line error is better approximated by an exponential rather than a gaussian distribution.

The first lesson of Anderson and Ellis's paper, an old one, but one that it is valuable to reassert strongly at all times, is that the gaussian distribution is inappropriate in a great many cases that occur in navigational practice.[11] It is the attempt to replace this distribution by a particular two-parameter family that I find irksome and restrictive. If the proposed form is put forward simply as an aid to understanding the structure of navigational distributions that do not confirm to the gaussian law, the size of the parameter α indicating the closeness or otherwise of the distribution to normal (large α—near normal; low α—much longer tailed, see Fig. 7 of ref. 1), it may help people interpret the nature of their data. But for general purposes I would favour, in navigational work, replacing the gaussian distribution, where it is patently inadequate, not by a multi-parameter family of distributions, however elegant, but by an open mind. Let the data speak for themselves, rather than subject them to a two-parameter strait jacket!

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Gaussian Logarithms and Navigation

D. H. Sadler

THE following comments on Captain C. H. Cotter's note (this *Journal*, Vol. 24, page 569) on the use of addition and subtraction logarithms in navigation may be of interest. Captain C. Carić was not the first to introduce gaussian logarithms to navigators, since their use was advocated, at least to Portuguese navigators, in

https://doi.org/10.1017/S0373463300022657 Published online by Cambridge University Press

1920 by Admiral Gago Coutinho (see my review of *Precision Astrolabe* by Francis M. Rogers, **25**, 135), and actually used by him on his pioneering transatlantic flight in 1922. They are included in several collections of nautical tables, including *Tábuas Náuticas* by Fontoura and Coutinho,* as well as those of Friocourt and Hoüel; but I have been unable to obtain the dates of their first inclusion. However, it is most unlikely that either Coutinho or Carić, who incidentally used quite different basic formulae, were the first to advocate the use of addition and subtraction logarithms, or possibly to publish methods and tables. They were in use to a small extent in astronomical calculations, and to a somewhat larger extent in surveying. Dr. L. J. Comrie, the computer and table-maker, who was Superintendent of H.M. Nautical Almanac Office (N.A.O.) from 1930 to 1936, was an enthusiastic advocate and designed many computational forms incorporating their use.

Although I must have seen Carić's tables when the N.A.O. library acquired its copy of the Italian edition shortly after the war, I had completely forgotten that he had made use of addition and subtraction logarithms. H. Bencker's 'Regimen of the Sea or Nautical Compendium' (*The Hydrographic Review*, Vol. XX, pages 91–170, 1943) gives only the barest information about them (and about the *Tábuas Náuticas*). The N.A.O. copy of Carić's tables, however, carries the following annotation (made before acquisition), undated and in an unknown hand:

> 'I tried to get published for yachtsmen Tables V and VI (and XXIV for "longitude method") + some Az. table shorter than X. Total 16 pages, incl. book of words.'

(Table V is the main table of log $\sin^2 \frac{1}{2}x$ and log $\cos^2 \frac{1}{2}x$; VI is addition logarithms, and XXIV subtraction logarithms; X is an *A*, *B*, *C* azimuth table.)

There is one other curious point. In addition to his articles referred to by Cotter, H. B. Goodwin published an article in *The Nautical Magazine* for July 1923 on 'New Notions in the Long Distance Navigation of the Air'. In this article he refers to the use by Coutinho of Hoüel's tables without apparently realizing that these were tables of addition and subtraction logarithms.

Perhaps I might be permitted to comment on the use of the words 'invention', 'first introduction', &c. in connection with the various formulae for the solution of the standard navigational spherical triangle? We are told that the fundamental cosine formula was discovered by Albatani about A.D. 900, and there can be no doubt that, by say 1880—after the publication and appreciation of the tables by J. T. Towson and Sir William Thomson—*all* essential principles and relationships concerning the spherical triangle were well-known. There are, however, well over 600 references in Bencker's 'Regimen' for the years between 1880 and 1940; some are duplications (different editions), some are not directly relevant to the solution of the spherical triangle, but to balance these the list is far from complete. It would be unrealistic to expect designers of nautical tables to study the published literature in order to be able to give references to prior use of their formulae and methods but, equally, an independent 'discovery' has an infinitesimal chance of priority. In designing tables for the practical solution of the navigational triangle, the formulae and methods used are only a small part of the

* My thanks are due to Captain Cotter for pointing out that Vitor Hugo de Azevedo Coutinho is not Gago Coutinho referred to earlier.

complete whole; excellent basic formulae (such as Carić's), can be ruined by poor presentation (why not use a parallel arrangement?), poor printing and the incidence of errors; poor basic formulae (e.g. the sine formulae) can sometimes be used effectively by the use of sophisticated tabulation techniques. Moreover, there can only be marginal advantages between different methods, which the navigator's natural preference for the method he was taught easily overrides.

