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ON THE ORDER OF ARC-STABILISERS IN ARC-TRANSITIVE GRAPHS, II

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Abstract

Let Γ be a G-vertex-transitive graph and let (u, v) be an arc of Γ. It is known that if the local action $G_v^{\Gamma(v)}$ (the permutation group induced by G_v on $\Gamma(v)$) is permutation isomorphic to the dihedral group of degree four, then either $|G_{uv}|$ is 'small' with respect to the order of Γ or Γ is one of a family of well-understood graphs. In this paper, we generalise this result to a wider class of local actions.

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1. Introduction

All graphs considered in this paper are finite and simple. A graph Γ is said to be *G-vertex-transitive* if G is a subgroup of $\mathrm{Aut}(\Gamma)$ acting transitively on the vertex set $\mathrm{V}(\Gamma)$ of Γ. Similarly, Γ is said to be *G-arc-transitive* if G acts transitively on the arc set $\mathrm{A}(\Gamma)$ of Γ. (An *arc* is an ordered pair of adjacent vertices.) When $G = \mathrm{Aut}(\Gamma)$, the prefix G in the above notation is sometimes omitted.

A very early and important result concerning arc-transitive graphs is the beautiful theorem of Tutte [5, 6] that, in a connected 3-valent arc-transitive graph, the order of an arc-stabiliser divides 16. To consider generalisations of this result, a very fruitful concept is that of local action, which we now define.

Definition 1.1. Let L be a permutation group, let Γ be a connected G-vertex-transitive graph and let $G_v^{\Gamma(v)}$ denote the permutation group induced by the action of G_v on the neighbourhood $\Gamma(v)$ of a vertex v. Then (Γ, G) is said to be *locally-L* if $G_v^{\Gamma(v)}$ is permutation isomorphic to L. More generally, if $\mathcal{P}$ is a permutation group property, then (Γ, G) will be called *locally-$\mathcal{P}$* provided that $G_v^{\Gamma(v)}$ possesses the property $\mathcal{P}$.

Note that, if Γ has valency d, then the permutation group $G_v^{\Gamma(v)}$ has degree d and, up to permutation isomorphism, does not depend on the choice of v. In [7], the author introduced the following notion: a transitive permutation group L is called *graph-restrictive* if there exists a constant $c(L)$ such that, for every locally-L pair (Γ, G) and an arc (u, v) of Γ, the inequality $|G_{uv}| \leq c(L)$ holds.

This definition makes it possible to give succinct formulations of many results and questions. For example, Tutte's theorem can be restated as follows: the symmetric group S_3 in its natural action on three points is graph-restrictive and the constant $c(S_3)$ can be chosen to be 16. The problem of determining which transitive permutation groups are graph-restrictive was also raised in [7]. A survey of the state of this problem can be found in [2].

If L is a transitive permutation group that is not graph-restrictive then, by definition, there exist locally-L pairs (Γ, G) with $|G_{uv}|$ arbitrarily large. In this case, the best that we can hope for is to obtain an upper bound on the order of G_{uv} in terms of the order of Γ.

The smallest transitive permutation group (either by degree or by order) which is not graph-restrictive is D_4, the dihedral group of degree four. The locally-D_4 case was thoroughly investigated in [3], where it was shown that there exists a sublinear function f such that, if (Γ, G) is a locally-D_4 pair, then $|G_{uv}| \leq f(|V(\Gamma)|)$, unless Γ is one of a family of well-understood graphs. Our goal in this paper is to generalise these results to a wider class of local actions, which we now define.

Definition 1.2. Let p be a prime. A transitive permutation group L on Ω is *weakly p-subregular* if there exist $x, y \in \Omega$ such that $|L_x| = p$ and $x^{\overline{L}} \cup y^{\overline{L}} = \Omega$, where $\overline{L} = \langle L_x, L_y \rangle$ and $x^{\overline{L}}$ (respectively, $y^{\overline{L}}$) denotes the orbit of x (respectively, y) under $\overline{L}$.

This definition generalises the notion of a p-subregular permutation group, which is the case when $\overline{L} = L$ (see [7, Definition 1.1]). Weakly p-subregular permutation groups are graph-restrictive if and only if they are p-subregular. (This follows from [2, Theorem 4 and Corollary 11].)

Some easy examples of weakly p-subregular permutation groups which are not graph-restrictive are dihedral groups of even degree and $\mathbb{Z}_p \operatorname{wr} \mathbb{Z}_2$ in its natural action on $\mathbb{Z}_p \times \mathbb{Z}_2$. There are also some nonsoluble examples, such as S_5 acting on its 5-cycles by conjugation. Our main goal in this paper is to prove the following theorem.

