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EVEN COVERS AND COLLECTIONWISE NORMAL SPACES

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1. Introduction. The concept of an even cover is introduced early in elementary topology courses and is known to be valuable. Among other facts it is known that X is paracompact if and only if every open cover of X is even. In this paper we introduce the concept of an n-even cover and show its usefulness. Using n-even we define an embedding that on closed subsets is equivalent to collectionwise normal. We also give sufficient conditions for a point finite open cover to have a locally finite refinement and also sufficient conditions for this refinement to be even. Finally we show that the collection of all neighborhoods of the diagonal of X is a uniformity if and only if every even cover is normal. This last result is particularly interesting in light of the fact that every normal open cover is even.

In order to prove these theorems we make use of several results as established in [**1**]. In Section 2 we state some of these basic definitions and fundamental results. However, throughout the paper we draw heavily from [**1**].

2. Definitions. In general we use the notation and terminology as in [**1**]. In particular if $\mathscr{G}$ and $\mathscr{H}$ are covers we write $\mathscr{H} < \mathscr{G}$ if $\mathscr{H}$ refines $\mathscr{G}$; i.e. if for every $H \in \mathscr{H}$ there is a $G \in \mathscr{G}$ such that $H \subset G$. We assume that $\bigcup \mathscr{H} = \bigcup \mathscr{G}$ if $\mathscr{H} < \mathscr{G}$.

If W is a neighborhood of the diagonal of X then we set $W(x) = \{y \in X : (x, y) \in W\}$. We will usually assume that W is open and symmetric ($W = W^{-1}$). We define $W \circ W = W^2 = \{(x, y) \in X \times X$: there exists $z \in X$ with $(x, z) \in W$ and $(z, y) \in W\}$ and we let $W^n = W^{n-1} \circ W$ for $n \in \mathbf{N}$, $n \neq 1$.

A sequence $(\mathscr{U}_n)_{n\in\mathbf{N}}$ of open covers of a topological space X is *normal* if for all $n \in N$, $\mathscr{U}_{n+1} <^* \mathscr{U}_n$ (i.e. $(\mathrm{st}(U, \mathscr{U}_{n+1}))_{U\in\mathscr{U}_{n+1}} < \mathscr{U}_n$). The cover $\mathscr{G}$ is *normal* if there is a normal sequence $(\mathscr{U}_n)_{n\in\mathbf{N}}$ of open covers such that $\mathscr{U}_1 < \mathscr{G}$.

2.1 If $\mathscr{G}$ is a normal open cover of a topological space X then there is a normal sequence of open covers $(\mathscr{U}_n)_{n\in\mathbf{N}}$ such that $\mathscr{U}_1$ refines $\mathscr{G}$. Furthermore there is a continuous pseudometric d on X that is associated with $(\mathscr{U}_n)_{n\in\mathbf{N}}$. In particular $(B(x, 1/2^3))_{x\in X}$ refines $\mathscr{G}$.

For a proof and discussion of these results the reader is referred to [**1**].

2.2 *Definition*. If (X, d) is a pseudometric space and if $\mathscr{G}$ is a cover of X we say that $\mathscr{G}$ is *Lebesgue* if there is a $\delta > 0$ such that $(B(x, \delta))_{x\in X}$ refines $\mathscr{G}$. (If $\mathscr{G}$ is Lebesgue it has an open refinement, hence we will usually assume our

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covers are open.) If $(X, \mathscr{U})$ is a uniform space, a cover $\mathscr{G}$ of X is said to be *Lebesgue* if there is a $U \in \mathscr{U}$ such that $(U(x))_{x\in X}$ refines $\mathscr{G}$.

Lebesgue covers are studied in [**4**] and [**8**] where they are shown to play an important role in covering dimension theory. Our major interest will be in the following definition.

2.3 *Definition.* If X is a topological space and $n \in \mathbf{N}$ we say that a cover $\mathscr{G}$ is *n-even* if there exist neighborhoods $W_1, \ldots, W_n$ of the diagonal of X such that $W_i^2 \subset W_{i-1}$ for $i = 2, \ldots, n$ and $(W_1(x))_{x\in X}$ refines $\mathscr{G}$. If there exists a sequence $(W_n)_{n\in\mathbf{N}}$ of neighborhoods of the diagonal of X such that $W_n^2 \subset W_{n-1}$ for all $n \in \mathbf{N}$, $n > 1$, and $(W_1(x))_{x\in X}$ refines $\mathscr{G}$ we say that $\mathscr{G}$ is $\aleph_0$-*even*. If $n = 1$ we write even instead of 1-even and note that this is the usual definition of even.

