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A NEW APPROACH TO THE DISTRIBUTION OF THE DURATION OF THE BUSY PERIOD FOR A $G/G/1$ QUEUEING SYSTEM

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Abstract

For a $G/G/1$ queueing system let X_t be the number of customers present at time t and $Y_t(Z_t)$ be the time elapsed since the last arrival of a customer (the last completion of a service) at time t. Let τ_l be the time until the number of customers in the system is reduced from j to $j-l$, given that $X_0 = j \geq l$, $Y_0 = y$, $Z_0 = z$. For the joint distribution of τ_1 and Y_{τ_1} and the Laplace transforms of the τ_l integral equations are derived. Under slight conditions these integral equations have unique solutions which can be determined by standard methods. Our results offer a method for calculating the busy period distribution which is completely different from the usual fluctuation theoretic approach.

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1. Introduction

For the $G/G/1$ queueing system the distribution of the duration of a busy period has been derived by Finch (1961) and Kingman (1961). The transform of the joint distribution of the number N of customers served during a busy period, its duration τ and the length of the subsequent idle period I is given in several textbooks (for example, Prabhu (1980)): for all $z \in (0,1)$, $\theta_1 \geq 0$, $\theta_2 \geq 0$,

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(1.1)
$$E(z^N e^{-\theta_1\tau-\theta_2 I}) = 1-\exp\left\{-\sum_{n=1}^{\infty}\frac{z^n}{n}\iint_{v-u\le 0} e^{-\theta_1 v+\theta_2(v-u)}\,dA^{*n}(u)\,dB^{*n}(v)\right\},$$

where A and B are the distribution functions of the interarrival times and the service times, and A^{*n} denotes n-fold convolution of A. The derivation of (1.1) is based on fluctuation theory applied to the underlying random walk of the queueing system.

In this paper a different approach is developed. For $t \ge 0$ let X_t be the number of customers in the system at time t, Y_t be the time elapsed at time t since the last arrival of a customer and Z_t be the time elapsed between t and the last completion of a service before t. Let the system start at time 0 with the condition $X_0 = j$, $Y_0 = y$, $Z_0 = z$ for some $j \in \mathbf{N}$ and $y, z \ge 0$. Clearly (X_t, Y_t, Z_t) is a Markov process. For $l \le j$ let τ_l be the time passing until the number of customers in the system is reduced from j to $j-l$. We shall derive an integral equation for

(1.2) $$\Psi_{y,z}(\alpha, E) := E(e^{-\alpha\tau_1}1_{\{Y_{\tau_1}\in E\}}|X_0 = j, Y_0 = y, Z_0 = z)$$

($\alpha > 0$, E a Borel subset of $[0,\infty)$) by some rather simple arguments based on the Markov character of the process (X_t, Y_t, Z_t). Under a slight condition this integral equation is seen to determine $(\alpha, E) \to \Psi_{y,z}(\alpha, E)$ uniquely and, moreover, turns out to be solvable by the method of successive approximations. This method provides a sequence $\Psi_{y,z}^{(n)}$ tending to $\Psi_{y,z}$ at an exponential rate of convergence uniformly with respect to (y, z). This will be useful for a numerical determination of the joint distribution of τ_1 and Y_{τ_1}. Note that τ_1, the time for decreasing the numbers of customers from j to $j-1$, is for $j = 1$ simply the ordinary busy period duration, while Y_{τ_1} is, for $j = 1$, the waiting time of the last customer served in the busy period under consideration.

The process (X_t, Y_t, Z_t) has also been studied by Keilson and Kooharian (1960, 1962) who rely on rather involved Wiener-Hopf techniques. The fairly straightforward approach given here is however sufficient to derive the conditional joint distribution of (τ_1, Y_{τ_1}) for an arbitrary initial condition on (X_0, Y_0, Z_0). Further it will be seen in Section 3 that our method can be applied to determine the Laplace transforms $\varphi_j(\cdot|y, z)$ of the τ_j, conditional on $X_0 = l \ge j$, $Y_0 = y$, $Z_0 = z$. Define their joint generating function by

(1.3) $$\Phi(x, u|y, z) = \sum_{j=1}^{\infty}\varphi_j(u|y, z)x^j, \qquad |x| < 1,\ u, y, z \ge 0.$$

We obtain a system of two Fredholm integral equations of the second kind for the functions $y \to \Phi(x, u|y, 0)$ and $z \to \Phi(x, u|0, z)$ and an equation which gives Φ in terms of these two functions. Under weak conditions this

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system of integral equations has the corresponding Neumann series as its unique solution.

