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Spectral mapping theorems for essential spectra and regularized functional calculi

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Gramsch and Lay [8] gave spectral mapping theorems for the Dunford-Taylor calculus of a closed linear operator T,

$$\widetilde{\sigma}_i(f(T)) = f(\widetilde{\sigma}_i(T)),$$

for several extended essential spectra $\widetilde{\sigma}_i$. In this work, we extend such theorems for the regularized functional calculus introduced by Haase [10, 11] assuming suitable conditions on f. At the same time, we answer in the positive a question made by Haase [11, Remark 5.4] regarding the conditions on f which are sufficient to obtain the spectral mapping theorem for the usual extended spectrum $\widetilde{\sigma}$. We use the model case of bisectorial-like operators, although the proofs presented here are generic, and are valid for similar functional calculi.

Keywords: Essential spectrum; holomorphic functional calculi; spectral mapping theorem

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1. Introduction

Let T be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$. A spectral singularity $\lambda \in \sigma(T)$, where $\sigma(T)$ denotes the spectrum of T, is said to be in the essential spectrum of T if λ is not an isolated eigenvalue of finite multiplicity, see [21].

If T is not self-adjoint, or if T is an operator on a Banach space X, most modern texts define the essential spectrum $\sigma_{ess}(T)$ of T in terms of Fredholm operators, that is, $\lambda \in \sigma_{ess}(T)$ iff $\lambda - T$ is not a Fredholm operator. Recall that a bounded operator T is a Fredholm operator if both its nullity $\operatorname{nul}(T)$ (dimension of its kernel) and its defect $\operatorname{def}(T)$ (codimension of its range) are finite. One of the main useful properties of the essential spectrum (defined this way) is that it is invariant under compact perturbations. As a matter of fact, $\sigma_{ess}(T) = \sigma(p(T))$, where $p(T)$ is the projection of T in the Calkin algebra, i.e., the quotient algebra of the bounded operators $\mathcal{L}(X)$ on X modulo the compact operators $\mathcal{K}(X)$ on X.

However, several different definitions for the essential spectrum were introduced in the 1950s and 60 s, especially in the framework of differential operators. For instance, if we denote by G^l (G^r) the semigroup of left (right) regular elements in the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$, Yood [22] studied the spectral sets of $T \in \mathcal{L}(X)$ given by $\sigma_2(T) := \{\lambda \in \mathbb{C} : \lambda - p(T) \notin G^l\}$ and $\sigma_3(T) := \{\lambda \in \mathbb{C} : \lambda - p(T) \notin G^r\}$. Indeed, he obtained the following characterizations of the semigroups $p^{-1}(G^l), p^{-1}(G^r)$:

$$p^{-1}(G^l) = \{T \in \mathcal{L}(X) \mid \mathrm{nul}(T) < \infty \text{ and } \operatorname{ran} T \text{ is complemented}\},$$
$$p^{-1}(G^r) = \{T \in \mathcal{L}(X) \mid \mathrm{def}(T) < \infty \text{ and } \ker T \text{ is complemented}\},$$

where $\operatorname{ran} T$, $\ker T$ denote the range space of T and the kernel of T respectively.

Alternatively, spectral sets associated with semi-Fredholm operators have also been referred to as essential spectra. More precisely, let Φ^+, Φ^- be given by

$$\Phi^- := \{T \in \mathcal{L}(X) \mid \mathrm{nul}(T) < \infty \text{ and } \operatorname{ran} T \text{ is closed}\},$$
$$\Phi^+ := \{T \in \mathcal{L}(X) \mid \mathrm{def}(T) < \infty\}.$$

Then Gustafson and Weidmann [9] used the term essential spectra for the spectral sets $\sigma_4(T) := \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi^-\}$, $\sigma_5(T) := \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi^+\}$, while Kato [13] considered the spectral set $\sigma_6(T) := \sigma_4(T) \cap \sigma_5(T)$, i.e., $\lambda \in \sigma_6(T)$ iff $\lambda - T$ is not in $\Phi^- \cup \Phi^+$. Note that $\sigma_2(T) = \sigma_4(T)$ and $\sigma_3(T) = \sigma_5(T)$ if T is an operator on a Hilbert space, but these identities do not hold in general in the framework of Banach spaces, see the work of Pietsch [18].

In another direction, Browder [3] defined the essential spectrum of T ($\sigma_8(T)$ here) as those spectral values of T which are not isolated eigenvalues of finite multiplicity of T nor isolated eigenvalues of finite multiplicity of the adjoint operator T^* (cf. [14]). It turns out that $\lambda \notin \sigma_8(T)$ iff λ is a pole of the resolvent of finite rank [3, Lemma 17]. With this in mind, Gramsch and Lay [8] considered the following additional essential spectrum, which is given by

$$\sigma_9(T) := \{\lambda \in \mathbb{C} \mid \text{the resolvent of } T \text{ is not meromorphic at } \lambda\}.$$

Nevertheless, the essential spectrum of Browder $\sigma_8(T)$ fails to be invariant under compact perturbations. In this regard, Schechter [19] defined the essential spectrum of T, $\sigma_7(T)$ here, as the largest subset of $\sigma(T)$ which is invariant under compact perturbations. Equivalently $\lambda \notin \sigma_7(T)$ iff $\lambda - T$ is Fredholm with index zero, i.e., $\mathrm{nul}(\lambda - T) = \mathrm{def}(\lambda - T) < \infty$. For a more systematic treatment of all these essential spectra, we refer to § 2.

In this work, we deal with spectral mapping theorems for all the different essential spectra described above, that is, identities of the form

$$\sigma_i(f(T)) = f(\sigma_i(T)). \tag{1.1}$$

There, f is a function in the domain of some functional calculus of a (possibly unbounded) operator T.

At this point, the first approach to a Banach space functional calculus of unbounded operators is the so-called Dunford-Taylor calculus. For this calculus,

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one considers functions f which are holomorphic in an open set containing the extended spectrum $\widetilde{\sigma}(T)$ of T, defined by $\widetilde{\sigma}(T) := \sigma(T) \cup \{\infty\}$ if T is unbounded and $\widetilde{\sigma}(T) := \sigma(T)$ otherwise. Then, for unbounded T, $f(T)$ is given by

$$f(T) := f(\infty) + \frac{1}{2\pi i} \int_{\Gamma} f(z)(z - T)^{-1}\, \mathrm{d}z, \tag{1.2}$$

where Γ is a suitable finite cycle that avoids $\widetilde{\sigma}(T)$. Moreover, the Dunford-Taylor formula above (1.2) still works when the curve Γ touches $\widetilde{\sigma}(T)$ at some points $a_1, \ldots, a_n$ and f is not holomorphic at $a_1, \ldots, a_n$, as long as f tends to a finite number at each point $a_1, \ldots, a_n$ fast enough to deal with the size of the resolvent at these points. In this case, we say that f has regular limits (at $a_1, \ldots, a_n$) and denote it by $\mathcal{E}(T)$.

Furthermore, in the setting of strip-type operators, Bade [**2**] introduced a 'regularization trick' in order to define $f(T)$ for functions which do not grow too fast at ∞. This 'regularization trick' was further developed, in the framework of sectorial operators, by McIntosh [**15**], Cowling *et al* [**4**] and Haase [**10**]. In particular, fractional powers and/or logarithms can be defined for suitable unbounded operators with this 'regularization trick'.

Here, we consider the 'regularized' functional calculus of meromorphic functions developed by Haase [**10**], which is based on the following idea. A meromorphic function f is in the domain of the regularized functional calculus of T, which we denote by $f \in \mathcal{M}(T)$, if there exists a holomorphic function $e \in \mathcal{E}(T)$ such that $e(T)$ is injective and $ef \in \mathcal{E}(T)$. In this case, one defines

$$f(T) := e(T)^{-1}(ef)(T), \tag{1.3}$$

which is a (possibly unbounded) closed operator on X.

Spectral mapping theorems, i.e., identities of the form (1.1), were proven by Gramsch and Lay [**8**] in the setting of the Dunford-Taylor calculus, for most (extended) essential spectra described here, see § 2 for their definitions. González and Onieva [**7**] used a unified approach and gave simpler proofs for these spectral mapping theorems. Their proofs are based on the following observations:

1) a closed operator T with non-empty resolvent set is essentially invertible iff, for $b \in \mathbb{C} \setminus \sigma(T)$, the bounded operator $T(b - T)^{-1}$ is essentially invertible [**7**, Lemma 1],

2) for f, g in the domain of the Dunford-Taylor calculus of T, one has $(fg)(T) = f(T)g(T) = g(T)f(T)$. As a consequence, $(fg)(T)$ is essentially invertible if/only if (see [**7**, Lemma 3]) both $f(T), g(T)$ are essentially invertible,

where we say that an operator A is essentially invertible (regarding the essential spectrum σ_i) if $0 \notin \sigma_i(A)$;

3) if f is in the domain of the Dunford-Taylor calculus of T, then one can assume that f has a finite number of zeroes of finite multiplicity.

It sounds sensible to ask whether these spectral mapping theorems can be extended to cover the functions in the domain of the regularized functional calculus given by (1.3). This is partly motivated by potential applications in Fredholm theory, in particular when considering fractional powers or logarithms of unbounded operators. For instance, we make use, in an ongoing work with L. Abadías, of the results presented here to describe the essential spectrum of fractional Cesàro operators and Hölder operators acting on spaces of holomorphic functions. However, there are two main difficulties for such an extension of the spectral mapping theorem. First, for $f, g \in \mathcal{M}(T)$, it is not true in general that $(fg)(T) = f(T)g(T) = g(T)f(T)$, so item 2) above fails. Indeed, one only has the inclusions $f(T)g(T), g(T)f(T) \subseteq (fg)(T)$, where $S \subseteq T$ means that $\operatorname{dom} S \subseteq \operatorname{dom} T$ with $Sx = Tx$ for every $x \in \operatorname{dom} S$. Secondly, since the function f may not be holomorphic at the points $a_1, \ldots, a_n$ where the integration path touches $\sigma(T)$, item 3) above also fails to be true.

Nevertheless, in the setting of sectorial operators, Haase [**11**] overcame these two problems for the usual extended spectrum $\widetilde{\sigma}$, and obtained the spectral mapping theorem

$$\widetilde{\sigma}(f(T)) = f(\widetilde{\sigma}(T)), \tag{1.4}$$

for a meromorphic function f in the domain of the regularized functional calculus, i.e., $f \in \mathcal{M}(T)$, such that f has almost logarithmic limits at the points $a_1, \ldots, a_n$ where the integration path Γ touches $\widetilde{\sigma}(T)$. This 'almost logarithmic' condition on the behavior of the limits of f is stronger than asking f to have regular limits at $a_1, \ldots, a_n$. As a matter of fact, Haase leaves open the question whether the hypothesis of f having regular limits is sufficient to obtain the spectral mapping theorem, see [**11**, Remark 5.4].

Still, it is far from trivial to extend the spectral mapping theorem (1.4) from the usual extended spectrum to the (extended) essential spectra described here. This extension, which is given here, is the main contribution of the paper. Even more, we obtain spectral mapping theorems for the essential spectra described above and for functions f with regular limits lying in the domain of the regularized functional calculus of meromorphic functions (1.3), answering in the positive Haase's conjecture on regular limits explained above.

