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# ON CO-FPF MODULES

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A ring R is called right co-FPF if every finitely generated cofaithful right R-module is a generator in mod-R. This definition can be carried over from rings to modules. We say that a finitely generated projective distinguished right R-module P is a co-FPF module (quasi-co-FPF module) if every P-finitely generated module, which finitely cogenerates P, generates $\sigma$[P] (P, respectively). We shall prove a result about the relationship between a co-FPF module and its endomorphism ring, and apply it to study some co-FPF rings.

## 1. Introduction

In this note all rings are associative with identities and all modules are unitary. Let $M_R$ be a right $R$-module. A module $N$ is called $M$-generated or $M$ generates $N$ if there exist a set $A$ and an epimorphism $M^{(A)} \to N$, where $M^{(A)}$ is the direct sum of $|A|$ copies of $M$ ($|A|$ denotes the cardinality of the set $A$). When $A$ is finite, we say that $N$ is $M$-finitely generated. $N$ is called $M$-cogenerated or $M$ cogenerates $N$ if there exist a set $A$ and a monomorphism $N \to M^A$, where $M^A$ is the direct product of $|A|$ copies of $M$. When $A$ is finite, we say that $N$ is $M$-finitely cogenerated. Let $M_R$ and $U_R$ be two modules. Then $M$ is called $U_R$-distinguished if for every nonzero homomorphism $h : X_R \to M$ from a module $X_R$ into $M$ there exists a homomorphism $g : U \to X$ so that $hg \neq 0$. A module $M_R$ is distinguished if $M$ is $M$-distinguished. For a module $M_R$, we denote by $\sigma[M]$ the full subcategory of mod-$R$ whose objects are submodules of $M$-generated modules (see [**11**]).

For a right $R$-module $M$, the trace ideal of $M$ in $R$ is denoted by trace$(M)$. By definition, $\text{trace}(M) = \sum\{\text{im}\,\varphi, \varphi \in \text{Hom}_R(M, R_R)\}$ (see [**11**, p.154]).

A module $M_R$ is called faithful if $\{a \in R; Ma = 0\} = 0$. Then $M$ is faithful if and only if $M$ cogenerates every projective right $R$-module. Dually, a module $M_R$ is called cofaithful if $M$ generates every injective right $R$-module. It follows that $M$ is cofaithful if and only if there exists a finite subset $\{m_1, \dots, m_n\}$ of elements of $M$ such that $\{x \in R, m_1 x = \dots = m_n x = 0\} = 0$.

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A ring $R$ is called right FPF if every finitely generated faithful right $R$-module is a generator in mod-$R$. FPF rings have been the subject of much research and two recent monographs, Faith and Page [**3**] and Faith and Pillay [**4**], have been devoted to them. We introduce the family of right co-FPF rings as a generalisation of the class of right self-injective rings and the class of right FPF rings: A ring $R$ is called right co-FPF if every finitely generated cofaithful right $R$-module is a generator in mod-$R$. Basic results about co-FPF rings were obtained in [**6**].

The definition of FPF ring was carried over from rings to modules by Page [**8**] as follows: A finitely generated projective distinguished $R$-module $P_R$ is called FPF if every $P$-finitely generated module, which cogenerates $P$, generates $P$. Also we say that a finitely generated projective distinguished $R$-module $P_R$ is a (quasi-) co-FPF module if every $P$-finitely generated module, which finitely cogenerates $P$, generates $\sigma[P]$ ($P$, respectively). Thus an FPF module is a quasi-co-FPF module, but the converse is not true in general (see [**6**, Example 2.3]). Since a self-generator is distinguished, a ring $R$ is right co-FPF if and only if $R_R$ is a quasi-co-FPF module if and only if $R_R$ is a co-FPF module.

Page [**8**, Theorem 4] proved a result about the relationship between a FPF module and its endomorphism ring. Motivated by this, we show in this paper that if $P$ is a finitely generated distinguished projective right $R$-module, then:

(i) If $P$ is a quasi-co-FPF module, then $\mathrm{End}_R(P)$ is a right co-FPF ring;
(ii) If $P$ is a self-generator and $\mathrm{End}_R(P)$ is a right co-FPF ring, then $P$ is a co-FPF module.