In the concluding Section 4 we use our technique to calculate $E(\exp\{-u\tau_j\}| X_0 = l, Z_0 = z)$ for the bulk-arrival queue $M^X/G/1$. The distribution of the busy period duration (i.e. of τ_1 given that $X_0 = 1$, $Z_0 = 0$) has already been derived in Cohen (1980, Chapter III, 2.3) using a different method.

Throughout the paper we assume that the distribution functions of the interarrival times and the service times possess densities $a(x)$ and $b(y)$.

2. The joint distribution of (τ_1, Y_{τ_1}) in a $G/G/1$ queueing system

For $y, z \geq 0$ and $j = 1, 2$ let $Q_{y,z,j}$ be the joint conditional distribution of (τ_j, Y_{τ_j}) given that $X_0 = j$, $Y_0 = y$, $Z_0 = z$. Let

$$\Psi_{y,z}(\alpha, E) := \int_0^\infty e^{-\alpha t} Q_{y,z,1}(dt, E), \tag{2.1}$$

where $\alpha > 0$ and E is a Borel subset of $[0, \infty)$. The duration of a busy period initiated by one customer arriving in the system at time 0 then has the Laplace transform $\alpha \to \Psi_{0,0}(\alpha, [0, \infty))$. We shall now derive an integral equation for $\Psi_{y,z}$.

THEOREM 1. *For all $u, y, z \geq 0$ and all $\alpha > 0$ we have*

(2.2)

$$\begin{aligned}\Psi_{y,z}(\alpha, [0, u]) = &\int_0^\infty e^{-\alpha t} \frac{(1 - A(y + t))b(z + t)}{(1 - A(y))(1 - B(z))} 1_{[0,\infty)}(u - y - t)\, dt \\ &+ \int_{t \geq 0} \int_{w' \geq 0} e^{-\alpha t} \frac{(1 - B(z + t))a(y + t)}{(1 - B(z))(1 - A(y))} \Psi_{0,z+t}(\alpha, dw') \Psi_{w',0}(\alpha, [0, u])\, dt.\end{aligned}$$

PROOF. Let $X_0 = j$, $Y_0 = y$, $Z_0 = z$. The first change of the queue size occurs at the time $\min(S_y, T_z)$, where S_y and T_z are independent random variables with distributions given by

$$P(S_y \geq v) = \frac{1 - A(y + v)}{1 - A(y)}, \qquad v \geq 0, \tag{2.3}$$

$$P(T_z \geq w) = \frac{1 - B(z + w)}{1 - B(z)}, \qquad w \geq 0. \tag{2.4}$$

($S_y(T_z)$ is the first positive arrival (departure) time.) If $S_y = v < T_z$, we have $X_t = j$ for $0 \leq t < v$ and $X_v = j + 1$, $Y_v = 0$, $Z_v = z + v$. The time *remaining* thereafter up to τ_j has the conditional distribution of τ_{j+1}, given that $X_0 = j + 1$, $Y_0 = 0$, $Z_0 = z + v$.

If $T_z = w < S_y$, the analogous relations are $X_t = j$ for $0 \leq t < w$, $X_w = j-1$, $Y_w = y+w$, $Z_w = 0$, so that the *remaining* time up to τ_j has the same distribution as τ_{j-1}, given that $X_0 = j-1$, $Y_0 = y+w$, $Z_0 = 0$.