To obtain these results, on the one hand we provide a slightly simpler proof for the spectral inclusion of the usual extended spectrum, i.e., $f(\widetilde{\sigma}(A)) \subseteq \widetilde{\sigma}(f(A))$, than the one given in [**11**]. As a matter of fact, we no longer make use of the composition rule of the functional calculus. Such a simplification allows us to weaken the condition on the function f from almost logarithmic limits to the (quasi-)regular limits, cf. [**11**, Remark 5.4]. On the other hand, the core of the paper contained in § 3 and 4, is devoted to address the items 2) and 3) above, so we cover all the essential spectra described here.

In this work, we use the model case of bisectorial-like operators, which is a family of operators that slightly generalizes the one of bisectorial operators, see for instance [**1**, **16**]. This is partly motivated by two reasons. On the one hand, we want our results to cover the case when T is the generator of an exponentially bounded group. This is because, in a forthcoming paper, we obtain spectral properties of certain integral operators via subordination of such operators in terms of an exponentially bounded group, namely, a weighted composition group of hyperbolic symbol. On the other hand, the another incentive to do this is the fact that the regularized functional calculus for bisectorial-like operators is easily constructed by mimicking the regularized functional calculus of sectorial operators [**10**, **12**]. Finally, bisectorial operators play an important role in the field of abstract inhomogeneous differential equations over the real line, so we are confident that our results have applications of interest in that topic.

Nevertheless, the proofs presented here are generic and are valid for similar functional calculi to the one presented here. Indeed, the abstract properties collected in lemmas 2.7–2.11 are the key to prove our results. In particular, our method also works for the regularized functional calculus of sectorial operators and the regularized functional calculus of strip-type operators, see Subsection 5.2.

The paper is organized as follows. The regularized functional calculus for bisectorial-like operators is detailed in § 2. In § 3, we give the spectral mapping theorems for a bisectorial-like operator A in the case the integration path Γ does not touch any point of $\widetilde{\sigma}(T)$. The general case is dealt with in § 4. We give some final remarks in § 5, such as the answer in the positive to Haase's conjecture [**11**, Remark 5.4].

2. Extended essential spectra and regularized functional calculus for bisectorial-like operators

Let us fix (and recall) the notation through the paper. X will denote an infinite dimensional complex Banach space. Let $\mathcal{L}(X)$, $C(X)$ denote the sets of bounded operators and closed operators on X, respectively. For $T \in C(X)$, let $\operatorname{dom} T$, $\operatorname{ran} T$, $\ker T$ denote the domain, range, null space of T, respectively. Moreover, we denote the nullity of T by $\operatorname{nul}(T)$, and the defect of T by $\operatorname{def}(T)$. The *ascent* of T, $\alpha(T)$, is the smallest integer n such that $\ker T^n = \ker T^{n+1}$, and the *descent* of T, $\delta(T)$, is the smallest integer n such that $\operatorname{ran} T^n = \operatorname{ran} T^{n+1}$.

Now we recall the definition of the different essential spectra described in the Introduction. Following the notation and terminology of [**7**, **8**], set

$$\begin{aligned}
\Phi_0 &:= \{T \in C(X) \,|\, \operatorname{nul}(T) = \operatorname{def}(T) = 0\},\\
\Phi_1 &:= \{T \in C(X) \,|\, \operatorname{nul}(T), \operatorname{def}(T) < \infty\},\\
\Phi_2 &:= \{T \in C(X) \,|\, \operatorname{nul}(T) < \infty, \operatorname{ran} T \text{ complemented}\},\\
\Phi_3 &:= \{T \in C(X) \,|\, \operatorname{def}(T) < \infty, \ker T \text{ complemented}\},\\
\Phi_4 &:= \{T \in C(X) \,|\, \operatorname{nul}(T) < \infty, \operatorname{ran} T \text{ closed}\},\\
\Phi_5 &:= \{T \in C(X) \,|\, \operatorname{def}(T) < \infty\},\\
\Phi_6 &:= \Phi_4 \cup \Phi_5,\\
\Phi_7 &:= \{T \in C(X) \,|\, \operatorname{nul}(T) = \operatorname{def}(T) < \infty\},\\
\Phi_8 &:= \{T \in \Phi_7 \,|\, \alpha(T) = \delta(T) < \infty\},\\
\Phi_9 &:= \{T \in C(X) \,|\, \alpha(T), \delta(T) < \infty\}.
\end{aligned}$$

We observe that these operator families satisfy the following spectral inclusions

$$\Phi_0 \subseteq \Phi_8 \subseteq \Phi_7 \subseteq \Phi_1 \begin{array}{c} \rotatebox{45}{$\subsetneq$} \quad \Phi_3 \subseteq \Phi_5 \quad \rotatebox{-45}{$\subsetneq$} \\ \\ \rotatebox{-45}{$\subsetneq$} \quad \Phi_2 \subseteq \Phi_4 \quad \rotatebox{45}{$\subsetneq$} \end{array} \Phi_6 \qquad \text{and} \qquad \Phi_0 \subseteq \Phi_8 \subseteq \Phi_9,$$

Then, the respective spectra $\sigma_i(T)$ are defined in terms of the above families by

$$\sigma_i(T) := \{\lambda \in \mathbb{C} \,|\, \lambda - T \notin \Phi_i\} \quad \text{for} \quad 0 \leqslant i \leqslant 9.$$

Note that $\sigma_0(T)$ is the usual spectrum $\sigma(T)$ and most modern text use the term essential spectrum to denote the set $\sigma_1(T)$. It is also worth saying that the works [**7**, **8**] also considered the essential spectrum $\sigma_{10}(T)$ defined in terms of normally solvable operators, i.e., operators with closed range, see [**5**]. However, there exist bounded operators S, T on Hilbert spaces for which $\sigma_{10}(T^2) \not\subseteq (\sigma_{10}(T))^2$ and $\sigma_{10}(S^2) \not\supseteq (\sigma_{10}(S))^2$, see [**8**, Section 5].

Next we define the extended essential spectra $\widetilde{\sigma}_i(T)$.

Definition 2.1. *Let $T \in C(X)$. We define*

$$\widetilde{\sigma}_i(T) := \begin{cases} \sigma_i(T) \ \textit{if} \ \begin{cases} \operatorname{dom} T = X, \ \textit{for } i \in \{0, 7, 8\}, \\ \operatorname{codim}(\operatorname{dom} T) < \infty, \ \textit{for } i \in \{1, 3, 5\}, \\ \operatorname{dom} T \ \textit{closed}, \ \textit{for } i \in \{4, 6\}, \\ \operatorname{dom} T \ \textit{complemented}, \ \textit{for } i = 2, \\ \operatorname{dom} T^n = \operatorname{dom} T^{n+1} \ \textit{for some } n \in \mathbb{N}, \ \textit{for } i = 9, \end{cases} \\ \sigma_i(T) \cup \{\infty\}, \ \textit{otherwise.} \end{cases}$$

Note that $\widetilde{\sigma}_0(T)$ is the usual extended spectrum $\widetilde{\sigma}(T)$. If the resolvent set $\rho(T)$ is not empty, $\widetilde{\sigma}_i(T)$ coincides with the extended essential spectrum introduced by González and Onieva [**7**], which satisfies that $\infty \in \widetilde{\sigma}_i(T)$ if and only if $0 \in \sigma_i((\mu - T)^{-1})$ for any $\mu \in \rho(T)$. In particular, if T has non-empty resolvent set, $\widetilde{\sigma}_i(T)$ are non-empty compact subsets of $\mathbb{C}_\infty$ except for $i = 9$ (see [**8**]), where $\mathbb{C}_\infty$ denotes the Riemann sphere $\mathbb{C} \cup \{\infty\}$. If T has empty resolvent set, $\sigma_i(T)$ is a closed subset of $\mathbb{C}$ for $i \in \{0, 1, 2, 4, 5, 6, 7\}$, see [**6**, Section I.3] and [**22**]. We do not know if $\widetilde{\sigma}_i(T)$ or $\sigma_i(T)$ are closed in the other cases.

Now we turn to the definition of the regularized functional calculus of bisectorial-like operators. Its construction is completely analogous to the one of the regularized functional calculus of sectorial operators given by Haase in [**10**, **12**], and the adaptation of it from the sectorial operators to the bisectorial-like operators is straightforward.

Given any $\varphi \in (0, \pi)$, we denote the sector $S_\varphi := \{z \in \mathbb{C} : |\arg(z)| < \varphi\}$. For any $\omega \in (0, \pi/2]$ and $a \geqslant 0$, we set the bisector

$$BS_{\omega,a} := \begin{cases} (-a + S_{\pi-\omega}) \cap (a - S_{\pi-\omega}) & \text{if } \omega < \pi/2 \text{ or } a > 0, \\ i\mathbb{R} & \text{if } \omega = \pi/2 \text{ and } a = 0. \end{cases}$$

see Fig. 1 for a sketch of such a bisectorial-like set.

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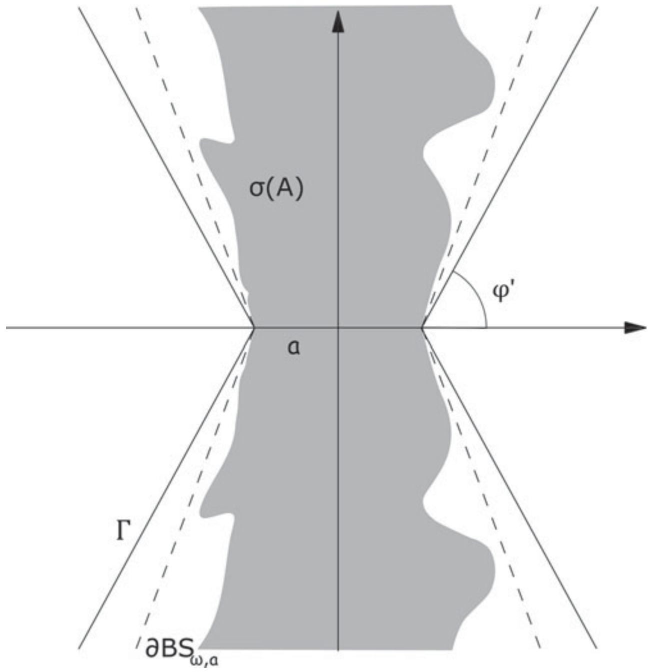

Figure 1. Spectrum of a bisectorial-like operator and integration path of the functional calculus.

DEFINITION 2.2. *Let* $(\omega, a) \in (0, \pi/2] \times [0, \infty)$ *and let* $A \in C(X)$*. We will say that* A *is a* **bisectorial-like operator** *of angle* ω *and half-width* a *if the following conditions hold:*

- $\sigma(A) \subseteq \overline{BS_{\omega,a}}$.
- *For all* $\omega' \in (0, \omega)$*,* A *satisfies the resolvent bound*

$$\sup\left\{ \min\{|\lambda - a|, |\lambda + a|\}\|(\lambda - A)^{-1}\| \,:\, \lambda \notin \overline{BS_{\omega',a}} \right\} < \infty.$$

We also set $M_A := \widetilde{\sigma}(A) \cap \{-a, a, \infty\}$*. For the rest of the paper,* (ω, a) *will denote a pair in* $(0, \pi/2] \times [0, \infty)$*.*

Given a Banach space X, we denote the set of all bisectorial-like operators on X of angle ω and half-width a in X by $\mathrm{BSect}(\omega, a)$. We omit an explicit mention to X for the sake of simplicity. Notice that $A \in \mathrm{BSect}(\omega, a)$ if and only if both $a + A, a - A$ are sectorial of angle $\pi - \omega$ in the sense of [**12**].