2.4 Suppose that X is a topological space and that $\mathscr{G}$ is a cover of X. If $\mathscr{G}$ is $\aleph_0$-even then $\mathscr{G}$ is n-even for any $n \in \mathbf{N}$.

3. Main results.

3.1 THEOREM. *Suppose that X is a topological space and that $\mathscr{G}$ is a cover of X. If $\mathscr{G}$ is either a*

(i) *normal open cover,*

(ii) *locally finite cozero-set cover, or*

(iii) *countable cozero-set cover,*

then $\mathscr{G}$ is $\aleph_0$-even.

Proof. Suppose that $\mathscr{G}$ is a normal open cover of X. By 2.1 there is a continuous pseudometric d associated with $\mathscr{G}$ such that $(B(x, 1/2^3))_{x\in X}$ refines $\mathscr{G}$. Let $(W_i)_{i\in\mathbf{N}}$ be defined by

$$W_i = \{(x, y) \in X \times X : d(x, y) < 1/2^{i+3}\}$$

for $i \in \mathbf{N}$. Then $W_i^2 \subset W_{i-1}$ for $i \in \mathbf{N}$, $i \neq 1$, and $(W_1(x))_{x\in X}$ refines $\mathscr{G}$. Therefore X is $\aleph_0$-even.

Since a locally finite cozero-set cover and a countable cozero-set cover is normal [**1**, 11.1 and 10.10], (b) and (c) hold.

Actually we can show that an $\aleph_0$-even cover is equivalent to a normal cover and furthermore, an $\aleph_0$-even cover is equivalent to a Lebesgue cover if X is completely regular and has the universal uniformity.

3.2 THEOREM. *If $\mathscr{G}$ is an open cover of a topological space X then $\mathscr{G}$ is normal if and only if $\mathscr{G}$ is $\aleph_0$-even.*

Proof. By 3.1 if $\mathscr{G}$ is normal then $\mathscr{G}$ is $\aleph_0$-even. Conversely if $\mathscr{G}$ is $\aleph_0$-even then there is a sequence $(W_n)_{n\in\mathbf{N}}$ of open symmetric neighborhoods of the diagonal of X such that $W_n^2 \subset W_{n-1}$ for all $n \in \mathbf{N}$, $n \neq 1$, and $(W_1(x))_{x\in X}$ refines $\mathscr{G}$. For all $n \in \mathbf{N}$, let $\mathscr{W}_n = (W_n(x))_{x\in X}$. We assert that $(\mathscr{W}_{2n})_{n\in\mathbf{N}}$ is a normal sequence of open covers such that $\mathscr{W}_1$ refines $\mathscr{G}$.

To show that it is a normal sequence of open covers such that $\mathscr{W}_{2n} <^* \mathscr{W}_{2n-2}$ for any $n \in \mathbf{N}$, $n \neq 1$, we observe that $\mathrm{st}(W_{2n}(x), \mathscr{W}_{2n}) \subset W_{2n-2}(x)$. In [**1**, 8.4] the existence of a continuous pseudometric d associated with $(\mathscr{W}_{2n})_{n\in\mathbf{N}}$ was shown. Thus by 2.1, $\mathscr{U} = (B(x, 1/2^3))_{x\in X}$ refines $\mathscr{G}$. Furthermore, in [**1**, 8.6] it was shown that $\mathscr{U}$ is normal, thus $\mathscr{G}$ is normal and the proof is complete.

3.3 THEOREM. *Suppose that X is a completely regular topological space, that $\mathscr{U}$ is the universal uniformity on X and that $\mathscr{G}$ is an open cover of X. Then $\mathscr{G}$ is $\aleph_0$-even if and only if $\mathscr{G}$ is Lebesgue relative to $(X, \mathscr{U})$.*

Proof. Suppose that $\mathscr{G}$ is $\aleph_0$-even. As in the proof of 3.2 there is a continuous pseudometric d such that $(B(x, 1/2^3))_{x\in X}$ refines $\mathscr{G}$. Thus if we let

$$W = \{(x, y) \in X \times X : d(x, y) < 1/2^3\}$$

then W is an element of $\mathscr{U}$ [**1**, 8.6] and $W(x) = B(x, 1/2^3)$ so $(W(x))_{x\in X}$ refines $\mathscr{G}$. Therefore $\mathscr{G}$ is Lebesgue relative to $(X, \mathscr{U})$.