Using these ideas for $j = 1$ it is seen that $Q_{y,z,1}$ satisfies

(2.5)
$$Q_{y,z,1}([0,t] \times [0,u]) = \int_{\substack{0 \leq s < t \\ y+s \leq u}} \frac{(1-A(y+s))b(z+s)}{(1-A(y))(1-B(z))}\,ds + \iiint_{\substack{s+v \leq t \\ w \leq u}} \frac{(1-B(z+s))a(y+s)}{(1-B(z))(1-A(y))} Q_{0,z+s,2}(dv,dw)\,ds.$$

Next we shall use the following relation between $Q_{y,z,1}$ and $Q_{y,z,2}$:

$$Q_{y,z,2}(E \times F) = \iint_{v'+v'' \in E} \int_{w' \geq 0} Q_{y,z,1}(dv',dw')Q_{w',0,1}(dv'',F) \tag{2.6}$$

for all Borel subsets E, F of $[0,\infty)$. To see (2.6), note that in order to reduce the queue size from 2 to 0, it must be first decreased to 1 which happens at some time v', say, and the time which is then elapsed since the last arrival can be any $w' \in [0,\infty)$. Thereafter the queue size has to be decreased from 1 to 0 after some time v''. Integrating with respect to (v', w') and v'' yields (2.6).

Inserting (2.6) into (2.5) we obtain

$$\begin{aligned} Q_{y,z,1}([0,t] \times [0,u]) &= \int_0^t \frac{(1-A(y+s))b(z+s)}{(1-A(y))(1-B(z))} 1_{[0,\infty)}(u-y-s)\,ds \\ &+ \iiint_{\substack{s+v'+v'' \leq t \\ s \geq 0}} \int_{w \leq u} \int_{w' \geq 0} \frac{(1-B(z+s))a(y+s)}{(1-B(z))(1-A(y))} \\ &\times Q_{0,z+s,1}(dv',dw')Q_{w',0,1}(dv'',dw)\,ds. \end{aligned} \tag{2.7}$$

Finally one has to take the Laplace transform of the measure

$$B \to Q_{y,z,1}(B \times [0,u]),$$

where u is fixed, to complete the proof.

For fixed $\alpha > 0$ the function $(y,z,E) \to \Psi_{y,z}(\alpha, E)$ is uniquely determined by equation (2.2) and the condition that $\Psi_{y,z}(\alpha,\cdot)$ is a subprobability measure, if

$$G(\alpha) := \sup_{y \geq 0} \int_0^\infty \frac{a(y+t)}{1-A(y)} e^{-\alpha t}\,dt < \frac{1}{2}. \tag{2.8}$$

For let $\mathscr{B}$ be the Banach space of all functions $\rho(y,z,E)$ such that $\rho(\cdot,\cdot,E)$: $[0,\infty)^2 \to \mathbf{R}$ is measurable for each Borel subset E of $[0,\infty)$, $\rho(y,z,\cdot)$ is a signed measure for each $(y,z) \in [0,\infty)^2$ and

$$\|\rho\| := \sup_{y,z \geq 0} |\rho(y,z,\cdot)| < \infty \tag{2.9}$$

($|\nu|$ denotes the total variation of a signed measure ν). Let $\mathscr{K} := \{\rho \in \mathscr{B} \mid \|\rho\| \le 1\}$ and define the operator $U_\alpha : \mathscr{K} \to \mathscr{K}$ by

$$(2.10)\quad (U_\alpha\rho)(y,z,E) := \int_{t\ge 0}\int_{w'>0} e^{-\alpha t}\frac{(1-B(z+t))a(y+t)}{(1-B(z))(1-A(y))} \times \rho(0,z+t,dw')\rho(w',0,E)\,dt.$$

If $\rho, \tilde\rho \in \mathscr{K}$, we have

(2.11)

$$\|U_\alpha\rho - U_\alpha\tilde\rho\| \le \sup_{y,z\ge 0}\int_{t\ge 0} e^{-\alpha t}\frac{(1-B(z+t))a(y+t)}{(1-B(z))(1-A(y))} \times \left\{\iint [|\rho(0,z+t,dw')\rho(w',0,dw) - \rho(0,z+t,dw')\tilde\rho(w',0,dw)| + |\rho(0,z+t,dw')\tilde\rho(w',0,dw) - \tilde\rho(0,z+t,dw')\tilde\rho(w',0,dw)|]\right\} dt$$
$$\le \sup_{y,z\ge 0} 2\|\rho - \tilde\rho\|\int_{t\ge 0} e^{-\alpha t}\frac{a(y+t)}{1-A(y)}\,dt = 2G(\alpha)\|\rho-\tilde\rho\|.$$

