BULL. AUSTRAL. MATH. SOC. 05C25, 20B25
VOL. 7 (1962), 131-134.

An algebraic characterization of symmetric graphs with a prime number of vertices

J.L. Berggren

A graph Γ is called *symmetric* if its automorphism group is transitive on its vertices and edges. Let p be an odd prime, $Z(p)$ the field of integers modulo p, and $Z^*(p) = \{a \in Z(p) \mid a \neq 0\}$, the multiplicative subgroup of $Z(p)$. This paper gives a simple proof of the equivalence of two statements:

(1) Γ is a symmetric graph with p vertices, each having degree $n \geq 1$;

(2) the integer n is an even divisor of $p - 1$ and Γ is isomorphic to the graph whose vertices are the elements of $Z(p)$ and whose edges are the pairs $\{a, a+h\}$ where $a \in Z(p)$ and $h \in H$, the unique subgroup of $Z^*(p)$ of order n.

In addition, the automorphism group of Γ is determined.

The results of this paper are not new, for they were conjectured in [5] and proved in [3]. However the proof given here is very much simpler than that of [3], for the author of that paper used results he proved in [2] by means of Schur's theory of simply-transitive permutation groups. The proof given in this paper, using methods similar to those in [1], shows the elementary nature of the main result of [3].

Received 22 March 1972.

In this paper, p will denote an odd prime, $Z(p)$ the field of integers modulo p, and $Z^*(p)$ the multiplicative subgroup of non-zero elements in $Z(p)$. Because $Z^*(p)$ is cyclic of order $p - 1$ it has, for each divisor n of $p - 1$, a unique subgroup of order n.

A graph Γ is called *symmetric* if its automorphism group G is transitive on its vertices and edges. Notice that each vertex of a symmetric graph has the same degree. In [5] it was proved that, of the following two statements, (ii) implies (i):

(i) Γ is a symmetric graph with p vertices each having degree $n \geq 1$;

(ii) the integer n is an even divisor of $p - 1$ and Γ is isomorphic to the graph whose vertices are the elements of $Z(p)$ and whose edges are the pairs $\{a, a+h\}$ where $a \in Z(p)$ and $h \in H$, the unique subgroup of $Z^*(p)$ of order n.

We now prove that (i) implies (ii).

Let Γ be as in (i) and G its automorphism group. If G is doubly-transitive on the vertices then Γ is obviously the complete graph on p vertices and hence isomorphic to the graph constructed as in (ii) with $H = Z^*(p)$. In this case $n = p - 1$. If G is not doubly-transitive on the vertices then by Theorem 7.3 of [4] we may suppose the vertices of Γ are the points of $Z(p)$ and that $G \leq \{T(a, b) \mid a \in Z^*(p), b \in Z(p)\} = S$, where $T(a, b)$ is the permutation of $Z(p)$ which maps x to $ax + b$.

Since G is transitive on the p vertices, $p \mid |G|$ and, since $K = \{T(1, b) \mid b \in Z(p)\}$ is *the* subgroup of order p in S, we conclude $G \geq K$. It is now easy to verify that $H = \{a \in Z^*(p) \mid T(a, 0) \in G\}$ is a subgroup of $Z^*(p)$ and that $G = \{T(a, b) \mid a \in H, b \in Z(p)\}$.

We can now easily finish the proof. For each $i, j \in Z(p)$ we know $T(1, -i-j) \in G$, so $\{i, j\}$ is an edge of Γ if and only if $\{-i, -j\}$ is an edge. Thus $T(-1, 0) \in G$, so $-1 \in H$, and H has even order. If $A(0)$ is the set of points joined to 0 by an edge then $A(0) = Hc_1 + \ldots + Hc_r$, for the stabilizer of 0 is $\{T(a, 0) \mid a \in H\}$. If $f \geq 2$ then there is $T(a, b) \in G$ so that $T(a, b)$ maps 0 to c_2

https://doi.org/10.1017/S0004972700044890 Published online by Cambridge University Press

and c_1 to 0 . This implies $b = c_2$ and $ac_1 = -c_2$, that is, $(-a)c_1 = c_2$. But (-1) and a are in H so $-a \in H$ and $Hc_1 = Hc_2$. This contradiction shows $r = 1$ so $n = |H|$, an even divisor of $p - 1$. We have seen the edges of Γ are the pairs $\{a, a+hc_1\}$, so the map $a \to ac_1^{-1}$ is an isomorphism from Γ onto the graph Γ' whose vertices are the points of $Z(p)$ and whose edges are the pairs $\{a, a+h\}$, $a \in Z(p)$ and $h \in H$. Also if G' is the automorphism group of Γ' then $G' = G$. For it is clear that $G' \geq G$ and, since Γ and Γ' are isomorphic, G' and G are isomorphic.

In conclusion, we have shown that any symmetric graph Γ with p vertices has even degree n . Further, $n \mid (p-1)$ and if H is the unique subgroup of $Z^*(p)$ of order n then Γ is isomorphic to the graph constructed from H as in (ii). Also, if $H \neq Z^*(p)$, the automorphism group of Γ is isomorphic, as a permutation group, to the group of permutations of $Z(p)$ given by $\{T(a, b) \mid a \in H \text{ and } b \in Z(p)\}$. That this group satisfies all the conclusions of Theorem 3 of [3] is clear.

References

[1] J.L. Berggren, "An algebraic characterization of finite symmetric tournaments", *Bull. Austral. Math. Soc.* 6 (1972), 53-59.

[2] Chong-yun Chao, "On groups and graphs", *Trans. Amer. Math. Soc.* 118 (1965), 488-497.

[3] Chong-yun Chao, "On the classification of symmetric graphs with a prime number of vertices", *Trans. Amer. Math. Soc.* 158 (1971), 247-256.

[4] Donald Passman, *Permutation groups* (W.A. Benjamin, New York, Amsterdam, 1968).

[5] James Turner, "Point-symmetric graphs with a prime number of points", *J. Combinatorial Theory* 3 (1967), 136-145.

Department of Mathematics,
Simon Fraser University,
Burnaby,
British Columbia,
Canada.

https://doi.org/10.1017/S0004972700044890 Published online by Cambridge University Press