Having written the above, I now point out, *without any claim for priority*, that separate tabulations of gaussian logarithms are often unnecessary in calculations involving trigonometrical functions, in particular the solution of the navigational spherical triangle. One of the standard solutions, given two sides and the included angle, is that provided by Delambre's analogies (which date from about 1806):

$$\begin{aligned}
\sin \tfrac{1}{2}z \sin \tfrac{1}{2}(A-C) &= \sin \tfrac{1}{2}(\varphi-\delta) \cos \tfrac{1}{2}P = k \\
\sin \tfrac{1}{2}z \cos \tfrac{1}{2}(A-C) &= \cos \tfrac{1}{2}(\varphi+\delta) \sin \tfrac{1}{2}P = l \\
\cos \tfrac{1}{2}z \sin \tfrac{1}{2}(A+C) &= \cos \tfrac{1}{2}(\varphi-\delta) \cos \tfrac{1}{2}P = m \\
\cos \tfrac{1}{2}z \cos \tfrac{1}{2}(A+C) &= \sin \tfrac{1}{2}(\varphi+\delta) \sin \tfrac{1}{2}P = n
\end{aligned}$$

in which: φ, δ are latitude and declination; P, z are used for hour angle and zenith distance (as in Cotter's note); and A, C are the azimuth and parallactic angles.

Clearly:

$$\begin{aligned}
\sin \tfrac{1}{2}z &= (k^2+l^2)^{\frac{1}{2}} = k \operatorname{cosec} \tfrac{1}{2}(A-C) = l \sec \tfrac{1}{2}(A-C) \\
\cos \tfrac{1}{2}z &= (m^2+n^2)^{\frac{1}{2}} = m \operatorname{cosec} \tfrac{1}{2}(A+C) = n \sec \tfrac{1}{2}(A+C)
\end{aligned}$$

with

$$\tan \tfrac{1}{2}(A-C) = k/l, \qquad \tan \tfrac{1}{2}(A+C) = m/n$$

For logarithmic calculation, put:

$$\begin{aligned}
K &= -\log k = \mathrm{S}(\varphi-\delta) + \mathrm{C}(P) \\
L &= -\log l = \mathrm{C}(\varphi+\delta) + \mathrm{S}(P) \\
M &= -\log m = \mathrm{C}(\varphi-\delta) + \mathrm{C}(P) \\
N &= -\log n = \mathrm{S}(\varphi+\delta) + \mathrm{S}(P) \\
\mathrm{T}(A-C) &= K-L, \text{ giving } (A-C) \\
\mathrm{T}(A+C) &= M-N, \text{ giving } (A+C)
\end{aligned}$$

where the functions $\mathrm{S}(\theta)$, $\mathrm{C}(\theta)$ and $\mathrm{T}(\theta)$ are used as abbreviations for $-\log \sin \tfrac{1}{2}\theta$, $-\log \cos \tfrac{1}{2}\theta$ and $-\log \tan \tfrac{1}{2}\theta$; it will be noted that $T(\theta) = \mathrm{S}(\theta) - \mathrm{C}(\theta)$.

Then:

$$\begin{aligned}
\mathrm{S}(z) &= K - \mathrm{S}(A-C) = L - \mathrm{C}(A-C) \\
\mathrm{C}(z) &= M - \mathrm{S}(A+C) = N - \mathrm{C}(A+C)
\end{aligned}$$

A single table, giving $\mathrm{S}(\theta)$, $\mathrm{C}(\theta)$, $\mathrm{T}(\theta)$ with argument θ, is all that is necessary for the complete (well-determined and unambiguous) solution for all three unknown elements of the triangle. For the complete solution the following steps are required:

1. Enter the table with arguments P, $\varphi-\delta$ and $\varphi+\delta$ and take out the six quantities $\mathrm{S}(P)$, $\mathrm{C}(P)$; $\mathrm{S}(\varphi-\delta)$, $\mathrm{C}(\varphi-\delta)$ and $\mathrm{S}(\varphi+\delta)$, $\mathrm{C}(\varphi+\delta)$.
2. Form K, L, M, N by addition, and $(K-L)$, $(M-N)$ by subtraction.