We denote by $\mathcal{O}(\Omega), \mathcal{M}(\Omega)$ the sets of holomorphic functions and meromorphic functions defined in an open subset $\Omega \subseteq \mathbb{C}$, respectively. For $A \in \mathrm{BSect}(\omega, a)$, let $U_A := \{-a, a, \infty\} \setminus \widetilde{\sigma}(A)$. If $\sigma(A) \neq \emptyset$, set

$$r_d := \begin{cases} \mathrm{dist}\{d, \sigma(A)\}, & \text{if d} \in \{-a, a\}, \\ r(A)^{-1}, & \text{if } d = \infty, \end{cases} \qquad \mathrm{d} \in U_A,$$

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where dist$\{\cdot,\cdot\}$ denotes the distance between two sets, and $r(A)$ the spectral radius of A. If $\sigma(A)=\emptyset$ (so $\widetilde{\sigma}(A)=\{\infty\}$ and $\infty\notin U_A$), set $r_a=r_{-a}:=\infty$.

For $\mathrm{d}\in U_A$ suppose that $s_d\in(0,r_d)$. Then, for $\varphi\in(0,\omega)$, set $\Omega(\varphi,(s_d)_{\mathrm{d}\in U_A})$ as follows. If $U_A=\emptyset$ (i.e., $M_A=\{-a,a,\infty\}$), we set $\Omega_\varphi:=BS_{\varphi,a}$. Otherwise, for each $\mathrm{d}\in U_A$, let $B_d(s_d)$ be a ball centred at d of radius s_d, where $B_\infty(r_\infty)=\{z\in\mathbb{C}\,|\,|z|>r_\infty^{-1}\}$. Then, we set $\Omega(\varphi,(s_d)_{\mathrm{d}\in U_A}):=BS_{\varphi,a}\setminus(\bigcup_{\mathrm{d}\in U_A}\overline{B_d(s_d)})$. Note that, if $\varphi<\varphi'<\omega$ and $s_d<s'_d<r_d$ for each $\mathrm{d}\in U_A$, then the inclusion $\Omega(\varphi',(s'_d)_{\mathrm{d}\in U_A})\subseteq\Omega(\varphi,(s_d)_{\mathrm{d}\in U_A})$ holds. Thus we can form the inductive limits

$$\mathcal{O}[\Omega_A]:=\bigcup\left\{\mathcal{O}(\Omega(\varphi,(s_d)_{\mathrm{d}\in U_A}))\,\middle|\,0<\varphi<\omega,\ 0<s_d<r_d\text{ for }\mathrm{d}\in U_A\right\},$$
$$\mathcal{M}[\Omega_A]:=\bigcup\left\{\mathcal{M}(\Omega(\varphi,(s_d)_{\mathrm{d}\in U_A}))\,\middle|\,0<\varphi<\omega,\ 0<s_d<r_d\text{ for }\mathrm{d}\in U_A\right\}.$$

Hence, $\mathcal{O}[\Omega_A],\mathcal{M}[\Omega_A]$ are algebras of holomorphic functions and meromorphic functions (respectively) defined on an open set containing $\widetilde{\sigma}(A)\setminus M_A$. Next, we define the following notion of regularity at M_A.

DEFINITION 2.3. *Let $f\in\mathcal{M}[\Omega_A]$. We say that f is regular at $\mathrm{d}\in\{-a,a\}\cap M_A$ if $\lim_{\mathrm{dom}\,f\ni z\to d}f(z)=:c_d\in\mathbb{C}$ exists and, for some $\varphi\in(0,\omega)$*

$$\int_{\partial BS_{\varphi',a},|z-d|<\varepsilon}\left|\frac{f(z)-c_d}{z-d}\right||\mathrm{d}z|<\infty,\quad\textit{for some }\varepsilon>0\textit{ and for all }\varphi'\in\left(\varphi,\frac{\pi}{2}\right],$$

where $\partial\Omega$ denotes the boundary of a subset $\Omega\subset\mathbb{C}$. If $\infty\in M_A$, we say that f is regular at ∞ if $\lim_{z\to\infty}f(z)=:c_\infty\in\mathbb{C}$ exists and

$$\int_{\partial BS_{\varphi',a},|z|>R}\left|\frac{f(z)-c_\infty}{z}\right||\mathrm{d}z|<\infty,\quad\textit{for some }R>0\textit{ and for all }\varphi'\in\left(\varphi,\frac{\pi}{2}\right].$$

We say that f is quasi-regular at $\mathrm{d}\in M_A$ if f or $1/f$ is regular at d. Finally, we say that f is (quasi-)regular at M_A if f is (quasi-)regular at each point of M_A.

REMARK 2.4. Note that if f is regular at M_A with every limit being not equal to 0, then $1/f$ is also regular at M_A. If f is quasi-regular at M_A, then $\mu-f$ and $1/f$ are also quasi-regular at M_A for each $\mu\in\mathbb{C}$. A function f which is quasi-regular at M_A has well-defined limits in $\mathbb{C}_\infty$ as z tends to each point of M_A.

Next, let $\mathcal{E}(A)$ be the subset of functions of $\mathcal{O}[\Omega_A]$ which are regular at M_A. Note that $\mathcal{E}(A)$ is a subalgebra of $\mathcal{O}[\Omega_A]$. Indeed, it is readily seen that it is a vector space, and the identity $f(z)g(z)-c_fc_g=g(z)(f(z)-c_f)+c_f(g(z)-c_g)$ yields that $\mathcal{E}(A)$ is closed under the point-wise product. Then, for any $b\in\mathbb{C}\backslash\overline{BS_{\varphi,a}}$, the set identity

$$\mathcal{E}(A)=\mathcal{E}_0(A)+\mathbb{C}\frac{1}{b+z}+\mathbb{C}\frac{1}{b-z}+\mathbb{C}\mathbf{1},\tag{2.1}$$

holds true, where $\mathbf{1}$ is the constant function with value 1, and

$$\mathcal{E}_0(A):=\left\{f\in\mathcal{O}[\Omega_A]\,:\,f\text{ is regular at }M_A\text{ with }\lim_{z\to d}f(z)=0\text{ for all }\mathrm{d}\in M_A\right\}.$$

Given a bisectorial-like operator $A \in \mathrm{BSect}(\omega, a)$, we define the mapping $\Phi : \mathcal{E}(A) \to \mathcal{L}(X)$ determined by setting $\Phi(1/b+z) = (b+A)^{-1}$, $\Phi(1/b-z) = (b-A)^{-1}$, $\Phi(\mathbf{1}) = I$, and

$$\Phi(f) := f(A) := \frac{1}{2\pi i}\int_{\Gamma} f(z)(z-A)^{-1}\,\mathrm{d}z, \quad f \in \mathcal{E}_0(A), \tag{2.2}$$

where Γ is the positively oriented boundary of $\Omega(\varphi', (s'_d)_{\mathrm{d}\in U_A})$ with $\varphi < \varphi' < \omega$ and $s_d < s'_d < r_d$, where $f \in \mathcal{O}(\Omega(\varphi, (s_d)_{\mathrm{d}\in U_A}))$.

LEMMA 2.5. *Let $a \geqslant 0$, $0 < \omega \leqslant \pi/2$ and $A \in \mathrm{BSect}(\omega, a)$. The mapping $\Phi : \mathcal{E}(A) \to \mathcal{L}(X)$ is a well-defined algebra homomorphism.*

Proof. The proof is analogous to the case of sectorial operators (see [**12**, Section 2.3]) with some minor changes. We give below a sketch of such a proof.

First, note that the integral (2.2) is well defined in the Bochner sense since $z \mapsto (z-A)^{-1}$ is analytic, so continuous, and $\int_\Gamma |f(z)|\|(z-A)^{-1}\|\,|\mathrm{d}z| < \infty$ for all $f \in \mathcal{E}_0(A)$. Also, such an integral is independent of the choice of $\varphi', (s'_d)_{\mathrm{d}\in U_A}$ by Cauchy's theorem. Fix $b \in \mathbb{C} \setminus \overline{BS_{\varphi,a}}$ for now. To see that Φ is well defined one has to prove that $\Phi(g)$ is independent of the decomposition of $g \in \mathcal{E}(A)$ via the set sum (2.1). So take $K, M, N \in \mathbb{C}$ such that the function given by

$$f(z) := K + \frac{M}{b+z} + \frac{N}{b-z}, \quad z \in \mathbb{C},$$

lies in $\mathcal{E}_0(A)$ (note that $K = 0$ if $\infty \in M_A$). Let Γ be an integration path as in (2.2). Assume $a \in M_A$ (i.e., $a \in \widetilde{\sigma}(A)$) and $a > 0$. Since $f \in \mathcal{E}_0(A)$, an application of Cauchy's theorem shows

$$\frac{1}{2\pi i}\int_{\Gamma} f(z)(z-A)^{-1}\,\mathrm{d}z = \frac{1}{2\pi i}\int_{\Gamma_\varepsilon} f(z)(z-A)^{-1}\,\mathrm{d}z, \quad \text{for all } \varepsilon > 0 \text{ small enough,}$$

where Γ_ε is the (suitable oriented) path given by

$$\Gamma_\varepsilon := \{z \in \Gamma \mid \Re(z) \leqslant 0\} \cup \{z \in \Gamma \mid \Re(z) > 0,\, |z-a| > \varepsilon\} \cup \{z \in \mathbb{C} \mid |z-a| = \varepsilon,\, |\arg(z-a)| \leqslant \varphi'\}.$$

After applying a similar trick to all the points in M_A, one can change Γ in the integral above by an integration path that does not touch $\widetilde{\sigma}(A)$ (even in the case $a = 0$). Thus,

$$\frac{1}{2\pi i}\int_{\Gamma} f(z)(z-A)^{-1}\,\mathrm{d}z = \widetilde{\Phi}(f) = K + M(b+A)^{-1} + N(b-A)^{-1},$$

where $\widetilde{\Phi}$ denotes the Dunford-Taylor functional calculus of A, see [**20**, Section V.8] for more details. We conclude that the mapping Φ is well defined, and a similar trick as above shows that Φ is independent of the choice of $b \in \mathbb{C} \setminus \overline{BS_{\varphi,a}}$.

On the other hand, reasoning as in the case of sectorial operators (see [**12**, Lemma 2.3.1]), several applications of Cauchy's theorem and the resolvent identity show

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that Φ is an algebra homomorphism when restricted to $\mathcal{E}_0(A)$. After that, a few computations with products of the type $(b-z)^{-1}(b+z)^{-1}$, $(b \pm z)^{-1}g(z)$, $g \in \mathcal{E}_0(A)$, show that indeed Φ is an algebra homomorphism from $\mathcal{E}(A)$ to $\mathcal{L}(X)$ (see also [**12**, Theorem 2.3.3]). □

Next, we follow the regularization method given in [**10**] to extend the functional calculus Φ to a regularized functional calculus (also denoted by Φ), which involves meromorphic functions.

