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ON SPREADS ADMITTING PROJECTIVE LINEAR GROUPS

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Introduction. In [**8**] Jha raised the following problem.

(*) Let Γ be a spread whose components are subspaces of $V_{2n}(GF(q))$. Suppose $G \leqq \operatorname{Aut} \Gamma$ leaves a set of $q + 1$ components invariant while acting transitively on $\Gamma \backslash \Delta$.

Find the possibilities for Γ or, more generally, the possibilities for (G, Γ, n, q).

Many special cases of (*) have been settled. For instance, Cohen *et al* [**1**] have shown that if G fixes two non-zero points of V, that do not both lie in the same component of Γ, then Γ is the spread associated with either a Hall plane or the Lorimer–Rahilly plane of order 16 (LR-16) [**14**], [**18**].

Another such result is given in [**8**]; there it is shown that if q is a prime number and G is a one-dimensional projective unimodular group then Γ is the spread associated with one of the following translation planes:

(1) the Desarguesian planes of order 4, 8, or 9;

(2) the nearfield plane of order 9;

(3) LR-16;

(4) the translation plane JW-16, obtained by transposing the slope maps of LR-16 [**19**].

The purpose of this article is to study (*) when G is a group of type $PSL(n, w)$ when $n \geqq 3$. Our object will be to show that under these conditions Γ is the spread associated with LR-16 or JW-16 and G is $PSL(3, 2)$. Thus, in terms of translation planes, our result may be stated as follows.

THEOREM A. *Let π^{l_∞} be a translation plane with order $>q$ but which admits a group of kern homologies of order $q - 1$. Suppose further that the translation complement of π^{l_∞} admits a group G such that;*

(i) *G leaves invariant a set Δ of $q + 1$ slopes and acts transitively on $l_\infty \backslash \Delta$; and*

(ii) *$G \cong PSL(n, w)$ for some prime power w and some $n \geqq 3$.*

Then $G \cong PSL(3, 2)$ and π^{l_∞} is either LR-16 *or* JW-16.

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There are two main stages in the proof of the above theorem.

Stage 1 (Section 2). The object here is to show that it is impossible for q and w to be relatively prime. This is achieved by using lower bounds for the degrees of the projective representations of $L_n(w)$ obtained by Harris and Hering [**3**].

Stage 2 (Section 3). Having shown that q and w are both powers of the same prime p, we refine and extend Jha's arguments in [**9**] to show that G has a large enough planar p-group to force π^{l_∞} to be of order 16. A theorem of Johnson and Ostrom then yields the conclusions of Theorem A.

The authors would like to thank the referee who pointed out one glaring error and made several pertinent remarks.

0. Preliminaries. We assume familiarity with the basic theory of linear groups [**6**, Kapitel II] and translation planes [**5**], [**16**]. Below we discuss some results that are especially relevant to this work, beginning with projective linear groups $PSL(n, w)$.

Mitchell and Hartley [**15**], [**2**] have determined all maximal subgroups of $PSL(3, w)$. By considering the indexes of these groups one obtains

1. RESULT. *$PSL(3, w)$ has no non-trivial representation as a permutation group of degree* $< w^2 + w + 1$.

A similar result, that applies to $PSL(n, w)$, may be deduced from [**3**].

2. RESULT. *Suppose $G = PSL(n, w)$ for $n \geqq 3$ and assume $G \not\subset PSL(3, 4)$. Let K be any field such that* $(w, \text{char } K) = 1$.

Assume that there is a non-trivial group homomorphism

$$\rho: G \to GL(N, K).$$

Then,

(a) $N \geqq (w^{n-1} - 1)/(n, w - 1)$; *and*

(b) $N \geqq w^{n-1} - 1$ *provided that* $(n, w) \neq (3, w)$ *when w is a power of* 2.