Thus if $G(\alpha) < 1/2$ and ρ, $\tilde\rho$ are two solutions of (2.2) satisfying $\|\rho\|, \|\tilde\rho\| \le 1$, (2.2) and (2.11) entail that

$$(2.12)\qquad \|\rho - \tilde\rho\| = \|U_\alpha\rho - U_\alpha\tilde\rho\| \le 2G(\alpha)\|\rho - \tilde\rho\|$$

so that $\rho = \tilde\rho$.

Especially if

$$(2.13)\qquad G(\alpha) < \tfrac{1}{2} \quad \text{for all } \alpha \ge \alpha_0$$

for some $\alpha_0 > 0$, equation (2.2) uniquely determines the Laplace transform of the measure $Q_{y,z,1}(\cdot, E)$ for every fixed triple (y,z,E). Moreover, the above considerations show that the method of successive approximations yields a sequence of $(\Psi^{(n)})_{n\ge 0}$ which converges to Ψ in the total variation distance with respect to E and uniformly with respect to y and z: we have

$$(2.14)\quad \sup_{y,z}|\Psi^{(n)}_{y,z}(\alpha,\cdot) - \Psi_{y,z}(\alpha,\cdot)| \le \frac{[2G(\alpha)]^n}{1-2G(\alpha)}\sup_{y,z}|\Psi^{(1)}_{y,z}(\alpha,\cdot) - \Psi^{(0)}_{y,z}(\alpha,\cdot)|.$$

We can take an arbitrary $\Psi^{(0)}_{y,z}(\alpha,E)$ belonging to $\mathscr{K}$ and then, for $n \ge 1$, have to find $\Psi^{(n)}_{y,z}(\alpha,E)$ recursively by

(2.15)

$$\Psi^{(n)}_{y,z}(\alpha,E) := \int_0^\infty e^{-\alpha t}\frac{(1-A(y+t))b(z+t)}{(1-A(y))(1-B(z))}1_{[0,\infty)}(u-y-t)\,dt + \int_{t\ge 0}\int_{w'\ge 0} e^{-\alpha t}\frac{(1-B(z+t))a(y+t)}{(1-B(z))(1-A(y))}\Psi^{(n-1)}_{0,z+t}(\alpha,dw')\Psi^{(n-1)}_{w',0}(\alpha,E)\,dt.$$

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Condition (2.13) is for example satisfied, if A has a bounded hazard rate $a(y)/(1-A(y))$. For if $a/(1-A) \leq K$,

$$G(\alpha) \leq \int_0^\infty \frac{a(y+t)}{1-A(y+t)} e^{-\alpha t}\, dt \leq K/\alpha. \tag{2.16}$$

3. The Laplace transform of τ_j

Next we consider

$$\varphi_j(u|y,z) := E(\exp\{-u\tau_j\}|X_0 = l, Y_0 = y, Z_0 = z), \tag{3.1}$$

where $u, y, z \geq 0$ and $l \geq j \geq 1$. Using the argument already employed at the beginning of the proof of Theorem 1, but now for arbitrary $j \geq 1$ and for the Laplace transforms instead of the distributions themselves, we obtain

(3.2)
$$\begin{aligned}\varphi_j(u|y,z) = &\int_0^\infty e^{-uv}\varphi_{j+1}(u|0,z+v)\frac{(1-B(z+v))a(y+v)}{(1-B(z))(1-A(y))}\,dv \\ &+ \int_0^\infty e^{-uw}\varphi_{j-1}(u|y+w,0)\frac{(1-A(y+w))b(z+w)}{(1-A(y))(1-B(z))}\,dw, \qquad j \geq 2,\end{aligned}$$

and

$$\begin{aligned}\varphi_1(u|y,z) = &\int_0^\infty e^{-uw}\frac{(1-A(y+w))b(z+w)}{(1-A(y))(1-B(z))}\,dw \\ &+ \int_0^\infty e^{-uv}\varphi_2(u|0,z+v)\frac{(1-B(z+v))a(y+v)}{(1-B(z))(1-A(y))}\,dv.\end{aligned} \tag{3.3}$$