From this it follows that if $R$ is a right co-FPF ring and $e$ a semicentral idempotent of $R$ (that is, $eR = eRe$), then $eR = eRe$ is a right co-FPF ring.

## 2. Results

First we list some known results used in this section.

**Lemma 1.** *Let $P_R$ be a finitely generated projective right $R$-module with $S = \mathrm{End}(P_R)$. Then:*

(i) *${}_SP_R$ is an $(S,R)$-bimodule, ${}_RP^*_S = \mathrm{Hom}_R(P_R, R)$ is an $(R,S)$-bimodule,*
(ii) *$F = -\otimes_R P^*$ is a functor from right $R$-modules to right $S$-modules, $G = -\otimes_S P$ is a functor from right $S$-modules to right $R$-modules, $H = \mathrm{Hom}_S(P^*, -)$ is a functor from right $S$-modules to right $R$-modules. The functors $G, F, H$ form an adjoint triple $(G, F, H)$ and there are natural transformations $\alpha : 1_S \longrightarrow FG$, $\alpha' : FH \longrightarrow 1_S$, $\beta : 1_R \longrightarrow HF$, and $\beta' : GF \longrightarrow 1_R$.*

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(iii) *There are the evalution homomorphisms:*

$$\begin{aligned} \nu : P^* \otimes_S P &\longrightarrow P \\ f \otimes p &\longmapsto f(p) \\ \theta : P \otimes_R P^* &\longrightarrow S \\ p \otimes f &\longmapsto pf() \end{aligned}$$

*and $\theta$ is an isomorphism.*

(iv) *Set $T = \operatorname{trace}(P)$, then $T = \operatorname{trace}(P^*), PT = P$, $TP^* = P^*$ and $P^*$ is a finitely generated projective left $R$-module.*

(v) *$\nu$ is an epimorphism if and only if $P_R$ is a generator.*

(vi) *$P_S$ is always a generator over $S$ and $P$ is finitely generated over $S$ if and only if $P$ finitely cogenerates $R$, that is, $P$ is a cofaithful right $R$-module.*

PROOF: (i), (ii), (iv) and (v) can be proved easily. The statements (iii) and (vi) are proved in [**1**, 11.19.1] and [**2**, 19.14B], respectively. ∎

LEMMA 2. (Kato [**5**, Lemma 3]) *Let $M_R$ and $U_R$ be two right $R$-modules. Then $M_R$ is $U$-distinguished if and only if for each $m \in M$, $m.\operatorname{trace}(U) = 0$ implies $m = 0$.*

Recall that a submodule $M'$ of $M$ is a pure submodule if the exact sequence

$$0 \to M' \to M \to M/M' \to 0$$

is pure, that is, $M/M'$ is flat.

LEMMA 3. (Zimmermann - Huisgen [**10**, Theorem 2.4]) *Let $P_R$ be a right $R$-module such that $P.\operatorname{trace}(P) = P$. Then $\operatorname{trace}(P)$ is left pure if and only if $P_R$ is a self generator.*

LEMMA 4. (Rutter [**9**]) *Let $P_R$ be a finitely generated projective module and $M$ be a right $R$-module. Then $M \otimes_R P^* = 0$ if and only if $M.\operatorname{trace}(P) = 0$. Moreover, if $M$ is injective and $P$ is distinguished, then $(M \otimes_R P^*)_S$ is injective .*

LEMMA 5. (Miller [**7**, Corollary 2.6 and Theorem 2.7]) *Let $P_R$ be a finitely generated projective module. If $\operatorname{trace}(P)$ is left flat, then $P^* \otimes_S P \simeq \operatorname{trace}(P)$ as an $(R,R)$-module.*

LEMMA 6. (Thuyet [**6**, Lemma 2.1]) *Let $M_R \in$ mod-$R$. Then the following conditions are equivalent:*