Proofs. [**3**, Theorems 4.2 and 4.3]. Since a permutation representation of G on N letters gives a group homomorphism of G into $GL(N, K)$, we have

3. COROLLARY. *Assume the hypothesis of the above result. Then G acts as a non-trivial permutation group of N letters only if*

$$N \geqq (w^{n-1} - 1)/(n, w - 1).$$

Moreover if $(n, w) \neq (3, w)$, when w is even, then $N \geqq w^{n-1} - 1$.

Convention. If G is a permutation group, then G_X is the elementwise stabilizer of the set X. In particular, if Q is a quasifield with a subset K, then $(\text{Aut } Q)_K$ is the elementwise stabilizer of K in the full automorphism group of Q.

The next result is an easy consequence of [**7**, Section 6] and is in fact explicitly established in [**10**, result 2.17].

4. RESULT. *Let Q be a quasifield with a subfield $K = GF(q)$ in its kern. Suppose* $\dim_K Q = r > 2$. *Then*

$$q^{r-1} \mid |(\mathrm{Aut}\ Q)_K| \text{ only if } |Q| = 16.$$

The following corollary to the above result is also an immediate consequence of [**7**, Proposition 6.8].

5. COROLLARY. *If Q is a quasifield of order q^r, then q^r does not divide* $|\mathrm{Aut}\ Q|$.

The following interesting result, due to Ostrom [**17**], will prove useful on a number of occasions. An indication of how this theorem may be proved is given in [**13**].

6. RESULT. *Let π^{l_∞} be an affine translation plane of order q^r with $GF(q)$ in its kern. Suppose S is an elementary abelian 2-group in the linear translation complement of π^{l_∞} such that at least one of the following conditions hold:*

(i) *all involutions in S are mutually conjugate in the full linear complement of π^{l_∞} and $|S| \geqq 4$; or*

(ii) *all involutions in S are Baer collineations.*

Then, if q is odd, $|S|$ divides r.

Definition. Let q be any prime power and r a positive integer greater than one. Then a prime number θ is called a *primitive divisor* of $q^r - 1$ if

(i) $\theta | q^r - 1$; and

(ii) $\theta \nmid q^s - 1$ whenever q^s is an integer satisfying $1 < q^s < q^r$.

(Note that we do not assume s to be an integer.)

The main fact about primitive divisors was proved by Zsigmondy [**20**].

7. RESULT. *Let q, r be as in the above definition. Then $q^r - 1$ has a primitive divisor unless one of the following hold:*

(i) $q^r = 64$,

(ii) *q is a Mersenne prime and $r = 2$.*

1. General case. Throughout this work we adopt the following

1. *Conventions.* (a) V is the $2r$ dimension vector space $V_{2r}(GF(q))$ and Γ is a spread of V that is left invariant by a subgroup G of $\Gamma L(V)$. Also p denotes the characteristic of Γ.

(b) $G \cong PSL(n, w)$ where w is some prime power and $n \geqq 3$.

(c) π^{l_∞} is an affine translation plane whose lines are the V-cosets of Γ.

(d) G leaves invariant a set Δ of $q + 1$ slopes of π^{l_∞} and acts transitively on $l_\infty \backslash \Delta$.

In terms of the above conventions, our object is to prove that $G = PSL(3, 2)$ and π^{l_∞} is LR-16 or JW-16. We begin with some elementary facts that are tacitly used on a number of occasions.

2. REMARKS.

(a) $q^r - q | w^{n(n-1)/2} \prod_{i=2}^{n} (w^i - 1)$.

(b) *G is in the linear translation complement of π^{l_∞} i.e., $G \not\subset GL(V)$.*

(c) *If q is odd then 4 divides r.*

Proof. (a) Use the transitivity of G on $l_\infty \backslash \Delta$.

(b) If this is false then G has a non-trivial homomorphic image in the cyclic group $\Gamma L(V)/GL(V)$. This contradicts the simplicity of G.

(c) Because of Ostrom's theorem (result 0.6), it is sufficient to show that G contains A_4. If w is odd, then

$$PSL(n, w) \supseteq PSL(3, w) \supset PSL(2, w) \supset A_4$$

[**6**, Hauptsatz 8.27, p. 213] and, if w is even, then

$$PSL(n, w) \supseteq PSL(3, 2) \cong PSL(2, 7)$$

[**6**, Satz 6.14(4)] and so we still have $G \supset A_4$. The result follows.