To solve this system we introduce the generating function

$$\Phi(x,u|y,z) := \sum_{j=1}^\infty \varphi_j(u|y,z)x^j, \qquad |x| < 1. \tag{3.4}$$

Summing (3.2) and (3.3) over j yields after some simple manipulations

(3.5)
$$\begin{aligned}x\Phi(x,u|y,z) = x^2 &\int_0^\infty e^{-uw}\frac{(1-A(y+w))b(z+w)}{(1-A(y))(1-B(z))}\,dw \\ &+ \int_0^\infty e^{-uw}\Phi(x,u|0,z+w)\frac{(1-B(z+w))a(y+w)}{(1-B(z))(1-A(y))}\,dw \\ &+ x^2\int_0^\infty e^{-uw}\Phi(x,u|y+w,0)\frac{(1-A(y+w))b(z+w)}{(1-A(y))(1-B(z))}\,dw \\ &- x\int_0^\infty e^{-uw}\varphi_1(u|0,z+w)\frac{(1-B(z+w))a(y+w)}{(1-B(z))(1-A(y))}\,dw.\end{aligned}$$

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Note that

$$\varphi_1(u|y,z) = \Psi_{y,z}(u,[0,\infty)), \tag{3.6}$$

and Ψ has been determined in the previous section. We arrive at the following result.

THEOREM 2. *Let $u \geq 0$ and $x \in \mathbf{R}$, $|x| < 1$, be fixed. The functions $\Phi(x,u|0,\cdot)$ and $\Phi(x,u|\cdot,0)$ satisfy the following system of Fredholm integral equations of the second kind:*

$$\begin{aligned}(1-B(z))x\Phi(x,u|0,z) &= x^2\int_0^\infty e^{-uw}(1-A(w))b(z+w)\,dw \\ &+ \int_0^\infty e^{-uw}\Phi(x,u|0,z+w)(1-B(z+w))a(w)\,dw \\ &+ x^2\int_0^\infty e^{-uw}\Phi(x,u|w,0)(1-A(w))b(z+w)\,dw \\ &- x\int_0^\infty e^{-uw}\varphi_1(u|0,z+w)(1-B(z+w))a(w)\,dw\end{aligned} \tag{3.7}$$

$$\begin{aligned}(1-A(y))x\Phi(x,u|y,0) &= x^2\int_0^\infty e^{-uw}(1-A(y+w))b(w)\,dw \\ &+ \int_0^\infty e^{-uw}\Phi(x,u|0,w)(1-B(w))a(y+w)\,dw \\ &+ x^2\int_0^\infty e^{-uw}\Phi(x,u|y+w,0)(1-A(y+w))b(w)\,dw \\ &- x\int_0^\infty e^{-uw}\varphi_1(u|0,w)(1-B(w))a(y+w)\,dw.\end{aligned} \tag{3.8}$$

For arbitrary $y, z \geq 0$ the function $\Phi(x,u|y,z)$ is then connected with $\Phi(x,u|0,\cdot)$, $\Phi(x,u|\cdot,0)$ and $\varphi_1(u|0,z)$ by (3.5).