(i) *$M_R$ is cofaithful;*

(ii) *There exists a finite set $\{m_1, \dots, m_n\}$ of elements of $M$ such that $\{x \in R, m_1 x = \dots = m_n x = 0\} = 0$;*

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(iii) *There exists a positive integer* $n$ *such that* $R_R$ *can be embedded into* $M^n$;

(iv) $M$ *generates every injective right* $R$*-module;*

(v) $\sigma[M] =$ *mod-*$R$*;*

(vi) *Cyclic submodules of* $M^{(\mathbb{N})}$ *form a set of generators in mod-*$R$.

Now we state a result about the relationship between a co-FPF module and its endomorphism ring.

**Theorem 7.** *Let* $P$ *be a finitely generated distinguished projective right* $R$*-module. Then :*

(i) *If* $P$ *is a quasi-co-FPF module, then* $S = \mathrm{End}_R(P)$ *is a right co-FPF ring.*

(ii) *If* $P$ *is a self-generator and* $S = \mathrm{End}_R(P)$ *is a right co-FPF ring, then* $P$ *is a co-FPF module.*

Proof: (i) Assume that $P_R$ is quasi-co-FPF and $S = \mathrm{End}_R(P)$. Let $M$ be a finitely generated cofaithful right $S$-module. Then we have an exact sequence in mod-$S$:

$$S^n \longrightarrow M \longrightarrow 0$$

for some postive integer $n$. Tensoring with ${}_SP_R$ gives an exact sequence

$$(S^n \otimes_S P)_R \to (M \otimes_S P)_R \to 0.$$

But it is clear that $P_R^n \simeq (S \otimes_S P)^n \simeq (S^n \otimes_S P)_R$. This proves that $P_R$ finitely generates $M \otimes_S P$.

Since $M_S$ is cofaithful, we have an exact sequence in mod-$S$:

$$0 \longrightarrow S \xrightarrow{g} M^l$$

for some positive integer $l$, and the homomorphism

$$P_R \simeq (S \otimes_S P)_R \xrightarrow{f=g\otimes id} M^l \otimes_S P$$

induces an exact sequence:

$$0 \longrightarrow \ker f \longrightarrow P \xrightarrow{f} M^l \otimes_S P.$$

Now by Lemma 1(iv), $P^*$ is a finitely generated projective left $R$-module, hence ${}_RP^*$ is flat. Hence the following sequence is exact:

$$0 \longrightarrow \ker f \otimes_R P^* \longrightarrow P \otimes_R P^* \longrightarrow \left(M^l \otimes_S P\right) \otimes_R P^*.$$

By Lemma 1 we have the commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \rightarrow & \ker f \otimes_R P^* & \xrightarrow{i\otimes id} & P \otimes_R P^* & \xrightarrow{f\otimes id} & M^l \otimes_S P \otimes_R P^* \\ & & \| & & \wr|\theta & & \wr|id\otimes\theta \\ 0 & \rightarrow & \ker f \otimes_R P^* & \xrightarrow{\theta o(i\otimes id)} & S & \xrightarrow{f\otimes\theta} & M^l \otimes S \\ & & \| & & \| & & \wr|\xi \\ 0 & \rightarrow & \ker f \otimes_R P^* & \xrightarrow{\theta o(i\otimes id)} & S & \xrightarrow{\xi o(f\otimes\theta)} & M^l \end{array}$$

where $\theta$ is defined in Lemma 1(iii), and $\xi$ is the canonical isomorphism. However $\ker(\xi o(f\otimes\theta)) = (f\otimes\theta)^{-1}\ker\xi = \ker(f\otimes\theta) = \operatorname{im} f \otimes \ker\theta = 0$, hence $0 = \operatorname{im}(\theta o(i\otimes id_{P^*})) = \theta(\operatorname{im}(i\otimes id)) = \theta(\ker f \otimes P^*)$. It follows that $\ker f \otimes P^* = 0$. Let $T = \operatorname{trace}(P)$. Then by Lemma 4, $\ker f.T = 0$ and since $P$ is distinguished, by Lemma 2, it follows that $\ker f = 0$, that is, we have an exact sequence:

$$0 \longrightarrow P_R \longrightarrow (M^l \otimes_S P)_R \simeq (M \otimes_S P)^l_R.$$