2. Case: w, q **are relatively prime.** In this section, we show that the following hypothesis leads to a contradiction.

(*) *Hypothesis.* $(w, q) = 1$.

The following relations will prove useful.

1. REMARKS. (a) $q^r - q > \frac{1}{2}q^r$.

(b) $w^{n^2-1} > \frac{1}{2}q^r$.

(c) $\ln x/(x^\alpha - a)$ *is a decreasing function of x for $x \geqq 3$, provided that $a \geqq 0$ and $\alpha \geqq 1$.*

Proofs. (a) $\frac{1}{2}q^r > q$ except when π^{l_∞} is a plane of order 4 or 2. However, in these cases $PSL(n, w)$, with $n \geqq 3$, does not act.

(b) This follows from part (a) and remark 1.2(a).

(c) This is obtained by differentiation.

The purpose of the next two lemmas is to show that (*) is false when $w = 2$.

2. LEMMA. *Suppose (*). Then w can be 2 only when $G = PSL(3, 2)$.*

Proof. $PSL(n, w)$ contains the full translation group of the affine Desarguesian space of dimension $n - 1$ over $GF(w)$. So when w is 2, G contains an elementary abelian group of order 2^{n-1} all of whose involutions

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are G-conjugate. Thus 2^{n-1} divides r by result 0.6. Now by remark 1(b) we further have

$$\tfrac{1}{2}q^{(2^{n-1})} < 2^{n^2-1}$$

and so

(i) $2^{n-1} \ln q < n^2 \ln 2$.

As q is odd $\ln q > 1$ and (i) yields $2^{n-1} < n^2$ (0.7). But by induction, this inequality does not hold if $n \geqq 6$. On the other hand, (i) fails for $n = 5$ and so, if the lemma is false, then $G = PSL(4, 2)$.

Now Remarks 1.2(a) and 1.2(b) lead to

$$q^r - q | 2^6(63)(7)(3) \text{ and } 4|r.$$

These conditions easily yield a contradiction, and so the lemma follows.

We now exclude $G = PSL(3, 2)$ by using an argument that also excludes $PSL(3, 2^k)$.

3. Lemma. *Suppose* (*). *Then G cannot be $PSL(3, w)$ when w is even.*

Proof. Suppose the lemma is false and w is even. Then (*) implies q is odd. Arguing as in the last lemma, we see that w^2 divides r and so by Remark 1(b) we get $\tfrac{1}{2}q^{w^2} < w^8$. On taking logs and noting that $q \geqq 3$, we find that

(i) $1 < \ln q < (8 \ln w + \ln 2)/w^2$.

But (i) is false for $w = 4$ and so by Remark 1(c), we get $w = 2$ (since w is assumed to be even). Thus (i) forces q to be 3 and the usual conditions

$$3^r - 3 \mid |PSL(2, 3)| \text{ and } 4|r$$

easily lead to a contradiction. The result follows.

The following inequality will be used in the next lemma.

4. Remark. $3^{n-1} > 2n^2$ *for* $n \geqq 5$.

Proof. Suppose the result is false. Then on taking logs, we find that

$$(\ln 2 + 2 \ln n)/(n - 1) \geqq \ln 3 \text{ for some } n \geqq 5.$$

But by Remark 1(c), the left hand side of the above inequality decreases as n increases and so

$$(\ln 2 + 2 \ln 5)/4 \geqq 3.$$

This is a contradiction and so the lemma is proved.

5. LEMMA. *Suppose* (*) *holds and that* $G \neq PSL(3, 3)$. *Then* $G|\Delta =$ *identity.*

Proof. Suppose the lemma is false. Then result 0.1 implies $q \geqq w^2 + w$, and so by remark 1(b)

(i) $\frac{1}{2}w^{2r} < w^{n^2-1}$.