Equations (3.7) and (3.8) can be written in the form

$$f = g + Kf, \tag{3.9}$$

where

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \quad Kf = \begin{pmatrix} K_{11}f_1 + K_{12}f_2 \\ K_{21}f_1 + K_{22}f_2 \end{pmatrix} \tag{3.10}$$

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and f_1, f_2, g_1, g_2 are defined as follows:

(3.11) $f_1(y) = (1 - B(y))\Phi(x, u|0, y)$,

(3.12) $f_2(y) = (1 - A(y)\Phi(x, u|y, 0)$,

(3.13) $$g_1(y) = x\int_0^\infty e^{-uw}(1 - A(w))b(y + w)\,dw - \int_0^\infty e^{-uw}\varphi_1(u|0, y + w)(1 - B(y + w))a(w)\,dw,$$

(3.14) $$g_2(y) = x\int_0^\infty e^{-uw}(1 - A(y + w))b(w)\,dw - \int_0^\infty e^{-uw}\varphi_1(u|0, w)(1 - B(w))a(y + w)\,dw.$$

The definition of the integral operators K_{ij}, $i, j = 1, 2$, is clear from (3.7) and (3.8); for instance,

(3.15) $$(K_{11}h)(z) = x^{-1}\int_0^\infty e^{-uv}h(z + v)a(v)\,dv.$$

For $y \geq 0$ let $\hat{a}_y$ and $\hat{b}_y$ be the Laplace transform of the functions $v \to a(y+v)$ and $v \to b(y+v)$. Then we have, for arbitrary bounded measurable functions $h\colon [0, \infty) \to \mathbf{R}$,

(3.16) $$|K_{11}h(z)| = \left|x^{-1}\int_0^\infty e^{-uv}h(z + v)a(v)\,dv\right| \leq |x|^{-1}\hat{a}_0(u)\|h\|_\infty,$$

where $\|h\|_\infty := \sup_{z\geq 0}|h(z)|$, and similarly

(3.17) $$|K_{12}h(z)| = \left|x\int_0^\infty e^{-uv}h(v)b(z + v)\,dv\right| \leq |x|\hat{b}_z(u)\|h\|_\infty,$$

(3.18) $|K_{21}h(z)| \leq |x|^{-1}\hat{a}_z(u)\|h\|_\infty$,

(3.19) $|K_{22}h(z)| \leq |x|\hat{b}_0(u)\|h\|_\infty$.

If we define $\|h\| := \|h_1\|_\infty + \|h_2\|_\infty$ for

$$h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} : [0, \infty) \to \mathbf{R}^2,$$

(3.16)–(3.18) yield

(3.20) $$\|Kh\| \leq |x|^{-1}\hat{a}_0(u)\|h_1\|_\infty + |x|\sup_{z\geq 0}\hat{b}_z(u)\|h_2\|_\infty + |x|^{-1}\sup_{z\geq 0}\hat{a}_z(u)\|h_1\|_\infty + |x|\hat{b}_0(u)\|h_2\|_\infty.$$

Let us assume that

(3.21) $$\lim_{u\to\infty}\sup_{z\geq 0}\hat{a}_z(u) = \lim_{u\to\infty}\sup_{z\geq 0}\int_0^\infty e^{-uv}a(z + v)\,dv = 0$$

and

(3.22) $$\lim_{u\to\infty} \sup_{z\geq 0} \hat{b}_z(u) = 0.$$

Equations (3.21) and (3.22) are not very restrictive conditions; they are for example satisfied, if a and b are monotone on $[T,\infty)$ for some $T \geq 0$. If (3.21) and (3.22) are valid, some standard arguments using (3.20) now show that, for sufficiently large u, (3.9) possesses a unique continuous solution which is given by the uniformly convergent Neumann series

(3.23) $$f = g + Kg + K^2 g + \cdots .$$

Thus for large u the functions $\Phi(x, u|0, \cdot)$ and $\Phi(x, u|\cdot, 0)$ are uniquely determined by (3.7) and (3.8), and the series (3.23) gives a way to approximate them exponentially fast. For arbitrary $\varepsilon \in (0, 1/2)$ this convergence is uniform with respect to $x \in (\varepsilon, 1-\varepsilon)$, if $u \geq u_0 = u_0(\varepsilon)$.