3. Enter the tables with $(K-L)$, $(M-N)$ in the column $T(\theta)$ and take out the corresponding values of θ, $S(\theta)$ and $C(\theta)$, for $\theta=(A-C)$ and $(A+C)$.
4. Form $S(z)$ and $C(z)$ by subtraction.
5. Enter the tables with $S(z)$ and/or $C(z)$ and take out z.
6. Form A and C from $(A-C)$ and $(A+C)$.

Although there is considerable duplication only six table entries (the absolute minimum) are required in a single table, to provide zenith distance (or altitude), azimuth and parallactic angles. The duplication provides a check on the arithmetic, but some of it can be avoided, particularly if only the zenith distance is wanted. In this case K, L only are required, the table is entered with $(K-L)$ in column $T(\theta)$ to give $S(A-C)$ or $C(A-C)$ from which $S(z)$ is formed to give z from the table.

This procedure is identical in principle with Carić's method as described by Cotter. Carić uses:

$$\log a=-2L, \qquad \log b=-2K$$
$$\log \sin^2 \tfrac{1}{2}z=\log a+\text{add. }\log(\log a-\log b)$$

as compared with

$$\log \text{cosec } z=L-\log \sec \tfrac{1}{2}(A-C)$$

where

$$-\log \tan \tfrac{1}{2}(A-C)=K-L=\tfrac{1}{2}(\log a-\log b)$$

The use of a separate table of addition logarithms is avoided by appeal to the relationships between $S(\theta)=\log \text{cosec } \frac{1}{2}\theta$, $C(\theta)=\log\sec \frac{1}{2}\theta$ and $T(\theta)=\log \cot \frac{1}{2}\theta$; these relationships take the form:

$$S(\theta_1)=\tfrac{1}{2}\log(1+x_1) \quad \text{where} \quad \tfrac{1}{2}\log x_1=T(\theta_1)$$
$$C(\theta_2)=\tfrac{1}{2}\log(1+x_2) \quad \text{where} \quad \tfrac{1}{2}\log x_2=-T(\theta_2)$$

and

$$T(\theta_3)=\tfrac{1}{2}\log(y_3-1) \quad \text{where} \quad \tfrac{1}{2}\log y_3=S(\theta_3)$$
$$-T(\theta_4)=\tfrac{1}{2}\log(y_4-1) \quad \text{where} \quad \tfrac{1}{2}\log y_4=C(\theta_4)$$

and

$$-S(\theta_5)=\tfrac{1}{2}\log(1-x_5) \quad \text{where} \quad \tfrac{1}{2}\log x_5=-C(\theta_5)$$
$$-C(\theta_6)=\tfrac{1}{2}\log(1-x_6) \quad \text{where} \quad \tfrac{1}{2}\log x_6=-S(\theta_6)$$

in which, in all cases, $S(\theta)-C(\theta)=T(\theta)$. Between them (and their many variants) they provide in principle all that could be required of gaussian logarithms for both addition and subtraction. The function $T(\theta)$ is not required if subtraction logarithms only are needed. The factor of $\frac{1}{2}$ arises because the more natural function $S(\theta)=\log \text{cosec } \frac{1}{2}\theta$ has been used instead of $\log \text{cosec}^2 \frac{1}{2}\theta$; it introduces undesirable complications unless x or y can be expressed as a square of products of trigonometrical functions. This can clearly be done with Carić's method, but not with, for example, the standard form of the cosine formula for altitude as in Coutinho's method. (See final paragraph.)