Conversely if $\mathscr{G}$ is Lebesgue relative to $(X, \mathscr{U})$ then there exists $U \in \mathscr{U}$ such that $(U(x))_{x\in X}$ refines $\mathscr{G}$. Because $\mathscr{U}$ is a uniformity there exists $U_1 \in \mathscr{U}$ such that $U_1^2 \subset U$. By induction, for any $n \in \mathbf{N}$ we can define $U_n \in \mathscr{U}$ such that $U_n^2 \subset U_{n-1}$. It follows that $\mathscr{G}$ is $\aleph_0$-even.

The second part of Theorem 3.3 actually proved the following.

COROLLARY. *If $(X, \mathscr{U})$ is a completely regular uniform space and if $\mathscr{G}$ is a Lebesgue cover then $\mathscr{G}$ is $\aleph_0$-even.*

We are now ready to prove several results concerning n-even covers. Our first result will show that a point-finite even cover has a locally finite refinement. This result is interesting in light of results of Michael (see [**7**] or [**1**, p. 132]) as discussed after Theorem 3.5.

3.4 THEOREM. *Suppose that X is a topological space and that $\mathscr{G} = (G_\alpha)_{\alpha\in I}$ is a point finite even cover. Then there exists a locally finite cover $\mathscr{F} = (F_\alpha)_{\alpha\in I}$ such that $F_\alpha \subset G_\alpha$ for all $\alpha \in I$.*

Proof. Since $\mathscr{G}$ is even there exists an open symmetric neighborhood U of the diagonal of X such that $(U(x))_{x\in X}$ refines $\mathscr{G}$. For each $\alpha \in I$ let $F_\alpha = \{x \in X : U(x) \subset G\}$ and let $\mathscr{F} = (F_\alpha)_{\alpha\in I}$. It is easy to show that $\mathscr{F}$ covers X, that $F_\alpha \subset G_\alpha$ and that $\mathscr{F}$ is locally finite.

If $\mathscr{G}$ is a point finite open cover then $\mathscr{G}$ has a locally finite refinement. We now give sufficient conditions for the refinement to be even and hence open.

3.5 THEOREM. *Suppose that X is a topological space and that $\mathscr{G} = (G_\alpha)_{\alpha\in I}$ is a 2-even point finite cover of X. Then there exists a locally finite even cover $\mathscr{F} = (F_\alpha)_{\alpha\in I}$ such that* cl $F_\alpha \subset G_\alpha$ *for all $\alpha \in I$.*

Proof. By hypothesis there exists W_1 and W_2 open symmetric neighborhoods of the diagonal of X such that $W_2^2 \subset W_1$ and $(W_1(x))_{x \in X}$ refines $\mathscr{G}$. For each $\alpha \in I$, let $F_\alpha = \bigcup \{W_2(x) : W_1(x) \subset G_\alpha\}$ and let $\mathscr{F} = (F_\alpha)_{\alpha \in I}$. We assert that $\mathscr{F}$ is a locally finite even cover of X such that cl $F_\alpha \subset G_\alpha$.

Clearly $\mathscr{F}$ is even since $(W_2(x))_{x \in X}$ refines $\mathscr{F}$. To see that $\mathscr{F}$ is locally finite, let $x \in X$. Since $\mathscr{G}$ is point finite there exists a finite subset K of I such that $x \notin G_\alpha$ if $\alpha \notin K$. If $y \in W_2(x) \cap F_\alpha$ then $(x, y) \in W_2$ and $y \in W_2(z)$ such that $W_1(z) \subset G_\alpha$. But then $(y, z) \in W_2$ so $(x, z) \in W_2^2 \subset W_1$ whence $x \in W_1(z) \subset G_\alpha$, hence $\alpha \in K$. It follows that $W_2(x) \cap F_\alpha = \emptyset$ if $\alpha \notin K$. Clearly $\mathscr{F}$ is a cover because $(W_1(x))_{x \in X}$ refines $\mathscr{G}$. Finally, if $x \in$ cl F_α then $W_2(x)$ is a neighborhood of x that meets F_α. By an argument similar to above, one shows that $x \in G_\alpha$.

