AN EXCLUSION THEOREM FOR TRI-DIAGONAL MATRICES

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An $n \times n$ matrix $A_n = (a_{ij})$ is *tri-diagonal* if $a_{ij} = 0$ for $|i-j| \geqq 2$. The latent roots of such matrices may be conveniently studied by forming the sequence of polynomials $\Psi_k(\lambda) = |\lambda I - A_k|$, where A_k is the principal sub-matrix of A_{k+1} obtained by deleting the last row and column of A_{k+1}, and then observing that these polynomials satisfy the following recurrence relation:

$$\Psi_0(\lambda) = 1, \quad \Psi_1(\lambda) = \lambda - a_{11},$$

$$\Psi_{k+1}(\lambda) = (\lambda - a_{k+1,\,k+1})\Psi_k(\lambda) - a_{k+1,\,k}a_{k,\,k+1}\Psi_{k-1}(\lambda),$$
$$k = 1, 2, \ldots, n-1.$$

In (**1**) Arscott obtained the following result. See also (**2**, p. 166) and the review of (**1**) in *Mathematical Reviews*, **25**, p. 407.

If A_n is real and the products $a_{k+1,\,k}a_{k,\,k+1}$ are all negative, then the real latent roots of A_n lie between the least and the greatest of the diagonal elements a_{ii}, these values included.

In his proof he first establishes that if the diagonal elements are all positive, the real latent roots are all positive. This observation, plus consideration of certain examples, prompts one to conjecture that more generally (when each $a_{k+1,\,k}a_{k,\,k+1} < 0$) the real parts of all the latent roots are positive, negative or zero if all the diagonal elements are respectively positive, negative or zero. We shall establish this result and then use it to generalize Arscott's theorem. This generalization is an exclusion theorem in Householder's terminology (**3**, p. 39); i.e., it specifies a region containing all the latent roots. When this region is intersected with that specified by Gershgorin's Theorem, an improved region is obtained.

Following Arscott's notation we denote the non-zero elements of our tri-diagonal matrix $A_n = (a_{ij})$ by $a_{ii} = b_i$, $i = 1, \ldots, n$; $a_{i,\,i+1} = -c_i$, $a_{i+1,i} = a_{i+1}$, $i = 1, \ldots, n-1$.

Thus the recurrence relation is

$$\Psi_0(\lambda) = 1, \quad \Psi_1(\lambda) = \lambda - b_1, \tag{1}$$

$$\Psi_{k+1}(\lambda) = (\lambda - b_{k+1})\Psi_k(\lambda) + a_{k+1}c_k\Psi_{k-1}(\lambda), \; k = 1, \ldots, n-1.$$

We shall make use of the identity

$$\Psi_{k+1}(\lambda)\overline{\Psi}_k(\lambda) + \overline{\Psi}_{k+1}(\lambda)\Psi_k(\lambda) \equiv (2\,\mathrm{Re}\,(\lambda) - 2b_{k+1})|\Psi_k(\lambda)|^2$$
$$+ a_{k+1}c_k(\Psi_k(\lambda)\overline{\Psi}_{k-1}(\lambda) + \overline{\Psi}_k(\lambda)\Psi_{k-1}(\lambda)), \tag{2}$$

$k = 1, \ldots, n-1$, which is easily established from (1). Here $\text{Re}(\lambda)$ denotes the real part of λ; $\overline{\Psi}_k$ is the complex conjugate of Ψ_k. Note that if λ is a zero of Ψ_{k+1}, the left side of (2) is zero, since λ is also a zero of $\overline{\Psi}_{k+1}$.

Theorem 1. *Suppose that in* (1) $a_{k+1}c_k>0$, $k = 1, \ldots, n-1$. *Then all the zeros of each* $\Psi_k(\lambda)$ *have negative, positive or zero real parts if the* b_k *are respectively all negative, all positive or all zero,* $k = 1, \ldots, n$.

Proof. Suppose first that each $b_k<0$. If $\text{Re}(\lambda) \geqq 0$, it then follows by induction on (2) that

$$\Psi_{k+1}(\lambda)\overline{\Psi}_k(\lambda)+\overline{\Psi}_{k+1}(\lambda)\Psi_k(\lambda)>0,\ k = 1, \ldots, n-1.$$

Thus no λ satisfying $\text{Re}(\lambda) \geqq 0$ can be a zero of $\Psi_{k+1}(\lambda)$, $k = 1, \ldots, n-1$, for the left side of (2) is different from zero for each k.