It must be emphasized that the method of solution is the standard one used universally with Delambre's analogies; no question of addition arises, as occurs when the altitude formula is used alone. But clearly the above relationships can be used instead of addition and subtraction logarithms even when the angle-argument is not itself required; suitable parallel tabulations (but usually of log cosec θ instead of log cosec $\frac{1}{2}\theta$) of the logarithms of trigonometrical functions are commonplace in collections of mathematical and nautical tables. Although convenient five-figure tables are available in, for example, Bowditch's Table 33, I have used those in Milne-Thomson and Comrie's *Standard Four-Figure Mathematical Tables* for the following complete solution of the numerical example given by Carić and illustrated by Cotter. The table entries have been adjusted (by taking complements &c.) and are in units of the fourth decimal.

	$\varphi-\delta$	20° 53′	a	$\varphi+\delta$	6° 07′	a
φ − 13° 30′ a	S	7417	a	S	12,728	b
δ + 7° 23′ a	C	72	b	C	6	a
P 63° 56′.5 a	C(P)	714	a	S(P)	2762	a
	K	8131	a	M	786	b
	L	2768	a	N	15,490	b
	$(K-L)$	+5363	a	$(M-N)$	−14,704	b
	S$(A-C)$	5540		S$(A+C)$	2	
	C$(A-C)$	176	a	C$(A+C)$	14,706	
	$(A-C)$	32° 26′	b	$(A+C)$	176° 08′	b
	$K-$S$(A-C)$	2591		$M-$S$(A+C)$	784	
	$L-$C$(A-C)$	2592	a	$N-$C$(A+C)$	784	
	z	66°50′	a	z	66° 48′	
	A	104° 17′	b	C	71° 51′	
	Az.	284°.3	b			

The quantities marked 'a' correspond precisely to those used in Cotter's illustration for the zenith distance only, except that the logarithms are halved and the precision is that much lower. The quantities marked 'b' are the *additional* quantities required to determine the azimuth (for which Carić uses separate A, B, C tables); note that only one additional table entry is needed. The remaining quantities are required only as checks on the arithmetic, since C(z) is always a poorer function for determining z than S(z); but if the azimuth is determined then it is a trivial extension to find C(z) as a check.

The obvious symmetry, apparent simplicity and elegance of the above equations certainly do not imply that they form the basis for a practical solution of the navigational triangle. After all they, and the corresponding tables—though not in the optimum form—have been available for more than 150 years! A rapid survey suggests that a single table of 30 pages would provide adequately for most navigational requirements, though careful attention would have to be given to matters of detailed design, including procedures concerning quadrants and signs (note that $(\varphi-\delta)$ can be negative, simply allowed for by interchanging φ and δ). The equations have been deliberately written to make most quantities positive, but much further thought would be needed to transform them to the optimum form; for example, the function $-\log\sin\frac{1}{2}\theta$ can be replaced by $-p\log(q\sin\frac{1}{2}\theta)$, where p and q can be chosen to provide the most suitable working unit. Even so, the number of quantities to be written (12 for the altitude alone) is larger than for some other methods (the cosine-haversine requires a comparable 9) although the number of tables and table-entries is less.

I have been unable to find any reference, in either mathematical or nautical text-books or tables, to the *precise* use of these relationships as the equivalent of addition and subtraction logarithms. However, Bowditch's *American Practical Navigator* (1958 edition, page 530) refers to tables proposed by George W. D. Waller in 1946 (but not published because of Waller's death) in which addition and subtraction logarithms were to be incorporated into standard tables of $-\log \sin \theta$ and $-\log \cos \theta$ by the addition of columns of $\log (1+\sin \theta)$ and $-\log (1-\sin \theta)$. The tables were then to be used with the standard cosine formula using the same principle as Coutinho.

I shall be most surprised if such a simple transformation of a sum or difference to a product had not been used in the days before desk calculating machines superseded the general use of logarithms. It has since been widely used, with natural numbers, in transformations of rectangular to polar coordinates:

$$r=(x^2+y^2)^{\frac{1}{2}}=x \sec \theta=y \operatorname{cosec} \theta$$

where θ, $\sec \theta$ and $\operatorname{cosec} \theta$ are taken out from tables with argument (y/x) in the $\tan \theta$ columns. And such tables are often used to provide ready values of $(1+x^{\pm 2})^{\pm \frac{1}{2}}$ (for example: $\sin \theta=x$, $\cos \theta=(1-x^2)^{\frac{1}{2}}$; $\tan \theta=x$, $\sec \theta=(1+x^2)^{\frac{1}{2}}$) to which there are, in general, corresponding logarithmic forms.