Remark. In [**7**] Michael has an example of a point finite open cover with no locally finite open refinement. By 3.5 the original cover is thus not 2-even. The following helps clarify 2-even and even.

3.6 THEOREM. *Suppose that* $\mathscr{G} = (G_\alpha)_{\alpha \in I}$ *is a point finite open cover. Then* (1) *implies* (2) *implies* (3).

(1) *The cover* $\mathscr{G}$ *is* 2-*even*.

(2) *There exists a locally finite open cover* $\mathscr{F} = (F_\alpha)_{\alpha \in I}$ *such that* cl $F_\alpha \subset G_\alpha$ *for each* $\alpha \in I$.

(3) *The cover* $\mathscr{G}$ *is even*.

Proof. (1) implies (2) is 3.5. To show (2) implies (3) let

$$V_\alpha = (G_\alpha \times G_\alpha) \cup ((X - F_\alpha) \times (X - F_\alpha))$$

and let $V = \bigcap_{k \in I} V_\alpha$. One can show that V is a neighborhood of the diagonal of X and that $(V(x))_{x \in X}$ refines $\mathscr{G}$.

We next consider countable covers and obtain the following.

3.7 THEOREM. *Suppose that* X *is a topological space and that* $\mathscr{G} = (G_n)_{n \in \mathbf{N}}$ *is a countable* 3-*even cover. Then there exists a countable locally finite even cover* $\mathscr{F} = (F_n)_{n \in \mathbf{N}}$ *such that* $F_n \subset G_n$ *for all* $n \in \mathbf{N}$.

Proof. Suppose that $\mathscr{G}$ is a 3-even cover so there exist open symmetric neighborhoods W_1, W_2 and W_3 of the diagonal of X such that $W_3^2 \subset W_2 \subset W_2^2 \subset W_1$ and $(W_1(x))_{x \in X}$ refines $\mathscr{G}$. Let $\mathscr{W}_i = (W_i(x))_{x \in X}$ for $i = 1, 2, 3$; and let $H_n = \{x \in X : W_1(x) \subset G_n\}$. Set

$$F_n = G_n - \bigcup_{m<n} \text{cl } (\text{st } (H_m, \mathscr{W}_3))$$

and let $\mathscr{F} = (F_n)_{n \in N}$. We assert that $\mathscr{F}$ is a locally finite open cover such that $F_n \subset G_n$ for all $n \in \mathbf{N}$.

To see that $\mathscr{F}$ is locally finite let $x \in X$ and let n be the first integer such that $x \in \text{st}(H_n, \mathscr{W}_3)$. Then $\text{st}(H_n, \mathscr{W}_3)$ is a neighborhood of x and $\text{st}(H_n, \mathscr{W}_3) \cap F_m = \emptyset$ if $m > n$. To show that $\mathscr{F}$ is a cover, let $x \in X$ and choose the first integer n such that $x \in \text{cl}(\text{st}(H_n, \mathscr{W}_3))$. One readily shows that $x \in F_n$.

Finally, to show that $\mathscr{F}$ is even we prove that $(W_3(x))_{x\in X}$ refines $\mathscr{F}$. Let $x \in X$ and choose the first integer n such that $W_3(x) \cap \mathrm{cl}(\mathrm{st}(H_n, \mathscr{W}_3)) \neq \emptyset$. Then $W_3(x) \subset X - \bigcup_{m<n} \mathrm{cl}(\mathrm{st}(H_m, \mathscr{W}_3))$. A standard argument shows that $W_3(x) \subset G_n$ whence $W_3(x) \subset F_n$ and therefore $\mathscr{F}$ is even.

Since a countable cozero-set cover of a topological space is normal [**1**, 11.2] it is $\aleph_0$-even, so we have the following.