Theorem A. *Let (Γ, G) be a locally-L pair and let (u, v) be an arc of Γ. If L is weakly p-subregular for some prime p then either:*

(1) $|G_{uv}| \leq |A(\Gamma)|^3$*; or*
(2) *L is permutation isomorphic to $\mathbb{Z}_p \operatorname{wr} \mathbb{Z}_2$ and $\Gamma \cong C(p, r, s)$ for some $r \geq 3$ and $1 \leq s \leq r - 1$.*

We now define the graphs $C(p, r, s)$ which appear in Theorem A. These were first studied by Praeger and Xu [4]. Let p be a prime and let r and s be positive integers with $r \geq 3$ and $1 \leq s \leq r - 1$. Let $C(p, r, 1)$ be the lexicographic product $C_r[pK_1]$ of a cycle of length r and an edgeless graph on p vertices. In other words, $V(C(p, r, 1)) = \mathbb{Z}_r \times \mathbb{Z}_p$ with (i, u) being adjacent to (j, v) if and only if $i - j \in \{-1, 1\}$. A path in $C(p, r, 1)$ is

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called *traversing* if it contains at most one vertex from $\{y\} \times \mathbb{Z}_p$ for each $y \in \mathbb{Z}_r$. For $s \geq 2$, let $\mathrm{C}(p, r, s)$ be the graph with vertices being the traversing paths in $\mathrm{C}(p, r, 1)$ of length $s - 1$ and with two such $(s - 1)$-paths being adjacent in $\mathrm{C}(p, r, s)$ if and only if their union is a traversing path in $\mathrm{C}(p, r, 1)$ of length s. Clearly, $\mathrm{C}(p, r, s)$ is a connected $2p$-valent graph with rp^s vertices.

There is an obvious action of the wreath product $H = \mathbb{Z}_p \operatorname{wr} \mathrm{D}_r$ as a group of automorphisms of $\mathrm{C}(p, r, 1)$ which induces a faithful arc-transitive action on $\mathrm{C}(p, r, s)$ for $s \leq r - 1$. It is easily seen that, in this case, the local action is $\mathbb{Z}_p \operatorname{wr} \mathbb{Z}_2$ in its natural action on $\mathbb{Z}_p \times \mathbb{Z}_2$, which is weakly p-subregular. Note that $|H| = 2rp^r$ and hence the order of the stabiliser of an arc of $\mathrm{C}(p, r, s)$ in H is p^{r-s-1}. Thus, if p and s are fixed, then the order of the stabiliser in H of an arc of $\mathrm{C}(p, r, s)$ grows exponentially with r and hence exponentially with the number of arcs of $\mathrm{C}(p, r, s)$. This behaviour contrasts sharply with the polynomial upper bound which is the other possibility in the conclusion of Theorem A.

2. Proof of Theorem A

In the course of the proof, we will need the following result of Glauberman.

Theorem 2.1 [1, Theorem 1]. *Let P be a subgroup of a finite group G, let $g \in G$, and let $P \cap P^g$ be a normal subgroup of prime index p in P^g. Let n be a positive integer, and let $\tilde{G} = \langle P, P^g, \ldots, P^{g^n} \rangle$. Assume that:*

(1) *g normalises no nonidentity normal subgroup of P; and*
(2) $P \cap Z(\tilde{G}) = 1$.

Then $|P| = p^t$ for some positive integer t for which $t \leq 3n$ and $t \neq 3n - 1$. Moreover, if $n = 2$, $p = 2$ and $t = 6$, then P contains a nonidentity normal subgroup of $\tilde{G}$.

We now note a few elementary facts about weakly p-subregular permutation groups. Let L be a permutation group on Ω. Let p be a prime and suppose that L is weakly p-subregular as witnessed by x and y. Let $\overline{L} = \langle L_x, L_y \rangle$. As $x^{\overline{L}} \cup y^{\overline{L}} = \Omega$, the group $\overline{L}$ contains all point-stabilisers of L and hence $\overline{L}$ is normal in L. Since L_x and L_y both have order p, L_{xy} is normal in both of them and hence L_{xy} is normal in $\overline{L}$. Since $\Omega = x^{\overline{L}} \cup y^{\overline{L}}$, it follows that $L_{xy} = 1$ and $L_x \neq L_y$. We are now ready to prove the following theorem about locally weakly p-subregular pairs.

Theorem 2.2. *Let p be a prime, let (Γ, G) be a locally weakly p-subregular pair and let (u, v) be an arc of Γ. Then either $|G_{uv}| \leq |\mathrm{A}(\Gamma)|^3$ or G contains an abelian normal subgroup intersecting G_{uv} nontrivially.*

Proof. Fix a vertex v of Γ. By definition, there exist $u, w \in \Gamma(v)$ such that $|G_{uv}^{\Gamma(v)}| = p$ and $\Gamma(v) = u^{\overline{X}} \cup w^{\overline{X}}$, where $\overline{X} = \langle G_{uv}^{\Gamma(v)}, G_{vw}^{\Gamma(v)} \rangle$. By the notes preceding this theorem, $\overline{X}$ is normal in $G_v^{\Gamma(v)}$, $G_{uvw}^{\Gamma(v)}$ is normal and of index p in both $G_{uv}^{\Gamma(v)}$ and $G_{vw}^{\Gamma(v)}$, and $G_{uv}^{\Gamma(v)} \neq G_{vw}^{\Gamma(v)}$.