Hence $(M\otimes_S P)_R$ finitely cogenerates $P$. By assumption, $M\otimes_S P$ generates $P$, but since $P$ is finitely generated, $M\otimes_S P$ finitely generates $P$, that is, there exists a positive integer $h$ such that

$$(M\otimes_S P)^h \longrightarrow P \longrightarrow 0$$

is exact in mod-$R$. This gives

$$\begin{array}{ccccc} (M\otimes_S P\otimes_R P^*)^h & \longrightarrow & P\otimes_R P^* & \longrightarrow & 0 \\ \wr|id\otimes\xi & & \wr|\xi & & \\ M^h_S & \longrightarrow & S_S & \longrightarrow & 0 \end{array}$$

in which the rows are exact. This shows that $M$ generates $S$. Thus $S$ is a right co-FPF ring.

(ii) Assume that $P$ is a self-generator and $S = \operatorname{End}_R(P)$ is a right co-FPF ring. We note that $P_R$ is a self-generator if and only if trace $P = T_R$ is pure in ${}_RR$ if and only if ${}_R(R/T)$ is flat (see [**10**, Theorem 2.4]). To prove that $P_R$ is quasi-co-FPF, let $N_R$ be a $P$-finitely generated right $R$-module and $N$ finitely cogenerate $P$. Then we have two exact sequences:

(1) $$P^m \longrightarrow N \longrightarrow 0, \quad m \in \mathbb{N},$$

(2) $$0 \longrightarrow P \longrightarrow N^l, \quad l \in \mathbb{N}.$$

Put $V_S = N\otimes_R P^*_S$. The proof of (ii) is divided into four steps.

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STEP 1. $V_S$ is cofaithful. In fact, since ${}_R P^*$ is flat, and from (2), we obtain:

$$\begin{array}{ccccc} 0 & \longrightarrow & P \otimes_R P^* & \longrightarrow & N^l \otimes_R P^* \\ & & \wr\|\xi & & \wr\| \\ 0 & \longrightarrow & S & \longrightarrow & (N \otimes_R P^*)^l \end{array}$$

with exact rows. This gives that $V_S$ is cofaithful.

STEP 2. $V$ is $S$-finitely generated. In fact, from (1), we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccc} P^m \otimes_R P^* & \longrightarrow & N \otimes_R P^* & \longrightarrow & 0 \\ \wr\| & & \| & & \\ S^m & \longrightarrow & (N \otimes_R P^*)_S & \longrightarrow & 0 \end{array}$$

that is, $V$ is $S$-finitely generated.

STEP 3. $N$ finitely generates $P$. From steps 1 and 2 and by assumption, $N \otimes_R P^*$ is a generator in mod-$S$, that is, there exists an exact sequence in mod-$S$:

$$(N \otimes_R P^*)^n \longrightarrow S \longrightarrow 0$$

and this yields

$$\begin{array}{ccccc} (N \otimes_R P^*)^n \otimes_R P_R & \longrightarrow & S \otimes_S P & \longrightarrow & 0 \\ \wr\|k & & \wr\| & & \\ (N \otimes_R T)^n & \longrightarrow & P_R & \longrightarrow & 0 \end{array}$$

where the existence of isomorphism $k$ is obtained from Lemma 5. From this to show that $N$ generates $P$ it is enough to show that $N \simeq N \otimes_R T$. We consider the exact sequence:

$$0 \longrightarrow T \longrightarrow R \longrightarrow R/T \longrightarrow 0.$$

Since T is left pure,

$$(3) \qquad 0 \longrightarrow N \otimes T \longrightarrow N \otimes R \longrightarrow N \otimes (R/T) \longrightarrow 0$$

is exact.