Now by Lemma 3, G is not of the form $PSL(3, 2^k)$. Hence, Result 0.2(b) may be used; this yields, because $\dim V = 2r$, the following inequality

(ii) $r \geqq (w^{n-1} - 1)/2$.

Now because of (i) $2r < n^2$, and so (ii) gives

(iii) $w^{n-1} - 1 < n^2$.

But by Lemmas 2 and 3, $w > 2$ and so (iii) yields $3^{n-1} < 2n^2$. Now Remark 4 shows that $n = 3$ or 4. But $n = 4$ contradicts (iii) and so $n = 3$. Now, if we recall that w is odd, (iii) forces $w = 3$. The lemma follows.

6. LEMMA. *If* (*) *holds, then* $G = PSL(3, 3)$ *and* $q = 2$.

Proof. Suppose if possible that (*) but $G \neq PSL(3, 3)$. Then by the last lemma, G fixes $q + 1$ components of the spread associated with $\pi^{l\infty}$ and of course acts faithfully on them. Thus G acts faithfully on a vector space of dimension r and $G \neq PSL(3, 2^k)$ by Lemma 3. Thus result 0.2(b) shows that $r \geqq w^{n-1} - 1$. Using this inequality together with $\frac{1}{2}q^r < w^{n^2-1}$ and taking logs, we get

(i) $\ln q < ((n^2 - 1) \ln 2)/(w^{n-1} - 1)$.

Since $w \geqq 3$, Remark 1(b) and (i) show that

(ii) $\ln q < ((n^2 - 1) \ln 3 + \ln 2)/(3^{n-1} - 1)$.

By differentiating the right hand side of (ii), we see that it is a decreasing function of n and so if $n \geqq 4$ then (ii) yields the contradiction:

$$\ln 2 < (15 \ln 3 + \ln 2)/26.$$

Thus n can only be 3 and now (i) becomes

(iii) $\ln q < (8 \ln w + \ln 2)/(w^2 - 1)$.

But now Remark 1(c), and the fact that (iii) is false for $w = 5$, shows that w must be 3. Thus (i) now becomes

$$\ln q < (8 \ln 3 + \ln 2)/8 = \ln 3(2^{1/8}) < \ln 4.$$

Thus, since $(w, q) = 1$, we must have $q = 2$. This completes the proof of the lemma.

We now verify that w and q are both powers of the same prime p.

7. PROPOSITION. *w is a power of p.*

Proof. If the proposition is false then the lemma above shows that q is 2 and $G = PSL(3, 3)$. But since $3^2 + 3 + 1 > 2 + 1$, Result 0.1 shows that $G|\Delta$ = identity. So again considering the action of G on a component and using Result 0.2(b), we get that $r \geqq 3^{3-1} - 1 = 8$. Also since $G|\Delta$ = identity all the involutions in G are Baer collineations. Hence r is an even number $\geqq 8$. Now 1.2(a) shows that $2^{2k-1} - 1$ divides the odd part of $|G|$ $(=3^3.13)$ for some $k \geqq 4$. This quickly gives a contradiction and the required result follows.

3. Final case: p divides q and w.

The following lemma guarantees that r is at least 4.

1. LEMMA. *r is an even integer and $r \geqq 4$. Also if w is odd, then 4 divides r.*

Proof. The second sentence is an immediate consequence of Remark 1.2(c). Thus we may assume q and w are both even.

Suppose, if possible, that r is odd. Now Result 0.6 shows that the involutions in G cannot be Baer collineations and must be affine elations. But, by a theorem of Hering [**4**], the only non-solvable groups generated by elations, in a plane of even characteristic, are groups of form $Sz(2^k)$ or $PSL(2, 2^s)$. Thus we have a contradiction since G itself is generated by the set of all its involutions (recall that G is a simple group). Hence we conclude that r is even.

Finally, to see that $r > 2$, we observe that $r = 2$ contradicts a theorem of Johnson and Ostrom [**11**, Theorem 3.27].

To avoid problems associated with primitive divisors, we begin by investigating the case $q^{r-1} = 64$.