3.8 COROLLARY. *If $\mathscr{G}$ is a countable cozero-set cover of a topological space X, then $\mathscr{G}$ has a locally finite even countable refinement.*

3.9 COROLLARY. *If X is Lindelof and if $\mathscr{G}$ is a 3-even cover then there exists a locally finite even cover $\mathscr{F}$ such that $\mathscr{F}$ refines $\mathscr{G}$.*

4. Applications. We know that there is a normal space that has a point finite open cover that is not 2-even. However since every countable point finite open cover of a normal space is a normal cover we have:

4.1 PROPOSITION. *If X is a normal topological space then every countable point finite open cover is even*

We can also observe the following

4.2 PROPOSITION. *If every countable point finite open cover is even, then X is countably metacompact if and only if X is countably paracompact.*

Proposition 4.2 apparently generalizes the result that a normal space is countably paracompact if and only if it is countably metacompact [**5**]. Using a proof similar to 3.5 one can show that if every binary open cover of a topological space X is 2-even then X is normal. We are now able to show how n-even covers related to other topological concepts. But first another definition.

4.3 *Definition.* Let S be a subset of a topological space X and let $n \in \mathbf{N}$. We say that S is *E^n-embedded in X* if every n-even cover of S has a refinement that can be extended to an n-even cover of X. We write E-embedded instead of E^1-embedded. The subset S is *$E^{\aleph_0}$-embedded in X* if every $\aleph_0$-even cover of S has a refinement that can be extended to an $\aleph_0$-even cover of X. The subset S is *weakly $E^{\aleph_0}$-embedded in X* in case every $\aleph_0$-even cover of S has a refinement that can be extended to an even cover of X. We say that S is *P-embedded in X* if every continuous pseudometric on S can be extended to a continuous pseudometric on X.

The reader is referred to [**1**] for a discussion of P-embedding. In particular it is shown that X is collectionwise normal if and only if every closed subset is P-embedded. Similar results hold for P^γ-embedding and γ-collectionwise normal where γ is an infinite cardinal number.

Remark. Although we do not have an example it seems unlikely that E^n-embedded implies E^m-embedded if $n \neq m$.

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In the next result we show that if every closed subset is E-embedded in X then X is collectionwise normal. To obtain the converse we need to have $E^{\aleph_0}$-embedded closed subsets (see 4.6). It is interesting to ask when is an even open cover necessarily $\aleph_0$-even (or equivalently normal). In 4.7 we give a necessary and sufficient condition for this to be true. This condition is stronger than collectionwise normality.

4.4 THEOREM. *If every closed subset of a topological space X is E-embedded in X, then X is collectionwise normal.*

Proof. To show that X is collectionwise normal let $(F_\alpha)_{\alpha \in I}$ be a closed discrete collection of subsets of X. Let $S = \bigcup_{\alpha \in I} F_\alpha$ and note that S is a closed subset of X. Now set $U = \bigcup_{\alpha \in I} (F_\alpha \times F_\alpha)$ and note that U is a neighborhood of the diagonal of S. Moreover, on S, $(U(x))_{x \in S}$ refines $\mathscr{F} = (F_\alpha)_{\alpha \in I}$. Since S is E-embedded in X there is an even cover $\mathscr{G}$ of X such that $\mathscr{G}$ restricted to S refines $\mathscr{F}$. Since $\mathscr{G}$ is even there exists an open symmetric neighborhood W of the diagonal of X such that $(W(x))_{x \in X}$ refines $\mathscr{G}$. Clearly $W(F_\alpha)$ is a neighborhood of F_α.

We must show that $(W(F_\alpha)_{\alpha \in I}$ are pairwise disjoint. If $y \in W(F_{\alpha_1}) \cap W(F_{\alpha_2})$ where $\alpha_1 \neq \alpha_2$ then there exist $x_1 \in F_{\alpha_1}$ and $x_2 \in F_{\alpha_2}$ such that $(x_1, y) \in W$ and $(x_2, y) \in W$. Since $(W(x))_{x \in X}$ refines $\mathscr{G}$ there exists $G \in \mathscr{G}$ such that $W(y) \subset G$. Furthermore on S, $\mathscr{G}$ refines $\mathscr{F}$ so $G \cap S \subset F_{\alpha_3}$ for some $\alpha_3 \in I$. If $\alpha_3 = \alpha_2$ then $x_1 \in W(y) \cap S \subset G \cap S \subset F_{\alpha_2}$. Therefore $F_{\alpha_1} \cap F_{\alpha_2} \neq \emptyset$, a contradiction. On the other hand if $\alpha_2 \neq \alpha_3$ then $x_2 \in W(y) \cap S \subset F_{\alpha_3}$ so $F_{\alpha_2} \cap F_{\alpha_3} \neq \emptyset$, a contradiction. Therefore X is collectionwise normal.