Note that $PT = T$. Thus $0 = (P \otimes R/T)T = PT \otimes R/T = R \otimes R/T$. By (1), we obtain the commutative diagram with exact rows:

$$\begin{array}{ccccc} P^m \otimes R/T & \longrightarrow & N \otimes R/T & \longrightarrow & 0 \\ \wr\| & & \| & & \\ 0 = (P \otimes R/T)^m & \longrightarrow & N \otimes R/T & \longrightarrow & 0. \end{array}$$

Thus $N \otimes R/T = 0$. From this and (3), we obtain the following exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & N \otimes T & \longrightarrow & N \otimes_R R & \longrightarrow & 0 \\ & & \| & & \wr\| & & \\ 0 & \longrightarrow & N \otimes T & \longrightarrow & N & \longrightarrow & 0. \end{array}$$

Hence $N \simeq N \otimes_R T$.

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STEP 4. By [11, 18.5], $P$ is a generator of $\sigma[P]$. Hence by step 3, $N$ is also a generator in $\sigma[P]$. This proves that $P$ is a co-FPF module. ∎

Now we have some applications to co-FPF rings.

**PROPOSITION 8.** *Let $R$ be a right co-FPF ring and $e$ an idempotent of $R$ such that $eR$ is distinguished. Then $eR$ is a quasi-co-FPF module and $eRe$ is a right co-FPF ring.*

PROOF: Let $M$ be a $eR$-finitely generated module such that it finitely cogenerates $eR$. Hence we have two exact sequences,

$$(1)\qquad (eR)^n \xrightarrow{f} M \longrightarrow 0$$

$$(2)\qquad 0 \longrightarrow eR \xrightarrow{g} M^l.$$

From (1) we obtain another exact sequence

$$R^n \longrightarrow M \longrightarrow 0.$$

It follows that $M$ is $R$-finitely generated. Now we set $U = M \oplus (1-e)R$, then $U$ is also $R$-finitely generated. And from (2) we construct a homomorphism $k$ as follows:

$$k : R \longrightarrow M^l \oplus (1-e)R$$

with $k = g \oplus id_{(1-e)R}$. It is easy to see that $k$ is a monomorphism. From this and the inclusion map $j : M^l \oplus (1-e)R \hookrightarrow \left(M \oplus (1-e)R\right)^l$ we obtain that $jk$ is a monomorphism from $R_R$ to $\left(M \oplus (1-e)R\right)^l$. This shows that $M \oplus (1-e)R$ is a cofaithful module. By assumption $U = M \oplus (1-e)R$ is a generator for mod-$R$, in particular $U$ generates $eR$. Thus

$$\operatorname{trace}_{eR}\left(M \oplus (1-e)R\right) = \operatorname{trace}_{eR} M + \operatorname{trace}_{eR}(1-e)R = \operatorname{trace}_{eR} M = eR,$$

that is, $M$ generates $eR$ or equivalently, $eR$ is a quasi-co-FPF module. This together with Theorem 7 shows that $\operatorname{End}_R(eR) \simeq eRe$ is right co-FPF. ∎

**LEMMA 9.** *If $e$ is a right semicentral idempotent, then $eR$ is distinguished.*

PROOF: For every $er \in eR$, if $er.\operatorname{trace}(eR) = 0$ then $erReR = 0$ and hence $ere = 0$. Since $e$ is a right semicentral idempotent, $er = ere = 0$. By Lemma 2, $eR$ is distinguished. ∎

**Corollary 10.** *Let $R$ be a right co-FPF ring, and $e$ be a right semicentral idempotent. Then $eRe$ is a right co-FPF ring.*

PROOF: By Proposition 8 and Lemma 9. ∎

**Corollary 11.** *If $e$ is an idempotent of $R$ such that $eR$ is distinguished, selfgenerator and $eRe$ is a right co-FPF ring, then $eR$ is a co-FPF module.*

PROOF: By Theorem 7. ∎

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