2. LEMMA. *Let $\mathfrak{G} = \{PSL(3, 2), PSL(3, 4), PSL(4, 2)\}$. Then*
(a) $q^{r-1} = 64 \Rightarrow G \in \mathfrak{G}$; *and*
(b) $G \in \mathfrak{G} \Rightarrow G|\Delta$ = *identity.*

Proof. (a) Suppose $q^{r-1} = 64$. Since r is even and at least 4, we can only have $(q, r) = (4, 4)$. Thus $|\Delta| = 5$ and $G \supseteq PSL(3, 2)$ (since w is also even). Now Result 0.1 shows that $G|\Delta$ = identity and hence the Sylow p-subgroups of G are planar groups. Thus by Corollary 0.5, we see that

$$w^{n(n-1)/2} < q^r = 256.$$

Hence (w, n) must correspond to the parameters of the groups in $\mathfrak{G}$.

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(b) Suppose, if possible, that $G|\Delta \neq$ identity.

Case (i). $G = PSL(4, 2)$. Now Result 0.1 shows $q \geqq 2^2 + 2$ and so $q \geqq 8$. Using the relation $\frac{1}{2}q^r < w^{n^2-1}$ now yields $2^{3r} < 2^{16}$. So $r < 6$ and Lemma 1 forces r to be 4. Thus $q^4 < 2^{16}$ and hence $q \leqq 8$. We therefore have $(q, r) = (8, 4)$, and this contradicts

$$q^{r-1} - 1 \mid |PSL(4, 2)|.$$

Case (ii). $G = PSL(3, 4)$. Now Result 0.1 shows that $q \geqq 32$. As we also have $r \geqq 4$ the inequality $\frac{1}{2}q^r < w^{n^2-1}$ is contradicted.

Case (iii). $G = PSL(3, 2)$. As in Case (ii), we have $(q, r) \geqq (8, 4)$ and again contradict $\frac{1}{2}q^r < w^{n^2-1}$.

This completes the proof of the lemma.

3. Lemma. *At least one of the following hold:*

(i) $q^{r-1} \leqq w^n$;

(ii) *G is $PSL(3, 2)$ or $PSL(4, 2)$, and $q^{r-1} = 64$.*

Proof. If $q^{r-1} = 64$, then the last lemma gives (ii) except when $G = PSL(3, 4)$. But in the latter case, (i) is satisfied.

Next suppose $q^{r-1} \neq 64$. Then by Remark 1.2(a) and Lemma 1 of this section, we get

$$q^{r-1} - 1 \,\Big|\, \prod_{i=2}^{n} (w^i - 1) \quad \text{and} \quad r - 1 > 2.$$

Now by Result 0.7, we get $w^n \geqq q^{r-1}$ since the primitive divisor of $q^{r-1} - 1$ divides one of the terms of type $(w^i - 1)$. This completes the proof of the lemma.

4. Lemma $G|\Delta =$ *identity.*

Proof. Suppose, if possible, that $G|\Delta \neq$ identity. Then by Lemma 2(b) the exceptional situations of Lemma 3(ii) do not arise. Hence

(i) $q^{r-1} \leqq w^n$.

Now consider first the case when

[A] $G \neq PSL(3, w)$ when w is even.

Now the final sentence of Corollary 0.3 shows that $q + 1 \geqq w^{n-1} - 1$. As r is at least 4 (Lemma 1), we now get, using (i), the following inequalities:

$$w^n \geqq q^{r-1} > q^2 \geqq (w^{n-1} - 2)^2,$$

and so

$$1 > w^{n-2} - 4w^{-1} + 4w^{-n}.$$

Thus $3 > w^{n-2}$ and so $(n, w) = (3, 2)$. This contradicts Lemma 2(b) and so case [A] cannot occur.

It remains to consider the 'complement' of case A viz.

[B] $G = PSL(3, w)$ and w is even.