DEFINITION 2.6. *Let $a \geqslant 0$, $0 < \omega \leqslant \pi/2$ and $A \in \mathrm{BSect}(\omega, a)$. Then, a function $f \in \mathcal{M}[\Omega_A]$ is called regularizable by $\mathcal{E}(A)$ if there exists $e \in \mathcal{E}(A)$ such that*

- *$e(A)$ is injective,*
- *$ef \in \mathcal{E}(A)$.*

For any regularizable $f \in \mathcal{M}[\Omega_A]$ with regularizer $e \in \mathcal{E}(A)$, we set

$$\Phi(f) := f(A) := e(A)^{-1}(ef)(A).$$

By [**10**, Lemma 3.2], this definition is independent of the regularizer e, and $f(A)$ is a well-defined closed operator. We denote by $\mathcal{M}(A)$ the subset of functions of $\mathcal{M}[\Omega_A]$ which are regularizable by $\mathcal{E}(A)$. As in the case for sectorial operators [**10**, Theorem 3.6], this regularized functional calculus satisfies the properties given in the lemma below. For $A, B \in C(X)$, we mean by $A \subseteq B$ that $\operatorname{dom} A \subseteq \operatorname{dom} B$ with $Ax = Bx$ for every $x \in \operatorname{dom} A$.

LEMMA 2.7. *Let $A \in \mathrm{BSect}(\omega, a)$ and $f \in \mathcal{M}(A)$. Then*

1. *If $T \in \mathcal{L}(X)$ commutes with A, that is, $TA \subseteq AT$, then T also commutes with $f(A)$, i.e. $Tf(A) \subseteq f(A)T$.*
2. *$\zeta(A) = A$, where $\zeta(z) = z$, $z \in \mathbb{C}$.*
3. *Let $g \in \mathcal{M}(A)$. Then*

   $$f(A) + g(A) \subseteq (f+g)(A), \qquad f(A)g(A) \subseteq (fg)(A).$$

   Furthermore, $\operatorname{dom}(f(A)g(A)) = \operatorname{dom}(fg)(A) \cap \operatorname{dom} g(A)$, and one has equality in these relations if $g(A) \in \mathcal{L}(X)$.
4. *Let $\lambda \in \mathbb{C}$. Then*

   $$\frac{1}{\lambda - f(z)} \in \mathcal{M}(A) \iff \lambda - f(A) \text{ is injective.}$$

   If this is the case, $(\lambda - f(z))^{-1}(A) = (\lambda - f(A))^{-1}$. In particular, $\lambda \in \rho(A)$ if and only if $(\lambda - f(z))^{-1} \in \mathcal{M}(A)$ with $(\lambda - f(A)^{-1}) \in \mathcal{L}(X)$.

Proof. The statement follows by straightforward applications of the Cauchy's theorem, the resolvent identity, and [**10**, Section 3]. □

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Let $\sigma_p(T)$ denote the point spectrum of a closed linear operator T.

LEMMA 2.8. *Let* $A \in \mathrm{BSect}(\omega, a)$, $\lambda \in \sigma_p(A)$ *and* $f \in \mathcal{M}(A)$. *Then* $f(\lambda) \in \mathbb{C}$ *and* $f(A)x = f(\lambda)x$ *for any* $x \in \ker(\lambda - A)$.

Proof. See [**11**, Proposition 3.1] for the analogous result for sectorial operators. □

LEMMA 2.9. *Let* $A \in \mathrm{BSect}(\omega, a)$, $f \in \mathcal{E}(A)$ *and* $\lambda \in (\mathrm{dom}\, f) \setminus M_A$ *such that* $f(\lambda) = 0$. *For each* $b \in \rho(A)$, *the function* $b - (\cdot)/\lambda - (\cdot) f$ *lies in* $\mathcal{E}(A)$.

Proof. The claim follows by the definitions of $\mathcal{E}(A)$ and regular limits. □

LEMMA 2.10. *Let* $A \in \mathrm{BSect}(\omega, a)$, $f \in \mathcal{M}(A)$ *and* $\lambda \in \widetilde{\sigma}(A) \setminus M_A$ *such that* $f(\lambda) \neq \infty$. *There is a regularizer* $e \in \mathcal{E}(A)$ *for* f *with* $e(\lambda) \neq 0$.

Proof. The proof is analogous to the case of sectorial operators, see [**11**, Lemma 4.3]. □

LEMMA 2.11. *Let* $A \in \mathrm{BSect}(\omega, a)$ *and* $f \in \mathcal{M}[\Omega_A]$. *Assume that* f *is regular at* M_A *and that all the poles of* f *are contained in* $\mathbb{C}\setminus\sigma_p(A)$. *Then,* $f \in \mathcal{M}(A)$. *Moreover, if every pole of* f *is contained in* $\rho(A)$, *then* $f(A) \in \mathcal{L}(X)$.

Proof. The proof is the same as in the case of sectorial operators, see [**11**, Lemma 6.2]. We include it here since we need it in the proof of Theorem 5.3.

Let $f \in \mathcal{M}[\Omega_A]$ be as required. That is, there exists $\varphi \in (0, \omega)$ and $s_d \in (0, r_d)$ for each $\mathrm{d} \in U_A$ such that $f \in \mathcal{M}(\Omega(\varphi, (s_d)_{\mathrm{d}\in U_A}))$. Since f has finite limits at M_A, we can assume that f has only finitely many poles by making $\varphi, (s_d)_{\mathrm{d}\in U_A}$ bigger. Thus, let λ_j for $j \in \{1, \ldots, N\}$ be an enumeration of those poles of f and let $n_j \in \mathbb{N}$ be the order of pole of f located at λ_j, for $j \in \{1, \ldots, N\}$. Then, the function $g(z) := f(z)\Pi_{j=1}^N (\lambda_j - z)^{n_j}/(b - z)^{n_j}$ has no poles, i.e. $g \in \mathcal{O}[\Omega_A]$, and is regular at M_A. Hence $g \in \mathcal{E}(A)$. Moreover, setting $r(z) := \Pi_{j=1}^N (\lambda_j - z)^{n_j}/(b - z)^{n_j}$, one has that the operator $r(A) = \Pi_{j=1}^N (\lambda_j - A)^{n_j}(b - A)^{-n_j}$ is bounded and injective, since by assumption $\{\lambda_1, \ldots, \lambda_n\} \subseteq \mathbb{C} \setminus \sigma_p(A)$. In short, f is regularized by r, so $f \in \mathcal{M}(A)$.

Now, assume that the poles of f lie inside $\rho(A)$. Then the operator $r(A)$ is not only bounded and injective, but invertible too, from which follows $f(A) = r(A)^{-1}(rf)(A) \in \mathcal{L}(X)$. □

3. Spectral mapping theorems for $M_A = \emptyset$

For $A \in \mathrm{BSect}(\omega, a)$, the spectral mapping theorems (1.1) given in [**7**, **8**] are applicable to every $f \in \mathcal{E}(A)$ whenever $M_A = \emptyset$. This section is devoted to extend these spectral mapping theorems to all $f \in \mathcal{M}(A)$ when $M_A = \emptyset$.

First, we proceed to state the spectral inclusion of the spectrum $\widetilde{\sigma}$.

PROPOSITION 3.1. *Let* $A \in \mathrm{BSect}(\omega, a)$, $f \in \mathcal{M}(A)$, *and assume that* f *is quasi-regular at* M_A. *Then*

$$\widetilde{\sigma}(f(A)) \subseteq f(\widetilde{\sigma}(A)).$$

Proof. The proof runs along the same lines as in the case of sectorial operators, see [**11**, Proposition 6.3]. As in lemma 2.11, we include the proof here since it will be needed in the proof of Theorem 5.3.

Take $\mu \in \mathbb{C}$ such that $\mu \notin f(\widetilde{\sigma}(A))$. Then $1/\mu - f \in \mathcal{M}[\Omega_A]$ is regular in M_A, and all of its poles are contained in $\rho(A)$. By lemma 2.11, we conclude that $(\mu - f)^{-1} \in \mathcal{M}(A)$ and that $(\mu - f)^{-1}(A)$ is a bounded operator. Thus, it follows that $\mu - f(A)$ is invertible, hence $\mu \notin \widetilde{\sigma}(f(A))$.

Assume now that $\mu = \infty \notin f(\widetilde{\sigma}(A))$. Then f is regular at M_A and its poles are contained in $\rho(A)$. Another application of lemma 2.11 yields that $f(A)$ is a bounded operator, so $\infty \notin \widetilde{\sigma}(f(A))$. □

Next, we give some technical lemmas.

Lemma 3.2. *Let $A \in \mathrm{BSect}(\omega, a)$, $e, f, h \in \mathcal{M}(A)$ and $0 \neq c \in \mathbb{C}$ with $fh = e - c$. Suppose that $e(A), h(A), f(A) \in \mathcal{L}(X)$. Then*

$$y \in \operatorname{ran} h(A) \iff e(A)y \in \operatorname{ran} h(A).$$

Proof. Implication $\Longrightarrow$ follows from $e(A)h(A) = h(A)e(A)$; and $\Longleftarrow$ follows from $I = 1/c(e(A) - h(A)f(A))$. □

The following lemma, which is a refinement of [**11**, Lemma 4.2], is crucial to deal with regularized functions $f \in \mathcal{M}(A)$ which are not in $\mathcal{E}(A)$.

Lemma 3.3. *Let $f \in \mathcal{M}(A)$ and $\lambda \in \sigma(A) \setminus M_A$ with $f(\lambda) = 0$, and let $b \in \rho(A)$. If $g(z) := f(z)(b - z)/(\lambda - z)$, then $g \in \mathcal{M}(A)$, $\operatorname{dom} g(A) = \operatorname{dom} f(A)$ and*

$$f(A) = (\lambda - A)(b - A)^{-1} g(A) = g(A)(\lambda - A)(b - A)^{-1}.$$

Proof. Let e be a regularizer for f with $c := e(\lambda) \neq 0$ (see lemma 2.10). Then $eg \in \mathcal{E}(A)$ by lemma 2.9 and hence e is a regularizer for g. Define $h := (\lambda - (\cdot))/(b - (\cdot)) \in \mathcal{E}(A)$. Since $f = hg$, it follows $\operatorname{dom} g(A) \subseteq \operatorname{dom} f(A)$.

For the converse, suppose that $x \in \operatorname{dom} f(A)$, so there is $y \in X$ such that $(ef)(A)x = e(A)y$, i.e.,

$$e(A)y = (ehg)(A)x = h(A)(eg)(A)x \in \operatorname{ran} h(A).$$

Observe that $e - c = \widetilde{f} h$ with

$$\widetilde{f}(z) := (b - z)\frac{e(z) - c}{\lambda - z}, \quad z \in \operatorname{dom} f.$$

By lemma 2.9, one has $\widetilde{f} \in \mathcal{E}(A)$. By lemma 3.2, there exists $v \in X$ such that $y = h(A)v$. This yields $h(A)(eg)(A)x = e(A)y = e(A)h(A)v = h(A)e(A)v$, that is, $(eg)(A)x - e(A)v \in \ker h(A)$. But $\ker h(A) = \ker(\lambda - A) \subseteq \ker(c - e(A)) \subseteq \operatorname{ran} e(A)$, see lemma 2.8. Hence $(eg)(A)x \in \operatorname{ran} e(A)$, so $x \in \operatorname{dom} g(A)$ as claimed.

Finally, the identity $f(A) = h(A)g(A) = g(A)h(A)$ follows by what we have already proven and lemma 2.7(3). □

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Remark 3.4. Let $T \in C(X)$ with non-empty resolvent set, and $\alpha(T), \delta(T) < \infty$. Then $\alpha(T) = \delta(T) =: p_T$ and $X = \ker T^{p_T} \oplus \operatorname{ran} T^{p_T}$, see for example [**20**, Theorem V.6.2].

