq W(t,x),\ (t,x)\in I\times\mathbf{R}^n$$

$$\dot W_{(2)}(t,x)\leqq -c_1(W(t,x)),\ (t,x)\in\bigcup_1^\infty\Omega_i$$

and for each $H>0$ there exists a constant $d(H)>0$ such that

$$|W(t,x_1)-W(t,x_2)|\leqq d(H)\|x_1-x_2\|,\ t\in I,\ x_1,\ x_2\in\mathbf{R}^n_H.$$

Then the zero solution of system (1) is globally asymptotically stable.

THEOREM 4. *Let the conditions of Theorem 3 be satisfied and suppose there exists a function* $C \in \mathcal{K}$ *such that*

$$V(t+0,x,y) \leqq C(\|x\|+\|y\|),\ t \in I,\ x \in \mathbf{R}^n,\ y \in \mathbf{R}^m.$$

Then the zero solution of system (1) is globally equi-asymptotically stable.

Example 1. Consider the system

$$(12)\qquad \begin{cases} \dot{x} = ax + \varphi(t)y \\ \dot{y} = Cx + f(t,y),\ t \neq t_i \\ \triangle x/_{t=t_i} = I_i(x(t_i)),\ \triangle y/_{t=t_i} = P_i(y(t_i)) \end{cases}$$

where $x, y \in \mathbf{R}$, the functions $f\colon I \times \mathbf{R} \to \mathbf{R}$, $\varphi\colon I \to \mathbf{R}$, I_i, $P_i\colon \mathbf{R} \to \mathbf{R}$ are continuous, $f(t,0) \equiv 0$, $I_i(0) = P_i(0) = 0$, $0 < t_1 < t_2 \ldots$ and $\lim\limits_{i\to\infty} t_i = \infty$.

Consider also the system

$$(13)\qquad \begin{cases} \dot{x} = ax,\ t \neq t_i \\ \triangle x/_{t=t_i} = I_i(x(t_i)) \end{cases}$$

Let

$$V(t,x,y) = (Cx - ay)^2 + 2\int_0^y [af(t,u) - C\varphi(t)u]du$$
$$W(t,x) = x^2$$

Then

$$\dot{V}_{(14)}(t,x,t) = -2[C\varphi(t) - a\frac{f(t,y)}{y}][a + \frac{f(t,y)}{y}]y^2 - C\dot{\varphi}(t)y^2 + 2\int_0^y af_t(t,u)du \quad \text{for} \quad t \neq t_i$$
$$\dot{W}_{(15)}(t,x) = 2ax^2 \quad \text{for} \quad t \neq t_i.$$

Let the following conditions hold:

a) $a < 0$;

b) the function $\varphi(t)$ is bounded in I and for each $H > 0$ we have $\sup\{|f(t,y)| : t \in I,\ |y| \leqq H\} < \infty$;

c) $a\frac{f(t,y)}{y} - C\varphi(t) \geqq \alpha(y) > 0$ for $t \in I$, $y \neq 0$, where $\alpha\colon \mathbf{R} \to [0,\infty)$ is continuous and $\alpha(0) = 0$;

d) $2[C\varphi(t) - \frac{f(t,y)}{y}][a + \frac{f(t,y)}{y}]y^2 - C\dot{\varphi}(t)y^2 - 2\int_0^y af_t(t,u)du \geqq \beta(y) > 0$ for $t \in I$, $y \neq 0$, where the function $\beta\colon \mathbf{R} \to [0,\infty)$ is continuous and $\beta(0) = 0$;

e) $[aP_i(y) - CI_i(x)][ay - Cx + \frac{1}{2}(aP_i(y) - CI_i(x))] \leqq 0$, $i = 1, 2, \dots$, $\int_y^{y+P_i(y)}[af(t,u) - C\varphi(t)u]du \leqq 0$

Then the conditions of Theorem 2 hold. Hence the zero solution of system (12) is uniformly stable and equi-asymptotically stable.

If, moreover, the following condition holds

f) $\int_0^y \alpha(u)udu \to \infty$ for $y \to \infty$,

then by Theorem 4 the zero solution of system (12) is globally equi-asymptotically stable.

References

[1] A.B. Dishliev and D.D. Bainov, 'Sufficient conditions for absence of "beating" in systems of differential equations with impulses', *Applicable Annal.* **18** (1985), 67–73.

[2] A.B. Dishliev and D.D. Bainov, 'Conditions for the absence of the phenomenon "beating" for systems of impulse differential equations', *Institute of Mathematics, Academia Sinica* **13** (1985), 237–256.

[3] S. Leela, 'Stability of measure differential equations', *Pacific J. Math.* **55** (1977), 489–498.

[4] S. Leela, 'Stability of differential systems with impulsive perturbations in terms of two measures', *Nonlinear Anal.* **1** (1977), 667–677.

[5] M. Marachkov, 'On a theorem on stability', *Bull. Soc. Phys. Math. Kazan* **12** (1940), 171–174. (in Russian) .

[6] A.D. Myshkis and A.M. Samoilenko, 'Systems with impulses in prescribed moments of the time', *Mat. Sb.* **74** (1967), 202-208. (in Russian).

[7] S.G. Pandit, 'On the stability of impulsively perturbed differential systems', *Bull. Austral. Math. Soc.* **17** (1977), 423–432.

[8] T. Pavlidis, 'Stability of systems described by differential equations containing impulses', *IEEE Trans. Automat. Control* **2** (1967), 43–45.

[9] M. Rama Mohana Rao and V. Sree Hari Rao, 'Stability of impulsively perturbed systems', *Bull. Austral. Math. Soc.* **16** (1977), 99–110.

[10] A.M. Samoilenko and N.A. Perestjuk, 'Stability of the solutions of differential equations with impulse effect', *Differentsial'niye Uravneniya* **11** (1977), 1981–1992. (in Russian) .

[11] P.S. Simeonov and D.D. Bainov, 'The second method of Liapunov for systems with impulse effect', *Tamkang J. Math.* **16** (1985), 19–40.

[12] P.S. Simeonov and D.D. Bainov, 'Stability with respect to part of the variables in systems with impulse effect', *J. Math. Anal. Appl.* **117** (1986), 247–263.

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