4. The bulk queue $M^X/G/1$

For the queueing system $M^X/G/1$ the above technique can also be applied to determine the conditional Laplace transform

(4.1) $$\varphi_j(u|z) := E(\exp\{-u\tau_j\}|X_0 = l, Z_0 = z), \qquad u, z \geq 0,\ l \geq j \geq 1,$$

of the first time instant τ_j at which the queue size is decreased from l to $l-j$. Let $A(x) = 1 - e^{-\lambda x}$, $x \geq 0$, for some $\lambda > 0$. At the time of the ith arrival in $(0,\infty)$ a group of A_i customers enters the system, where $A_1, A_2, \ldots$ are assumed to be independent random variables having the common distribution $P(A_i = n) = p_n$, $n = 1, 2, \ldots$, and the generating function $p(s) := \sum_{n=1}^{\infty} p_n s^n$. For $M^X/G/1$ obviously (X_t, Z_t) is a Markov process. It is not difficult to see that

(4.2) $$\varphi_j(u|z) = \varphi_1(u|z)\varphi_1(u|0)^{j-1}$$

so that it suffices to compute $\varphi_1(u|z)$. Now given that $X_0 = 1$, $Z_0 = z$, the following possibilities can be distinguished. If no new customers enter before the next service is completed at time x, say (an event of probability $[b(z+x)/(1-B(z))]e^{-\lambda x}\,dx$), we have $\tau_1 = x$. If $j \geq 1$ arrivals take place before the next service completion time x and the number of new customers entering the system in $(0, x]$ is equal to n, we have $\tau_1 = x + \tilde{\tau}_n$, where $\tilde{\tau}_n$ has the same distribution as τ_n, given that $X_0 = n$, $Z_0 = 0$. This possibility occurs with probability

$$e^{-\lambda x}\frac{(\lambda x)^j}{j!} P\left(\sum_{i=1}^{j} A_i = n\right) \frac{b(z+x)}{1-B(z)}\,dx.$$

For $\varphi_1(u|z)$ these considerations yield

(4.3)

$$\begin{aligned}\varphi_1(u|z) &= \int_0^\infty e^{-ux}\frac{b(z+x)}{1-B(z)}e^{-\lambda x}\,dx \\ &\quad + \sum_{j=1}^\infty\sum_{n=j}^\infty\left[\int_0^\infty e^{-ux}e^{-\lambda x}\frac{(\lambda x)^j}{j!}P\left(\sum_{i=1}^j A_i = n\right)\frac{b(z+x)}{1-B(z)}\,dx\right]\varphi_1(u|0)^n \\ &= \frac{1}{1-B(z)}\left[\int_0^\infty e^{-(u+\lambda)x}b(z+x)\,dx\right. \\ &\qquad \left. + \sum_{j=1}^\infty\int_0^\infty e^{-(u+\lambda)x}b(z+x)\frac{(\lambda x)^j}{j!}p(\varphi_1(u|0))^j\,dx\right] \\ &= \frac{1}{1-B(z)}\int_0^\infty b(z+x)\exp\{\lambda x p(\varphi_1(u|0)) - (u+\lambda)x\}\,dx.\end{aligned}$$

Equations (4.3) show how to compute $\varphi_1(u|z)$, if $\varphi_1(u|0)$ is known. For $z=0$, (4.3) can be written as

$$\varphi_1(u|0) = f(u+\lambda-\lambda p(\varphi_1(u|0))), \tag{4.4}$$

where f is the Laplace transform of $b(x)$.

Equation (4.4) is a generalization of the well-known Takács equation (Feller (1971), pages 441–442 and 473) which comes out for $p(x)=x$. As in the classic case the following lemma is easily proved.

LEMMA. *Assume that* $1/\mu := \int_0^\infty xb(x)\,dx < \infty$ *and* $\nu := \sum_{n=1}^\infty np_n < \infty$. *The equation*

$$\varphi(u) = f(u+\lambda-\lambda p(\varphi(u))), \qquad u>0, \tag{4.5}$$

possesses a unique solution $\varphi(u)$ *which is the Laplace transform of a distribution which is proper if* $\lambda\nu/\mu \le 1$ *and defective otherwise.*