Because of Lemma 2, we may also assume that $w > 4$. Thus we may again apply Corollary 0.3; this time, using the first sentence of the corollary, we get

$$q + 1 \geqq (w^2 - 1)/(3, w - 1) \geqq (w^2 - 1)/3.$$

Now Lemma 3 shows that

$$w^3 \geqq q^{r-1} \geqq q^3 \geqq \left(\frac{w^2 - 4}{3}\right)^3 = \left(\frac{w - 2}{3}\right)^3 (w + 2)^3.$$

This is certainly a contradiction if $w > 4$. The lemma follows.

We now prove the main result of this article viz. Theorem A of the introduction.

5. THEOREM. *π^{l_∞} is* LR-16 *or* JW-16 *and $G = PSL(3, 2)$.*

Proof. Let P denote any Sylow p-subgroup of G. Then by the previous lemma we see that $F(P)$ is a subplane of π^{l_∞}. Since P is in the linear complement of π^{l_∞}, the plane $F(P)$ contains a kern plane of order q (i.e., $F(P)$ contains a subplane $\pi_K{}^{l_\infty}$ that has order q and is invariant under a group of $(q - 1)$ kern homologies).

Now coordinatize π^{l_∞} by a quasifield Q obtained by choosing coordinate axes in π_K. It follows that:

[A] $|P| \mid |(\text{Aut } Q)_K|$ where $GF(q) = K \subseteq \text{kern } (Q)$.

The next step is to show that $|Q| = 16$. When $G = PSL(3, 2)$, Remark 1.2(a) shows $q^{r-1} - 1|21$ and this quickly leads to $q^r = 16$. In all other cases (including $G = PSL(4, 2)$), Lemma 3 shows that

[B] $q^{r-1} \leqq w^n \leqq w^{n(n-1)/2} = |P|$.

By [A] and [B], we see that

$q^{r-1} \mid (\text{Aut } Q)_K$ where $\dim_K Q = r$.

Since also $r > 2$ (Lemma 1), Result 0.4 shows that $|Q| = 16$. Thus w is also even and $G \supseteq PSL(3, 2)$. But a result of [**12**] states that the only translation planes of order 16 which admit $PSL(3, 2)$ in their complements are LR-16 and JW-16. Hence π^{l_∞} is LR-16 or JW-16. But as neither plane admits $PSL(3, 4)$ in their complements [**19**], the proof of the theorem is complete.

4. Some additional remarks. The method of proof used to prove Theorem A extends easily to handle other linear groups. For example, if we replace condition (ii) of Theorem A with the condition

(ii′) $G \cong SL_n(w)$ with $n \geqq 3$ and w a prime power

the same conclusion holds. (Note that $PSL(3, 2) = SL(3, 2)$.) Indeed, since in general $SL_n(w)$ has a slightly larger order than $L_n(w)$ the counting arguments are easier.

Condition (i) of Theorem A is a natural one to assume since it is satisfied by the group $G = SL(3, 2) = PSL(3, 2)$ in both the Lorimer–Rahilly and the Johnson–Walker planes. For both planes, the invariant q is 2 (that is, the kernel is $GF(2)$) and the group G fixes $q + 1 = 3$ points on l_∞ and is transitive on the remaining 14 points. A reasonable question to ask is: If, instead of giving the action on l_∞ in terms of the invariant q, the action is given in terms of w, do other possibilities occur? The answer is: No. If condition (i) of Theorem A is replaced with

(i′) G leaves invariant a set Δ of $w + 1$ slopes and acts transitively on $l_\infty \backslash \Delta$,

the same conclusion holds. Again the proof is somewhat simplified. Lemma 2.5 is immediate, even without the restriction that $G \neq PSL(3, 3)$ and inequality (b) of 2.1 is replaced with the inequality $w^l > q^r$ where $l = \frac{1}{2}n(n + 1)$.

In fact, choosing any one of the conditions (i), (i′) and choosing any one of conditions (ii), (ii′) gives rise to a theorem with the same conclusion as Theorem A. What remains to be done is to show that condition (i) (or condition (i′)) can be dropped.

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