Suppose next that each $b_k>0$. If $\text{Re}(\lambda) \leqq 0$, it now follows by induction on (2) that

$$\Psi_{k+1}(\lambda)\overline{\Psi}_k(\lambda)+\overline{\Psi}_{k+1}(\lambda)\Psi_k(\lambda)<0,\ k = 1, \ldots, n-1.$$

Hence no λ satisfying $\text{Re}(\lambda) \leqq 0$ can be a zero of $\Psi_k(\lambda)$, $k = 1, \ldots, n$.

Finally, suppose each $b_k = 0$. A proof that the zeros of $\Psi_k(\lambda)$ all have zero real parts can also be based on (2), but the following approach yields more information. Consider the sequence defined by

$$\theta_0(t) = 1, \quad \theta_1(t) = t,$$

$$\theta_{k+1}(t) = t\theta_k(t)-a_{k+1}c_k\theta_{k-1}(t),\ k = 1, \ldots, n-1.$$

A well-known theorem asserts that the zeros of each θ_k are real and simple and they interlace those of θ_{k+1}, since $a_{k+1}c_k>0$. Now let $\lambda = it$ ($i = \sqrt{-1}$). It is easy to verify by induction that under this change of variable

$$\Psi_k(\lambda) = i^k\theta_k(t),$$

where $\Psi_k(\lambda)$ is generated by (1) with each $b_k = 0$. Hence t^* is a zero of $\theta_k(t)$ if and only if $\lambda^* = it^*$ is a zero of $\Psi_k(\lambda)$. Consequently for each k the zeros of $\Psi_k(\lambda)$ all lie on the imaginary axis, they are simple and interlace those of $\Psi_{k+1}(\lambda)$.

Theorem 2. *Suppose that in* (1) $a_{k+1}c_k>0$, $k = 1, \ldots, n-1$; b_k *is real,* $k = 1, \ldots, n$. *Let* b_m, b_M *denote respectively the smallest and largest of the* b_k. *Then all the zeros of each* $\Psi_k(\lambda)$ *lie in the strip* $b_m \leqq \text{Re}(\lambda) \leqq b_M$.

Proof. Choose arbitrary $\varepsilon>0$ and set $t = \lambda-(b_m-\varepsilon)$, $\beta_k = b_k-(b_m-\varepsilon)$, $k = 1, \ldots, n$. Consider the recurrence relation

$$\theta_0(t) = 1, \quad \theta_1(t) = t-\beta_1,$$

$$\theta_{k+1}(t) = (t-\beta_{k+1})\theta_k(t)+a_{k+1}c_k\theta_{k-1}(t),\ k = 1, \ldots, n-1.$$

Under the change of variable $t = \lambda-(b_m-\varepsilon)$ it is easy to verify that $\theta_k(t) = \Psi_k(\lambda)$ where $\Psi_k(\lambda)$ is generated by (1). Since each $\beta_k>0$, it follows from Theorem 1 that the zeros of each $\theta_k(t)$ all have positive real parts. But t^* is a zero of some $\theta_k(t)$ if and only if $\lambda^* = t^*+(b_m-\varepsilon)$ is a zero of $\Psi_k(\lambda)$. Hence

$$\text{Re}(\lambda^*)-(b_m-\varepsilon) = \text{Re}(t^*)>0$$

for every $\varepsilon > 0$, and so $b_m \leqq \text{Re}(\lambda^*)$.

If we next set $s = \lambda - (b_M + \varepsilon)$, $\gamma_k = b_k - (b_M + \varepsilon)$, $k = 1, \ldots, n$, and consider the sequence

$$\theta_0(s) = 1, \quad \theta_1(s) = s - \gamma_1,$$

$$\theta_{k+1}(s) = (s - \gamma_{k+1})\theta_k(s) + a_{k+1}c_k\theta_{k-1}(s), \; k = 1, \ldots, n-1,$$

we can show in an analogous fashion that $\text{Re}(\lambda^*) \leqq b_M$, since each $\gamma_k < 0$.

The Gershgorin disks for the matrix A_n have their centres on the real line at $b_1, b_2, \ldots, b_n$. In particular, the disks with centres at b_m, b_M do not lie entirely within the strip $b_m \leqq \text{Re}(\lambda) \leqq b_M$; the following theorem is thus an immediate consequence.

Theorem 3. *Let G denote the union of the Gershgorin disks for the tri-diagonal matrix A_n whose elements satisfy the hypotheses of Theorem 2. Let S denote the strip $b_m \leqq \text{Re}(\lambda) \leqq b_M$. Then $S \cap G$ is properly contained in both G and S and contains all the latent roots of A_n. In this sense $S \cap G$ is an improvement over both S and G.*

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