Remark 3.5. Let $T \in C(X)$ with non-empty resolvent set, and let $b \in \rho(T)$ and $\lambda \in \mathbb{C}$. Then $\lambda - T \in \Phi_i$ if and only if $(\lambda - T)(b - T)^{-1} \in \Phi_i$ for all $0 \leqslant i \leqslant 9$, see for example [**7**, Lemma 1].

Lemma 3.6. *Let $A \in \mathrm{BSect}(\omega, a)$, $f, g \in \mathcal{M}(A)$ with f, g quasi-regular at M_A, $0 \notin g(\widetilde{\sigma}(A))$ and such that*

$$f(z) := g(z) \prod_{j=1}^{N} \left(\frac{\lambda_j - z}{b - z} \right)^{n_j},$$

for some $b \in \rho(A)$, $\lambda_j \in \sigma(A) \setminus M_A$, and $n_j \in \mathbb{N}$ for $j = 1, \ldots, N$. Then

(a) *if $f(A) \in \Phi_i$, then $\lambda_j - A \in \Phi_i$ for all $j = 1, \ldots, N$ and for all i except $i = 7$;*

(b) *if $\lambda_j - A \in \Phi_i$ for all $j = 1, \ldots, N$, then $f(A) \in \Phi_i$ for all i except $i = 6$.*

Proof. Set $r(z) := \Pi_{j=1}^{N} (\lambda_j - z)^{n_j} / (b - z)^{n_j}$, so $r(A) \in \mathcal{L}(X)$. Several applications of lemma 3.3 imply $\operatorname{dom} g(A) = \operatorname{dom} f(A)$ and $f(A) = r(A)g(A) = g(A)r(A)$. Moreover, proposition 3.1 yields $0 \notin \widetilde{\sigma}(g(A))$, so $g(A)$ is surjective and injective. Therefore $\ker f(A) = \ker r(A)$ and $\operatorname{ran} f(A) = \operatorname{ran} r(A)$, so $f(A) \in \Phi_i$ iff $r(A) \in \Phi_i$ for $0 \leqslant i \leqslant 7$. By replacing f, g, r by powers f^n, g^n, r^n with $n \in \mathbb{N}$, we cover the cases $i = 8, 9$.

Since the bounded operators $(\lambda_j - A)(b - A)^{-1}$ commute with each other, we have:

1. if $r(A) \in \Phi_i$, then $(\lambda_j - A)(b - A)^{-1} \in \Phi_i$ for all $j = 1, \ldots, N$, and for all i except $i = 7$,
2. If $(\lambda_j - A)(b - A)^{-1} \in \Phi_i$ for all $j = 1, \ldots, N$, then $r(A) \in \Phi_i$, for all i except $i = 6$.

see for example [**7**, Lemma 3] and [**8**, Lemma 5(c)]. Hence, the claim follows from remark 3.5. □

We give now the main result of this section.

Proposition 3.7. *Let $A \in \mathrm{BSect}(\omega, a)$, $f \in \mathcal{M}(A)$, where f is quasi-regular at M_A. Then*

(a) *$f(\widetilde{\sigma}_i(A)) \setminus f(M_A) \subseteq \widetilde{\sigma}_i(f(A))$ for all i except $i = 7$.*

(b) *$\widetilde{\sigma}_i(f(A)) \subseteq f(\widetilde{\sigma}_i)(A) \cup f(M_A)$ for all i except $i = 6$.*

Proof. Take $i \neq 7$ and let $\mu \in \mathbb{C}$ be such that $\mu \in f(\widetilde{\sigma}_i(A)) \setminus f(M_A)$. By considering the function $f - \mu$ instead of f, we can assume without loss of generality that $\mu = 0$. As $0 \notin f(M_A)$, $f^{-1}\{0\} \cap \widetilde{\sigma}(A)$ must be finite. Let $\lambda_1, \ldots, \lambda_N$ be all the points in

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$f^{-1}\{0\} \cap \widetilde{\sigma}(A)$ (so $\lambda_j \in \widetilde{\sigma}_i(A)$ for some $j \in \{1, \ldots, N\}$), and let n_j be the order of the zero of f at λ_j. Let $b \in \rho(A)$ and set

$$g(z) := f(z) \prod_{j=1}^{N} \left(\frac{b - z}{\lambda_j - z} \right)^{n_j} . \tag{3.1}$$

Then $0 \notin g(\widetilde{\sigma}(A))$ and is g quasi-regular at M_A. Several applications of lemma 3.3 imply $g \in \mathcal{M}(A)$, and lemma 3.6(a) yields $f(A) \notin \Phi_i$.

Take now $i \neq 6$ and let $\mu \in \mathbb{C}$ be such that $\mu \notin f(\widetilde{\sigma}_i(A)) \cup f(M_A)$. We prove that $\mu \notin \widetilde{\sigma}_i(f(A))$. We can assume $\mu = 0$. Again, $f^{-1}\{0\} \cap \sigma(A)$ has finite cardinal, so let g be as given in (3.1). Since $\lambda_j - A \in \Phi_i$ for all $j = 1, \ldots, n$, applications of lemmas 3.3 and 3.6(b) yield $f(A) \in \Phi_i$, as we wanted to show.

Assume now that $\mu = \infty$. If $\rho(f(A)) \neq \emptyset$ take $b \in \rho(f(A))$. An application of what we have already proven to the function $1/b - f(z)$ shows the claim, see the paragraph below definition 2.1. Hence, all that is left to prove is that we can assume without loss of generality that $\rho(f(A)) \neq \emptyset$. Take $\nu \in \mathbb{C} \setminus f(M_A)$, so $f^{-1}\{\nu\} \cap \widetilde{\sigma}(A)$ has finite cardinality. Let $\nu_1, \ldots, \nu_M$ be all the points in $f^{-1}\{\nu\} \cap \sigma(A)$, and let m_j be the order of the zero of $f - \nu$ at ν_j. Let $b \in \rho(A)$ and set

$$h(z) := (f(z) - \nu) \prod_{j=1}^{M} \left(\frac{b - z}{\nu_j - z} \right)^{m_j} . \tag{3.2}$$

Lemma 3.3 yields $h \in \mathcal{M}(A)$ with $\operatorname{dom} f(A) = \operatorname{dom} h(A)$, and using (3.2) it is readily seen that $\operatorname{dom} f(A)^n = \operatorname{dom} h(A)^n$ for all $n \in \mathbb{N}$. In particular, $\infty \in \widetilde{\sigma}_i(f(A))$ if and only if $\infty \in \widetilde{\sigma}_i(h(A))$. Since $0 \notin h(\widetilde{\sigma}(A))$, proposition 3.1 implies $0 \in \rho(h(A))$. Therefore, we can assume that $\rho(f(A)) \neq \emptyset$, and the proof is done. □

4. General case

In this section we deal with the case $M_A = \{-a, a, \infty\} \cap \widetilde{\sigma}(A) \neq \emptyset$. The difficulty of this setting arises from the fact that f is not necessarily either holomorphic or meromorphic at M_A, so the factorization techniques used in § 3 do not apply here. Also, note that item 3) in the Introduction is not true if f has a zero in M_A. To address this issue, we apply different techniques depending on the topological properties (relative to $\widetilde{\sigma}(A)$) of M_A. If these points are isolated points of $\widetilde{\sigma}(A)$, we provide useful properties of the spectral projections associated with such points in lemmas 4.4 and 4.5. If otherwise, these points are limit points of $\widetilde{\sigma}(A)$, we make use of a mixture of topological properties shared by all the essential spectra considered here, and of the algebraic properties of the regularized functional calculus given in propositions 4.6 and 4.7.

First, we give some remarks about M_A. These are key for the proof of the spectral mapping theorems.

Remark 4.1. Let $T \in C(X)$ with non-empty resolvent set, $\mathrm{d} \in \widetilde{\sigma}(T)$ with d an accumulation point of $\rho(T)$, and $i \neq 0, 9$. The following statements about the essential spectrum are well-known, see for example [**6**, Sections I.3 & I.4], [**13**, Chapter 4§5] and [**20**, Section V.6].

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(a) If d is also an accumulation point of $\widetilde{\sigma}(T)$, then $\mathrm{d} \in \widetilde{\sigma}_i(T)$.

(b) If $\mathrm{d} \in \widetilde{\sigma}_i(T)$ and d is not an accumulation point of $\widetilde{\sigma}_i(T)$, then there is a neighborhood Ω of d such that $\widetilde{\sigma}(T) \cap \Omega$ consists of d and a countable (possibly empty) set of eigenvalues of T with finite dimensional eigenspace, which are isolated between themselves.

(c) If $\mathrm{d} \notin \widetilde{\sigma}_i(T)$, then d is an isolated point of $\widetilde{\sigma}(T)$. Moreover, $\mathrm{d} \in \sigma_p(T)$ with $\mathrm{nul}(d-T) = \mathrm{def}(d-T) < \infty$, $\alpha(d-T) = \delta(d-T) < \infty$, and $\dim(\cup_{n \geqslant 1} \ker(d-T)^n) < \infty$.

LEMMA 4.2. *Let $A \in \mathrm{BSect}(\omega, a)$, $\mathrm{d} \in M_A$ and $i, j \neq 0, 9$. Then*

- $\mathrm{d} \in \widetilde{\sigma}_i(A)$ *if and only if* $\mathrm{d} \in \widetilde{\sigma}_j(A)$,
- *if $\infty \in \widetilde{\sigma}(A)$, then $\infty \in \widetilde{\sigma}_i(A)$.*

Proof. If $\mathrm{d} \in \widetilde{\sigma}_6(A)$, then $\mathrm{d} \in \widetilde{\sigma}_i(A)$ since $\widetilde{\sigma}_6(A) \subseteq \widetilde{\sigma}_i(A)$ for any $i \neq 0, 9$. If $\mathrm{d} \notin \widetilde{\sigma}_6(A)$, then remark 4.1(c) implies $\mathrm{d} \notin \widetilde{\sigma}_i(A)$ for $i \neq 0, 9$, and the first item follows.

Now, assume $\infty \in \widetilde{\sigma}(A)$. If ∞ is an accumulation point of $\widetilde{\sigma}(A)$, remark 4.1(a) implies $\infty \in \widetilde{\sigma}_i(A)$ (note that ∞ is an accumulation point of the resolvent set of a bisectorial-like operator). Suppose then that ∞ is an isolated point of $\widetilde{\sigma}(A)$ and take $b \in \rho(A)$. Then 0 is an isolated point of $\widetilde{\sigma}((b-A)^{-1})$. Since $(b-A)^{-1}$ is an injective operator, $0 \in \widetilde{\sigma}_i((b-A)^{-1})$ by remark 4.1(c), and the claim follows. □

Take $T \in C(X)$ with non-empty resolvent set, and let Λ be a subset of $\widetilde{\sigma}(T)$ which is open and closed in the relative topology of $\widetilde{\sigma}(T)$ (i.e. Λ is the union of some components of $\widetilde{\sigma}(T)$). If $\infty \notin \Lambda$, the spectral projection P_Λ of T is given by

$$P_\Lambda := \int_\Gamma (z-A)^{-1} \, \mathrm{d}z, \tag{4.1}$$

where Γ is a finite collection of paths contained in $\rho(T)$ such that Γ has index 1 with respect to every point in Λ, and has index 0 with respect to every point in $\sigma(T) \setminus \Lambda$. If $\infty \in \Lambda$, then the spectral projection P_Λ of T is given by $P_\Lambda := I - P_{\widetilde{\sigma}(T)\setminus\Lambda}$, where $P_{\widetilde{\sigma}(T)\setminus\Lambda}$ is as in (4.1).