4.5 THEOREM. *If S is a subset of a topological space X then S is $E^{\aleph_0}$-embedded in X if and only if S is P-embedded in X.*

Proof. Since a cover is $\aleph_0$-even if and only if it is normal, the result follows from [**1**, 14.7].

4.6 THEOREM. *If X is a topological space then the following statements are equivalent.*

(1) *The space X is collectionwise normal.*
(2) *Every closed subset of X is $E^{\aleph_0}$-embedded in X.*
(3) *Every closed subsct of X is weakly $E^{\aleph_0}$-embedded in X.*

Proof. The equivalence of (1) and (2) follows from 4.5 and the fact that X is collectionwise normal if and only if every closed subset is P-embedded in X.

Clearly (2) implies (3). The proof of (3) implies (1) is similar to the proof of 4.4 once we note that if $(F_\alpha)_{\alpha \in I}$, S and U are as in that proof, then $U^2 \subset U$ hence $\mathscr{F}$ is $\aleph_0$-even.

A partial converse of 4.4 is the following result.

4.7 THEOREM. *If X is a normal topological space then the following statements are equivalent.*

(1) *X is collectionwise normal.*

(2) *For every closed subset S of X every point finite 2-even open cover of S has a refinement that can be extended to a point finite 2-even open cover of X.*

Proof. The result follows from 3.5 and the fact [**1**, 11.7] that in a normal space every locally finite open cover is normal and therefore $\aleph_0$-even.

If $\mathscr{G}$ is an open cover of a topological space we know that $\mathscr{G}$ is normal if and only if $\mathscr{G}$ is $\aleph_0$-even (3.2). Hence a normal open cover is always even. Our final result gives necessary and sufficient conditions for the converse to be true. The condition stated in 4.8(1) lies strictly between paracompact and collectionwise normality (see [**2**] and [**3**]).

4.8 Theorem. *If X is a completely regular topological space then the following statements are equivalent.*

(1) *The collection of all neighborhoods of the diagonal is a uniformity (and therefore the universal uniformity).*

(2) *Every even open cover is normal.*

Proof. (1) implies (2): Suppose that $\mathscr{G}$ is an even cover of X. Thus there exists a neighborhood W of the diagonal of X such that $(W(x))_{x \in X}$ refines $\mathscr{G}$. But then $\mathscr{G}$ is a Lebesgue cover relative to the uniformity of all neighborhoods of the diagonal and hence by 3.3, $\mathscr{G}$ is $\aleph_0$-even and therefore by 3.2, $\mathscr{G}$ is normal.

(2) implies (1): Since every entourage in the universal uniformity is a neighborhood of the diagonal we need only show that every neighborhood W of the diagonal is in the universal uniformity. If W is a neighborhood of the diagonal then $\mathscr{W} = (W(x))_{x \in X}$ is an even cover and hence by (2) $\mathscr{W}$ is a normal cover. By 3.2, $\mathscr{W}$ is $\aleph_0$-even, hence in particular there exists a symmetric open neighborhood U of the diagonal of X such that $U^2 \subset W$. Note that $\mathscr{U} = (U(x))_{x \in X}$ is an even cover of X and is therefore normal. Hence by 2.1 there is a continuous pseudometric d on X such that $(B(x, 1/2^3))_{x \in X}$ refines $\mathscr{U}$. By [**1**, 8.6], $V = \{(x, y) \in X \times X : d(x, y) < 1/2^3\}$ is an element of the universal uniformity. If $(x, y) \in V$ then $d(x, y) < 1/2^3$. Since $(B(x, 1/2^3))_{x \in X}$ refines $\mathscr{U}$ there exists $z \in X$ such that $B(x, 1/2^3) \subset U(z)$. Thus $x \in U(z)$ and $y \in U(z)$ so that $(x, y) \in U^2 \subset W$. We thus have $V \subset W$ and therefore W is in the universal uniformity.

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