As an example, let us consider the case when $B(x) = 1-e^{-\mu x}$, $x\ge 0$, for some $\mu>0$. Equation (4.4) for $\varphi=\varphi_1(\cdot|0)$ takes the form

$$\varphi(u) = \frac{\mu}{\mu+\lambda+u-\lambda p(\varphi(u))} = \frac{\mu}{\mu+\lambda+u} + \frac{\lambda}{\mu+\lambda+u}\varphi(u)p(\varphi(u)). \tag{4.6}$$

We note that φ can be expanded into ascending powers of $(\lambda+\mu+u)^{-1}$ in the form

$$\varphi(u) = \sum_{n=1}^\infty q_n[(\lambda+\mu)/(\lambda+\mu+u)]^n, \qquad u\ge 0, \tag{4.7}$$

where $q_n \ge 0$ for all $n\ge 1$. To derive (4.7), let X_i be the time between the $(i-1)$th and the ith jump of the queue size and let Y_i be the size of the ith

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jump. Then if $j_1, j_2, \dots, j_n \in \{-1, 1, 2, 3, \dots\}$ satisfy $j_1 + \cdots + j_m > -1$ for $m = 1, \dots, n-1$ and $j_1 + \cdots + j_n = -1$, it is easily seen that

$$\int_{\{Y_1=j_1,\dots,Y_n=j_n\}} e^{-u(X_1+\cdots+X_n)}\,dP = \prod_{m=1}^{n} \int_{\{Y_m=j_m\}} e^{-uX_m}\,dP \tag{4.8}$$

and

$$\int_{\{Y_m=-1\}} e^{-uX_m}\,dP = \frac{\mu}{\lambda+\mu+u}, \tag{4.9}$$

$$\int_{\{Y_m=j_m\}} e^{-uX_m}\,dP = \frac{\lambda}{\lambda+\mu+u}, \quad \text{if } j_m \geq 1, \tag{4.10}$$

since X_m can be represented as the minimum of two exponential variables S_m and T_m, say, with means $1/\lambda$ and $1/\mu$, respectively, and $Y_m = 1$ if and only if $S_m < T_m$. Obviously $\varphi(u)$ can be written as a series of terms of the form (4.8). Inserting (4.9) and (4.10) into (4.8) shows (4.7).

Let $v := (\lambda+\mu)/(\lambda+\mu+u)$, $\tilde{\varphi}(v) := \varphi(u)$ if $v \in (0, 1]$ and $\tilde{\varphi}(0) := 0$. From (4.6) it follows that

$$\tilde{\varphi}(v) = \frac{\mu}{\mu+\lambda}v + \frac{\lambda}{\mu+\lambda}v\tilde{\varphi}(v)p(\tilde{\varphi}(v)). \tag{4.11}$$

Inserting (4.7) into (4.11) and comparing the coefficients at both sides gives the following recursive relation for the q_n:

$$\begin{aligned} &q_1 = \mu/(\mu+\lambda), \quad q_2 = 0, \\ &q_{n+1} = \frac{\lambda}{\mu+\lambda} \sum \binom{i_1+\cdots+i_n}{i_1,\dots,i_n} p_{i_1+\cdots+i_n-1} q_1^{i_1} \cdots q_n^{i_n}, \qquad n > 1, \end{aligned} \tag{4.12}$$

where the sum is taken over all n-tuples $(i_1, \dots, i_n)$ of nonnegative integers for which $\sum_{j=1}^{n} j i_j = n$. To check (4.9), it is convenient to use the formula

$$\frac{d^n}{dv^n}(F \circ \varphi)(v) = \sum \frac{n!}{i_1! \cdots i_n!} F^{(i_1+\cdots+i_n)}(\varphi(v)) \prod_{j=1}^{n} \left(\frac{\varphi^{(j)}(v)}{j!}\right)^{i_j} \tag{4.13}$$

where the sum is extended over the same set of n-tuples as in (4.12) (see Gradshteyn and Ryzhik (1980), page 19, formulae 0.430).

Equation (4.7) can be inverted term-by-term. Thus the density of τ_1 is given by

$$e^{-(\mu+\lambda)t} \sum_{n=1}^{\infty} \frac{(\mu+\lambda)^n}{(n-1)!} q_n t^{n-1}. \tag{4.14}$$

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