We collect in the form of a lemma some well-known results about spectral projections, see for instance [**5**, Section V.9].

LEMMA 4.3. *Let T, Λ be as above. Then*

1. *P_Λ is a bounded projection commuting with T;*
2. *$\widetilde{\sigma}(T_\Lambda) = \Lambda$, where $T_\Lambda : \mathrm{ran}\, P_\Lambda \to \mathrm{ran}\, P_\Lambda$ is the part of T in $\mathrm{ran}\, P_\Lambda$.*

As a consequence, for $\lambda \in \Lambda \cap \mathbb{C}$, $\ker(\lambda - T) \subseteq \mathrm{ran}\, P_\Lambda$ and $\mathrm{ran}(I - P_\Lambda) \subseteq \mathrm{ran}(\lambda - T)$. Also, if $\infty \notin \Lambda$, then $\mathrm{ran}\, P_\Lambda \subseteq \mathrm{dom}\, T$.

We also need the following two lemmas.

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LEMMA 4.4. *Let $A \in \mathrm{BSect}(\omega, a)$, and let $\Lambda \subseteq \widetilde{\sigma}(A)$ be an open and closed subset in the relative topology of $\widetilde{\sigma}(A)$. Then $A_\Lambda \in \mathrm{BSect}(\omega, a)$ with $\mathcal{M}(A) \subseteq \mathcal{M}(A_\Lambda)$, and one has*

$$f(A_\Lambda) = f(A)|_{\operatorname{ran} P_\Lambda}, \quad \textit{and} \quad \widetilde{\sigma}_i(f(A_\Lambda)) \subseteq \widetilde{\sigma}_i(f(A)),$$

for every $f \in \mathcal{M}(A)$ and $i \neq 7, 8$.

Proof. It follows by lemma 4.3 that $\widetilde{\sigma}(A_\Lambda) = \Lambda \subseteq \overline{BS(\omega, a)} \cup \{\infty\}$. Moreover, it is readily seen that $(z - A_\Lambda)^{-1} = (z - A)^{-1}|_{\operatorname{ran} P_\Lambda}$ for every $z \in \rho(A)$. As a consequence, one gets that A_Λ is indeed a bisectorial-like operator on $\operatorname{ran} P_\Lambda$ of angle ω and half-width a, and that $\mathcal{E}(A) \subseteq \mathcal{E}(A_\Lambda)$ with $f(A)|_{\operatorname{ran} P_\Lambda} = f(A_\Lambda)$ for all $f \in \mathcal{E}(A)$. Thus, if $e \in \mathcal{E}(A)$ is a regularizer for $f \in \mathcal{M}(A)$, then e is also a regularizer for f with respect to A_Λ, so $\mathcal{M}(A) \subseteq \mathcal{M}(A_\Lambda)$.

Now, we have $P_\Lambda f(A) \subseteq f(A) P_\Lambda$ for every $f \in \mathcal{M}(A)$ by lemma 2.7. From this and the above properties, it is not difficult to get, for every $f \in \mathcal{M}(A)$, $f(A_\Lambda) = f(A)|_{\operatorname{ran} P_\Lambda}$, with $\operatorname{dom} f(A_\Lambda) = \operatorname{dom} f(A) \cap \operatorname{ran} P_\Lambda$, $\ker f(A_\Lambda) = \ker f(A) \cap \operatorname{ran}(P_\Lambda)$, $\operatorname{ran}(f(A_\Lambda)) = \operatorname{ran} f(A) \cap \operatorname{ran} P_\Lambda$. Hence $\widetilde{\sigma}_i(f(A_\Lambda)) \subseteq \widetilde{\sigma}_i(f(A))$ for $i \in \{0, 1, 4, 5, 6\}$.

Thus, all that is left to prove are the spectral inclusions for $\widetilde{\sigma}_2, \widetilde{\sigma}_3$ and $\widetilde{\sigma}_9$. Regarding the first case, assume $f(A) \in \Phi_2$, i.e., $\operatorname{ran} f(A) \oplus W = X$ for some closed linear subspace W. Then $\operatorname{ran} f(A_\Lambda) \oplus (\operatorname{ran} f(A) \cap \operatorname{ran}(I - P_\Lambda)) \oplus W = X$ since $\operatorname{ran} f(A) = \operatorname{ran} f(A_\Lambda) \oplus (\operatorname{ran} f(A) \cap \operatorname{ran}(I - P_\Lambda))$. Thus, there exists a bounded projection $\widetilde{P}$ from X onto $\operatorname{ran} f(A_\Lambda)$. Then, $\widetilde{P}|_{\operatorname{ran} P_\Lambda}$ is a bounded projection from $\operatorname{ran} P_\Lambda$ onto $\operatorname{ran} f(A_\Lambda)$, that is, $\operatorname{ran} f(A_\Lambda)$ is complemented in $\operatorname{ran} P_\Lambda$, so $f(A_\Lambda) \in \Phi_2$. An analogous reasoning with $\operatorname{dom} f(A)$ and $\operatorname{dom} f(A_\Lambda)$ shows that, if $\infty \in \widetilde{\sigma}_2(f(A_\Lambda))$, then $\infty \in \widetilde{\sigma}_2(f(A))$. Hence, the inclusion $\widetilde{\sigma}_2(f(B)) \subseteq \widetilde{\sigma}_2(f(A))$ holds for any $f \in \mathcal{M}(A)$.

Similar reasoning proves the inclusion for $\widetilde{\sigma}_3$. For $\widetilde{\sigma}_9$, the inclusion follows from $\alpha(f(A)) = \max\{\alpha(f(B)), \alpha(f(A)|_Z)\}$ and $\delta(f(A)) = \max\{\delta(f(B)), \delta(f(A)|_Z)\}$, see [**20**, Problem V.6]. □

Let $A \in \mathrm{BSect}(\omega, a)$. Recall that $M_A \setminus \widetilde{\sigma}_i(A) = M_A \setminus \widetilde{\sigma}_j(A)$ for $i, j \neq 0, 9$, see lemma 4.2. Then, by remark 4.1(c), $M_A \setminus \widetilde{\sigma}_i(A)$ is an open and closed subset of $\widetilde{\sigma}(A)$ (in the relative topology of $\widetilde{\sigma}(A)$) for $i \neq 0, 9$. Also,

LEMMA 4.5. *Let $A \in \mathrm{BSect}(\omega, a)$, and let $\Lambda = \widetilde{\sigma}(A) \setminus (M_A \setminus \widetilde{\sigma}_j(A))$, with $j \neq 0, 9$. Then $M_{A_\Lambda} \subseteq \widetilde{\sigma}_i(A_\Lambda)$ for all $0 \leqslant i \leqslant 6$ and*

$$\widetilde{\sigma}_i(f(A)) = \widetilde{\sigma}_i(f(A_\Lambda)), \quad f \in \mathcal{M}(A), \quad 1 \leqslant i \leqslant 6.$$

Also, $\operatorname{codim}(\operatorname{ran} P_\Lambda) < \infty$.

Proof. The inclusions $\widetilde{\sigma}_i(f(A_\Lambda)) \subseteq \widetilde{\sigma}_i(f(A))$ are given in lemma 4.4. Let us show that the inclusions $\sigma_i(f(A)) \subseteq \sigma_i(f(A_\Lambda))$ also hold. To do this, we prove the following claims for all $f \in \mathcal{M}(A)$,

(1) If $\operatorname{nul}(f(A_\Lambda)) < \infty$, then $\operatorname{nul}(f(A)) < \infty$.

(2) If $\operatorname{def}(f(A_\Lambda)) < \infty$, then $\operatorname{def}(f(A)) < \infty$.

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(3) If $\operatorname{ran} f(A_\Lambda)$ is closed/complemented in $\operatorname{ran} P_\Lambda$, then $\operatorname{ran} f(A)$ is closed/complemented in X.

(4) If $\ker f(A_\Lambda)$ is complemented in $\operatorname{ran} P_\Lambda$, then $\ker f(A)$ is complemented in X.

Set $\Omega = \widetilde{\sigma}(A) \setminus \Lambda$, so $\Omega = M_A \setminus \widetilde{\sigma}_i(A)$ for any $i \in \{1, \ldots, 6\}$. Lemma 4.3 implies $\widetilde{\sigma}_i(A|_\Omega) = \emptyset$ for all $1 \leqslant i \leqslant 6$. Note that, given a closed operator T with $\rho(T) \neq \emptyset$ on a Banach space Y, the essential spectra $\sigma_i(T)$, $1 \leqslant i \leqslant 8$, are empty if and only if Y is finite-dimensional. This implies that $\operatorname{ran} P_\Omega$ is finite dimensional (see e.g. [**8**, Section 5] and remark 3.5), i.e., $\operatorname{codim} \operatorname{ran} P_\Lambda < \infty$ since $\operatorname{ran} P_\Lambda$ and $\operatorname{ran} P_\Omega$ are complementary subspaces. Since $\ker f(A_\Lambda) = \ker f(A) \cap \operatorname{ran} P_\Lambda$ and $\operatorname{ran} f(A_\Lambda) = \operatorname{ran} f(A) \cap \operatorname{ran} P_\Lambda$ (see the proof of lemma 4.4), we conclude that claims (1) and (2) hold true.

For the claim regarding closedness in (3), assume $\operatorname{ran} f(A_\Lambda)$ is closed in $\operatorname{ran} P_\Lambda$, so $\operatorname{ran} f(A_\Lambda)$ is closed in X too. Since $\operatorname{ran} f(A) = \operatorname{ran} f(A_\Lambda) \oplus \operatorname{ran} f(A_\Omega)$, we have that $\operatorname{ran} f(A)/\operatorname{ran} f(A_\Lambda)$ is finite-dimensional in $X/\operatorname{ran} f(A_\Lambda)$, hence closed. Thus $\operatorname{ran} f(A)$ is closed in X. For the claim regarding complementation in (3), assume $\operatorname{ran} f(A_\Lambda) \oplus U = \operatorname{ran} P_\Lambda$ for some closed subspace U. Note that $\operatorname{ran} f(A_\Omega) \oplus V = \operatorname{ran} P_\Omega$ for some finite-dimensional closed subspace since $\dim \operatorname{ran} P_\Omega < \infty$. Therefore $\operatorname{ran} f(A) \oplus (U \oplus V) = X$, and (3) follows. An analogous reasoning proves the claim (4).

Now, a similar reasoning as above with subspaces $\operatorname{dom} f(A), \operatorname{dom} f(A_\Lambda)$ shows that, if $\infty \in \widetilde{\sigma}_i(f(A))$, then $\infty \in \widetilde{\sigma}_i(f(A_\Lambda))$ for all $1 \leqslant i \leqslant 6$. Therefore, $\widetilde{\sigma}_i(f(A)) \subseteq \widetilde{\sigma}_i(f(A_\Lambda))$, as we wanted to show.