The principle of using the relationships between the main tabulations as the equivalents of addition and subtraction logarithms cannot conveniently be used with the cosine formula for the altitude in its standard form; but this may be transformed into

$$\begin{aligned}\cos z &= \cos (\varphi-\delta)-\cos \varphi \cos \delta(1-\cos P)\\ &=(1-x) \cos(\varphi-\delta)\end{aligned}$$

In practical computational form this becomes

$$F(z)=F(\varphi-\delta)+F(\theta)$$

where

$$G(\theta)=F(\varphi)+F(\delta)-F(\varphi-\delta)+G(P)$$

and

$$F(\theta)=\log \sec \theta \quad \text{and} \quad G(\theta)=-\log(1-\cos \theta)$$

are the only two functions required. Another form is

$$\begin{aligned}\cos z &= -\cos(\varphi+\delta)+\cos \theta \cos \delta(1+\cos P)\\ &=(y-1) \cos(\varphi+\delta)\end{aligned}$$

requiring tabulations of the functions $\log \sec \theta$ and $-\log (1+\cos \theta)$. These forms are presumably the basis of Waller's method, and both require the tabulation of a non-standard function instead of (but not in addition to) a standard function.*

* Captain Cotter has kindly sent me a copy of the specification of Waller's method, from which it is clear that the above forms are not the basis of his method; as with Coutinho's form, which it closely resembles, it requires the tabulation of both addition and subtraction logarithms, However, he points out that the method introduced by Abel Fontoura de Costa in the first (1907) edition of Fontoura and Coutinho's *Tâbuas Náuticas* is essentially the same as the first of the two forms given. I have still not seen the *Tâbuas Náuticas*, but I now find that Rogers, in *Precision Astrolabe*, quotes the formulae as those in current use by Portuguese navigators!

The form

$$\sin^2 \tfrac{1}{2}z = \sin^2 \tfrac{1}{2}(\varphi - \delta) + \cos\theta \cos\delta \sin^2 \tfrac{1}{2}P$$
$$= (1 + x) \sin^2 \tfrac{1}{2}(\varphi - \delta)$$

requires a tabulation of log cos θ (which may be available for other purposes) in addition to log cosec² θ and log cot² $\tfrac{1}{2}\theta$. But there are many variants—and, in every case, details (such as the procedure when some quantities become very large) that require clarification before these methods can be used in practice.

International Charts

L. N. Pascoe

In his Presidential Address Admiral Ritchie[1] referred to the two series of charts in 79 small-scale sheets which 16 member states of the International Hydrographic Organization have undertaken to produce; some of them will be published this year. This new development, and how it will affect mariners who now use our Admiralty charts, may be of some interest.

The two small-scale charts for ocean and offshore navigation require a comparatively limited effort and with the enthusiasm and interest shown by all participating nations it is expected that both could be completed within a reasonable period, probably before the end of 1974. When these charts are published it will be possible to withdraw many small-scale Admiralty charts, some of which were originally published more than a century ago and have subsequently received only partial correction, and much criticism as being out-of-date. But it will not usually be possible to replace them chart for chart as each individual new International Chart is published, because replacement of existing charts can best be made when the whole scheme or at least a complete regional area has been completed. This applies not only to the United Kingdom but to all the world charting authorities and will result in their still having to maintain many existing charts until the new coverage is complete. This problem of 'block' replacement, rather than individual chart replacement, will assume serious proportions in the subsequent larger-scale stages of the International Chart.

For the small-scale series, symbolization is limited and the agreed specifications are relatively few, but it will be interesting to see what national variants arise, especially in the generalization of depth contours. When the first series has been completed, the saving of effort in recompilation and re-editing will be welcomed by all authorities who publish small-scale charts and 5-year or 10-year cyclic revision should be possible. The greatest international saving in effort will, however, be in approach and coastal charts and port and harbour plans, where much duplication exists at present and where continuous, heavy correctional maintenance is necessary.

The concept of the International Chart, which was once expected to involve facsimile reproduction by 'printer' nations, is now that of a modified facsimile, where most printer nations will necessarily translate the title and memoir into the national language and make other language alterations necessary to conform to national charting practice. It is also anticipated that printer nations will add

https://doi.org/10.1017/S0373463300022657 Published online by Cambridge University Press