Finally, to prove that $M_{A_\Lambda} \subseteq \widetilde{\sigma}_i(A_\Lambda)$ (for all $1 \leqslant i \leqslant 6$), note that $M_A \setminus \widetilde{\sigma}_i(A) = \widetilde{\sigma}(A) \setminus \Lambda \subseteq \rho(A_\Lambda)$ by lemma 4.3. □

We are now ready to prove the spectral mapping theorems for most of the extended essential spectra considered in the Introduction. For the sake of clarity, we separate the proof of each inclusion into two different propositions.

Proposition 4.6. *Let $A \in \operatorname{BSect}(\omega, a)$ and let $f \in \mathcal{M}(A)$ be quasi-regular at M_A. Then*

$$\widetilde{\sigma}_i(f(A)) \subseteq f(\widetilde{\sigma}_i(A)), \quad \textit{for all } i \textit{ except } i \neq 6, 9.$$

Proof. The inclusion for $\widetilde{\sigma}_0$ is already given in Proposition 3.1. For $1 \leqslant i \leqslant 5$, we can assume $M_A \subseteq \widetilde{\sigma}_i(A)$ without loss of generality by lemma 4.5. Thus $\widetilde{\sigma}_i(f(A)) \subseteq f(\widetilde{\sigma}_i(A))$ for $1 \leqslant i \leqslant 5$ by proposition 3.7(b).

Now, we show the inclusions for $\widetilde{\sigma}_7, \widetilde{\sigma}_8$. So take $i \in \{7, 8\}$, and let $\mu \in \mathbb{C} \setminus f(\widetilde{\sigma}_i(A))$. Note that we can assume $\mu = 0$. If $0 \notin f(M_A)$, proposition 3.7(b) implies $0 \notin \widetilde{\sigma}_i(f(A))$. So assume $0 \in f(M_A)$. As $0 \notin f(\widetilde{\sigma}_i(A))$, lemma 4.2 and remark 4.1(c) imply that $f^{-1}\{0\} \cap \widetilde{\sigma}(A)$ is a finite set. Let $\lambda_1, \ldots, \lambda_N$ be an enumeration of $(f^{-1}\{0\} \cap \widetilde{\sigma}(a)) \setminus M_A$, and let $n_1, \ldots, n_N$ be the multiplicity of f at $\lambda_1, \ldots, \lambda_N$ respectively. Set $r(z) := \Pi_{j=1}^N (\lambda_j - z)^{n_j}/(b - z)^{n_j}$ for some $b \in \rho(A)$, and $g(z) := f(z)/r(z)$ for $z \in \mathbb{D}$. Several applications of lemma 3.6 yield $g \in \mathcal{M}(A)$ and $f(A) = r(A)g(A) = g(A)r(A)$ with $\operatorname{dom} f(A) = \operatorname{dom} g(A)$. Also, one has $g^{-1}\{0\} \cap \widetilde{\sigma}(A) \subseteq M_A \setminus \widetilde{\sigma}_i(A)$. Then, $g^{-1}\{0\} \cap \Lambda = \emptyset$ where $\Lambda := \widetilde{\sigma}(A) \setminus (M_A \setminus \widetilde{\sigma}_i(A))$. Thus, $g(A_\Lambda)$ is invertible by proposition 3.1 and lemma 4.4. On the other hand, $\dim \operatorname{ran}(I - P_\Lambda) < \infty$ by lemma 4.5 (see also lemma 4.2), so $g(A_{\widetilde{\sigma}(A)\setminus\Lambda}), r(A_{\widetilde{\sigma}(A)\setminus\Lambda}) \in \Phi_i$. Since $r(A) \in \Phi_i$ (see e.g. [**7**, Lemma 3] and [**8**, Lemma 5(c)]), we conclude that $r(A_\Lambda) \in \Phi_i$. Thus, reasoning with f, g, r and f^n, g^n, r^n as in the proof of lemma 3.6, we obtain $f(A_\Lambda) \in \Phi_i$. Using again that $\dim \operatorname{ran}(I - P_\Lambda) < \infty$, one gets $f(A) \in \Phi_i$ as claimed, that is $0 \notin \widetilde{\sigma}_i(f(A))$.

Now, for $i = 7, 8$, assume $\infty \notin f(\widetilde{\sigma}_i(A))$. Reasoning as at the end of the proof of proposition 3.7, we can assume $\rho(f(A)) \neq \emptyset$ without loss of generality. So let $\nu \in \rho(f(A))$. An application of what we have already proven to the function $1/(\nu - f)$ shows that $0 \notin \widetilde{\sigma}_i((\nu - f(A))^{-1})$, that is $\infty \notin \widetilde{\sigma}_i(f(A))$ for $i = 7, 8$, and the proof is finished. □

The proof of the proposition below contains the answer to the question/open problem posed in [**11**], where the inclusion $f(\widetilde{\sigma}(A)) \subseteq \widetilde{\sigma}(f(A))$ was established only under additional assumptions on f regarding its behaviour at the points in M_A. This is done via the following simple observation (missing in [**11**]), which in the end is very helpful to prove the spectral mapping theorem for the essential spectra: since the inclusion $f(\widetilde{\sigma}(A) \setminus M_A) \subseteq \widetilde{\sigma}(f(A))$ has already been established in [**11**, Corollary 4.5] (see also proposition 3.7) and $\widetilde{\sigma}(f(A))$ is closed, one may suppose without loss of generality that each point in M_A is isolated in $\widetilde{\sigma}(A)$. Passing to a subspace via spectral projections, one may suppose that $\widetilde{\sigma}(A)$ consists only of a single point of M_A, say d. Now, the inclusion theorem [**11**, Proposition 6.3] (see also proposition 4.6) tells $\widetilde{\sigma}(f(A)) \subseteq f(\widetilde{\sigma}(A)) = \{f(d)\}$. Since the spectral projection is non-trivial, the set $\widetilde{\sigma}(f(A))$ must be non-empty, hence equals $\{f(d)\}$. This tells $f(d) \in \widetilde{\sigma}(f(A))$, the missing information.

Proposition 4.7. *Let $A \in \mathrm{BSect}(\omega, a)$ and let $f \in \mathcal{M}(A)$ be quasi-regular at M_A. Then*

$$f(\widetilde{\sigma}_i(A)) \subseteq \widetilde{\sigma}_i(f(A)), \quad \textit{for all } i \textit{ except } i \neq 7, 9.$$

Proof. Note that $\widetilde{\sigma}_6(f(A)) \subseteq \widetilde{\sigma}_i(f(A))$ for each $i \neq 7, 9$. Thus, proposition 3.7 and lemma 4.2 yield that it is enough to prove the claim for $i \in \{0, 6\}$. Hence, we assume $i \in \{0, 6\}$ from now on.

Let $\mu \in f(\widetilde{\sigma}_i(A))$ with $\mu \neq \infty$, so we can assume $\mu = 0$ without loss of generality. If $0 \in f(\widetilde{\sigma}_i(A)) \setminus f(M_A)$, then $0 \in \widetilde{\sigma}_i(f(A))$ by proposition 3.7(a). So assume $0 \in f(\widetilde{\sigma}_i(A))$ with $0 \in f(M_A)$. If any point in $f^{-1}\{0\} \cap \widetilde{\sigma}_i(A)$ is an accumulation point of $\widetilde{\sigma}_i(A)$ (and we rule out the trivial case where f is constant), then 0 is an accumulation point of $f(\widetilde{\sigma}_i(A)) \setminus f(M_A) \subseteq \widetilde{\sigma}_i(f(A))$ (see proposition 3.7(a)), thus $0 \in \widetilde{\sigma}_i(f(A))$ since $\sigma_i(T)$ is closed for any $T \in C(X)$. So assume that each point in $f^{-1}\{0\} \cap \widetilde{\sigma}_i(A)$ is an isolated point in $\widetilde{\sigma}_i(A)$, and set

$$V_A := \{\mathrm{d} \in f^{-1}\{0\} \cap \widetilde{\sigma}_i(A) \,|\, \mathrm{d} \text{ is not an isolated point of } \widetilde{\sigma}(A)\},$$

which is a finite set by remark 4.1(c).

Assume first that V_A is not empty (thus $i = 6$). One has that, for each $\mathrm{d} \in V_A$, there is some neighbourhood Ω_d of d such that $\Omega_d \cap \widetilde{\sigma}(A) = \{d, \lambda_1^d, \lambda_2^d, \ldots\}$, where $\lambda_j^d \in \sigma_p(A) \setminus \sigma_6(A)$, each λ_j^d is an isolated point of $\sigma(A)$, and $\lambda_j^d \xrightarrow[j \to \infty]{} d$. Thus, $(f^{-1}\{0\} \cap \widetilde{\sigma}(A)) \setminus (M_A \cup (\cup_{\mathrm{d} \in V_A} \Omega_d))$ is a finite set. Let $\kappa_1, \ldots, \kappa_N$ be the

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elements of this set, let $n_1, \dots, n_N$ be the multiplicity of the zero of f at $\kappa_1, \dots, \kappa_N$ respectively, and set $g(z) := f(z)\Pi_{j=1}^N (b-z)^{n_j}/(\kappa_j - z)^{n_j}$. Several applications of lemma 3.3 yield $g \in \mathcal{M}(A)$ with $\operatorname{dom} g(A) = \operatorname{dom} f(A)$, and

$$f(A) = \left(\prod_{j=1}^{N} \left((\kappa_j - A)(b-A)^{-1}\right)^{n_j}\right) g(A) = g(A) \prod_{j=1}^{N} \left((\kappa_j - A)(b-A)^{-1}\right)^{n_j}, \tag{4.2}$$

where in the last term we regard $(\kappa_j - A)(b-A)^{-1}$ as bounded operators on $\operatorname{dom} f(A)$. Let us show that $0 \in \widetilde{\sigma}_6(g(A))$, from which follows $0 \in \widetilde{\sigma}_6(f(A))$, see for example [6, Theorem I.3.20]. Note that $g^{-1}\{0\} \cap \widetilde{\sigma}(A) \subseteq M_A \cup (\cup_{\mathrm{d} \in V_A} \Omega_d)$, and that if $\mathrm{d} \in M_A \setminus \widetilde{\sigma}_6(A)$ then d is an isolated point of $\widetilde{\sigma}(A)$, see remark 4.1(a). As a consequence, 0 is an accumulation point of $\mathbb{C} \setminus g(\widetilde{\sigma}(A))$. Thus, proposition 3.1 implies that 0 is an accumulation point of $\rho(g(A))$. If 0 is also an accumulation point of $\widetilde{\sigma}(g(A))$, then $0 \in \widetilde{\sigma}_6(g(A))$ by remark 4.1. So assume that 0 is not an accumulation point of $\widetilde{\sigma}(g(A))$. Since $\sigma_p(g(A)) \subseteq g(\sigma_p(A))$ (lemma 2.8), and $\lambda_j^d \in \sigma_p(A)$ with $\lambda_j^d \xrightarrow[j\to\infty]{} d$ for each $\mathrm{d} \in V_A$, it follows $g(\lambda_j^d) = 0$ for all but finitely many pairs $(j, d) \in \mathbb{N} \times V_A$. Hence, the set $g^{-1}\{0\} \cap \sigma_p(A)$ has infinite cardinal, so $\operatorname{nul}(g(A)) \geqslant \sum_{\lambda \in g^{-1}\{0\} \cap \sigma_p(A)} \operatorname{nul}(\lambda - A) = \infty$. Then remark 4.1(c) yields $0 \in \widetilde{\sigma}_6(g(A))$, as we wanted to prove.

Now, assume $V_A = \emptyset$ for $i = 0, 6$, so each $\mathrm{d} \in f^{-1}\{0\} \cap \widetilde{\sigma}_i(A)$ is an isolated point of $\widetilde{\sigma}(A)$. Set $\Lambda := f^{-1}\{0\} \cap M_A \cap \widetilde{\sigma}_i(A)$. Note that, in the case $i = 6$, then $\dim \operatorname{ran} P_\Lambda = \infty$ as a consequence of lemma 4.3. Since $f(\Lambda) = \{0\}$, we have $\widetilde{\sigma}(f(A_\Lambda)) \subseteq \{0\}$ by proposition 3.1. Hence, $\widetilde{\sigma}_i(f(A_\Lambda)) = \{0\}$ since $\widetilde{\sigma}_i(f(A_\Lambda))$ cannot be the emptyset (at least for any operator with non-empty resolvent set, see for example [8, Section 5] and remark 3.5). Therefore, $0 \in \widetilde{\sigma}_i(f(A))$ by lemma 4.4, as we wanted to show.

Finally, we deal with the case $\mu = \infty$. Reasoning as at the end of the proof of proposition 3.7, we can assume that $\rho(f(A)) \neq \emptyset$. Take any $\nu \in \rho(f(A))$, so $\infty \in \widetilde{\sigma}_i(f(A))$ if and only if $0 \in \widetilde{\sigma}_i((\nu - f(A))^{-1})$. But $0 \in \widetilde{\sigma}_i((\nu - f(A))^{-1})$ by applying what we have already proven to the function $1/(\nu - f)$, and the proof is finished. □

As a consequence, we have the following

THEOREM 4.8. *Let $A \in \mathrm{BSect}(\omega, a)$ and $f \in \mathcal{M}(A)$ quasi-regular at M_A. Then*

$$\begin{aligned}
&\widetilde{\sigma}_i(f(A)) = f(\widetilde{\sigma}_i(A)), \qquad \textit{for all } i \textit{ except } i \neq 6, 7, 9,\\
&f(\widetilde{\sigma}_6(A)) \subseteq \widetilde{\sigma}_6(f(A)),\\
&\widetilde{\sigma}_7(f(A)) \subseteq f(\widetilde{\sigma}_7(A)).
\end{aligned}$$

Proof. Immediate consequence of propositions 4.6 and 4.7. □

It is known that the spectral mapping theorem does not hold (in general) for $\widetilde{\sigma}_6, \widetilde{\sigma}_7$, see for instance [6, Section 3]. However, we do not know if it holds for $\widetilde{\sigma}_9$ for the regularized functional calculus considered here. Indeed, it holds if $M_A = \emptyset$, see proposition 3.7.

5. Final remarks

5.1. Operators with bounded functional calculus

A natural question is whether the condition of quasi-regularity can be relaxed in Theorem 4.8. For instance, one may think of a function $f \in \mathcal{M}(A)$ with well-defined limits (through the domain of f) at M_A. The properties of bounded functional calculi (see for instance [**4**, **12**, **17**]) are very helpful.

DEFINITION 5.1. *Let $A \in \mathrm{BSect}(\omega, a)$. We say that the regularized functional calculus of A is bounded if $f(A) \in \mathcal{L}(X)$ for every bounded $f \in \mathcal{M}(A)$.*

LEMMA 5.2. *Let $A \in \mathrm{BSect}(\omega, a)$ and $f \in \mathcal{M}(A)$. Then f is regular at $\sigma_p(A) \cap M_A$.*

Proof. The proof is analogous to the case of sectorial operators, see [**10**, Lemma 4.2]. □

THEOREM 5.3. *Let $A \in \mathrm{BSect}(\omega, a)$ such that the regularized functional calculus of A is bounded, and let $f \in \mathcal{M}(A)$ with (possibly ∞-valued) limits at M_A. Then*

$$\widetilde{\sigma}_i(f(A)) = f(\widetilde{\sigma}_i(A)), \quad \textit{for all } i \textit{ except } i \neq 6, 7, 9,$$
$$f(\widetilde{\sigma}_6(A)) \subseteq \widetilde{\sigma}_6(f(A)),$$
$$\widetilde{\sigma}_7(f(A)) \subseteq f(\widetilde{\sigma}_7(A)).$$

Proof. The proof of this claim is completely analogous to the path followed in this paper to prove Theorem 4.8. Indeed, the quasi-regularity notion is only explicitly needed in the proofs of lemma 2.11 and proposition 3.1. This creates a 'cascade' effect, and all following results need the quasi-regularity assumption in order to apply proposition 3.1. Therefore, we will prove the claim once we prove the following version of proposition 3.1:

"Let $A \in \mathrm{BSect}(\omega, a)$ such that the regularized functional calculus of A is bounded, and let $f \in \mathcal{M}(A)$ with (possibly ∞-valued) limits at M_A. Then $\widetilde{\sigma}(f(A)) \subseteq f(\widetilde{\sigma}(A))$."

We outline the proof of this claim. Let $\mu \in \mathbb{C}_\infty$ with $\mu \notin f(\widetilde{\sigma}(A))$, and set $f_\mu = 1/(\mu - f)$ if $\mu \in \mathbb{C}$ or $f_\mu = f$ if $\mu = \infty$, and we will show that $f_\mu \in \mathcal{M}(A)$ with $f_\mu(A) \in \mathcal{L}(X)$. Note that f_μ has finite limits at M_A. Even more, f_μ is regular at $\sigma_p(A) \cap M_A$ by lemma 5.2. Proceeding as in the proof of lemma 2.11, we can assume that f_μ has finitely many poles, all of them contained in $\rho(A)$. Let $r(z) := \Pi_{j=1}^n (\lambda_j - z)^{n_j}/(b - z)^{n_j}$, where λ_j, n_j are the poles of f_μ and their order, respectively. Hence, rf_μ has no poles, is regular at $\sigma_p(A) \cap M_A$ and has finite limits at M_A, thus rf_μ is bounded. For any $b \in \rho(A)$, the function $h(z) := 1/(b - z)\, \Pi_{\mathrm{d} \in \{-a, a\} \setminus \sigma_p(A)} (z - a)/(b - z)$ regularizes rf_μ, so $rf_\mu \in \mathcal{M}_A$. Since the regularized functional calculus of A is bounded, then $rf_\mu(A) \in \mathcal{L}(X)$. Moreover, $r(A)$ is bounded and invertible. Therefore, hr regularizes f_μ with $f_\mu(A) = r(A)^{-1}(rf_\mu)(A) \in \mathcal{L}(X)$, and the claim follows. □

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5.2. Spectral mapping theorems for sectorial operators and strip-type operators

Lemmas 2.7–2.11 are the key properties of the regularized functional calculus of bisectorial-like operators that we use in the proofs given in § 3 and 4. Therefore, for similar regularized functional calculi satisfying such properties, one may prove spectral mapping theorems for essential spectra analogous to the ones given in Theorem 4.8. In particular, we obtain the following results for the regularized functional calculus of sectorial operators and the regularized functional calculus of strip-type operators considered in [**11**].

THEOREM 5.4. *Let A be a sectorial operator of angle $\phi \in [0, 2\pi)$, and let f be a function in the domain of the regularized functional calculus of A that is quasi-regular at $\{0, \infty\} \cap \widetilde{\sigma}(A)$. Then*

$$\begin{aligned} &\widetilde{\sigma}_i(f(A)) = f(\widetilde{\sigma}_i(A)), \quad \textit{for all } i \textit{ except } i \neq 6, 7, 9, \\ &f(\widetilde{\sigma}_6(A)) \subseteq \widetilde{\sigma}_6(f(A)), \\ &\widetilde{\sigma}_7(f(A)) \subseteq f(\widetilde{\sigma}_7(A)). \end{aligned}$$

THEOREM 5.5. *Let A be a strip-type operator of height $h \geqslant 0$, and let f be a function in the domain of the regularized functional calculus of A that is quasi-regular at $\{\infty\} \cap \widetilde{\sigma}(A)$. Then*

$$\begin{aligned} &\widetilde{\sigma}_i(f(A)) = f(\widetilde{\sigma}_i(A)), \qquad \textit{for all } i \textit{ except } i \neq 6, 7, 9, \\ &f(\widetilde{\sigma}_6(A)) \subseteq \widetilde{\sigma}_6(f(A)), \\ &\widetilde{\sigma}_7(f(A)) \subseteq f(\widetilde{\sigma}_7(A)). \end{aligned}$$

5.3. A spectral mapping theorem for the point spectrum

To finish this paper, we give a spectral mapping theorem for the point spectrum. To prove it, we need to restrict to functions $f \in \mathcal{M}_A$ satisfying the following condition:

(**P**) For each d $\in M_A$ such that $f(d) \notin f(\sigma_p(A)) \cup \{\infty\}$, there is some $\beta > 0$ for which

- if d $\in \mathbb{C}$, then $|f(z) - c_d| \gtrsim |z - d|^\beta$ as $z \to d$, or
- if $d = \infty$, then $|f(z) - c_d| \gtrsim |z|^{-\beta}$ as $z \to d$,

where c_d denotes the limit of $f(z)$ as $z \to d$.

PROPOSITION 5.6. *Let $A \in \mathrm{BSect}(\omega, a)$ and $f \in \mathcal{M}(A)$ such that f is quasi-regular at M_A. Then*

$$f(\sigma_p(A)) \subseteq \sigma_p(f(A)) \subseteq f(\sigma_p(A)) \cup f(M_A).$$

If, furthermore, f satisfies condition (**P**) *above, then $f(\sigma_p(A)) = \sigma_p(f(A))$.*

Proof. The proof of the inclusions $f(\sigma_p(A)) \subseteq \sigma_p(f(A)) \subseteq f(\sigma_p(A)) \cup f(M_A)$ runs the same as for sectorial operators, see [**11**, Proposition 6.5]. Assume now that

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f satisfies (**P**). All that is left to prove is that, if $\mu \in f(M_A)\backslash f(\sigma_p(A))$, then $\mu \notin \sigma_p(f(A))$. The statement is trivial if $\mu = \infty$, so assume $\mu \in \mathbb{C}\backslash f(\sigma_p(A))$, and consider the function $g := 1/(\mu - f)$, which is quasi-regular at M_A. Note that poles of g are precisely $f^{-1}\{\mu\} \subseteq \mathbb{C}\backslash\sigma_p(A)$. Moreover, g is regular at $M_A \cap \sigma_p(A)$, since by assumption $\mu \notin f(\sigma_p(A))$. Let now

$$h_{l,m,n}(z) := \frac{(z-a)^m(z+a)^n}{(b-z)^{l+m+n}}, \quad z \in \mathbb{C}, \quad l,m,n \in \mathbb{N}, \quad b > a.$$

Then, by the assumptions made on f, $h_{l,m,n}g$ is regular for some m, n, l large enough, and where $l, m, n = 0$ if $\infty, a, -a \notin \widetilde{\sigma}(A)\backslash\sigma_p(A)$ respectively. Since $h_{l,m,n}(A) = (A-a)^m(A+a)^n(b-A)^{-(m+n+l)}$ is bounded and injective, $h_{l,m,n}$ regularizes g. Hence $g \in \mathcal{M}(A)$, which by lemma 2.7(4) implies that $\mu - f$ is injective, as we wanted to show. □

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