The fact that L is transitive implies that Γ is G-arc-transitive and hence there exists $g \in G$ such that $(u, v)^g = (v, w)$. Let $P = G_{uv}$ and $X = \langle P, P^g \rangle = \langle G_{uv}, G_{vw} \rangle$. Both P and

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P^g contain the kernel of the action of G_v on Γ_v. It follows from the previous paragraph that X is normal in G_v, that $\Gamma(v) = u^X \cup w^X$, that $P \cap P^g$ is normal and of index p in both P and P^g, and that $P \neq P^g$.

For any vertex $\alpha \in \mathrm{V}(\Gamma)$, there exists $f \in G$ such that $v^f = \alpha$. Let $X[\alpha] = X^f$. Note that since X is normal in G_v, $X[\alpha]$ only depends on α and not on the choice of f. Moreover, $X[\alpha]$ is a normal subgroup of G_α. Finally, let $G^* = \langle g, P\rangle$. We prove the theorem by a sequence of claims.

Claim 1. G^* is vertex-transitive.

First, note that $G^* = \langle g, X\rangle$. For $i \in \mathbb{Z}$, let $v_i = v^{g^i}$ and let $X_i = X^{g^i} = X[v_i]$. Note that $(v_{-1}, v_0, v_1) = (u, v, w)$ and hence $\Gamma(v_i) = v_{i-1}^{X_i} \cup v_{i+1}^{X_i}$ for any $i \in \mathbb{Z}$. Let $X^* = \langle X_i \mid i \in \mathbb{Z}\rangle$ and let $S = v^{\langle g\rangle} = \{v_i \mid i \in \mathbb{Z}\}$. Note that $X^* \leq G^*$, hence it suffices to show that $S^{X^*} = \mathrm{V}(\Gamma)$.

Suppose, for a contradiction, that there exists a vertex not in S^{X^*} and choose one with minimum distance to S. Call this vertex α and let $(p_0 = \alpha, \ldots, p_{n-1}, p_n = v_i)$ be a shortest path from α to a vertex of S. Since $\Gamma(v_i) = v_{i-1}^{X_i} \cup v_{i+1}^{X_i}$, it follows that there exists $\sigma \in X_i \leq X^*$ such that $p_{n-1}^\sigma \in \{v_{i-1}, v_{i+1}\} \subseteq S$. Since α is not in S^{X^*}, neither is α^σ, but α^σ is closer to S than α is, which is a contradiction.

Claim 2. g normalises no nonidentity normal subgroup of P.

Any normal subgroup N of P that is normalised by g must be normalised by $\langle g, P\rangle = G^*$ which is vertex-transitive. Since N fixes v, it must fix every vertex and hence be trivial.

For $i \geq 0$, let $H_i = \langle P, P^g, \ldots, P^{g^i}\rangle$. Let n be the smallest integer such that $H_n = H_{n+1}$.

Claim 3. H_n is normal in G.

First, note that $(H_n)^g \leq H_{n+1} = H_n$ and hence g normalises H_n. Recall that $P \neq P^g$, which implies that $n \geq 1$ and $H_n = \langle X, X^g, \ldots, X^{g^{n-1}}\rangle$. Since H_n is normalised by $\langle g, P\rangle = G^*$ which is vertex-transitive, it follows that $H_n = \langle X[\alpha] \mid \alpha \in \mathrm{V}(\Gamma)\rangle$. Since $X = X[v]$ is normal in G_v, it follows that $\langle X[\alpha] \mid \alpha \in \mathrm{V}(\Gamma)\rangle$ is normal in G.

Let N be the normal closure of $P \cap \mathrm{Z}(H_n)$ in G. Since H_n is normal in G, so is $\mathrm{Z}(H_n)$. It follows that $N \leq \mathrm{Z}(H_n)$ and hence N is abelian. If $N \neq 1$, then $N \cap P \neq 1$ and N is the required normal subgroup of G intersecting G_{uv} nontrivially.

We may thus assume that $N = 1$ and hence $P \cap \mathrm{Z}(H_n) = 1$. As we noted earlier, $P \cap P^g$ is normal of index p in both P and P^g. We can use Theorem 2.1 with $\tilde{G} = H_n$ to conclude that $|P| = p^t$ for some $t \leq 3n$. In particular, P is a p-group.

Claim 4. $|H_n| \geq p^n|P|$.

Let $i \in \{1, \ldots, n\}$. If P^{g^i} is a subgroup of H_{i-1}, then $H_{i-1} = H_i$, contradicting the minimality of n. It follows that P^{g^i} is a subgroup of H_i but not of H_{i-1}. Since P is a p-group, so is P^{g^i} and hence $|H_i| \geq p|H_{i-1}|$. The claim then follows by induction.

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Since $|G_{uv}||\mathrm{A}(\Gamma)| = |G| \geq |H_n| \geq p^n|P|$, it follows that $|\mathrm{A}(\Gamma)| \geq p^n \geq p^{t/3}$, concluding the proof. □

One of the reasons why Theorem 2.2 is interesting is that the condition of admitting an abelian normal subgroup which intersects an arc-stabiliser nontrivially is surprisingly strong. In fact, as we will see in the rest of this section, in the weakly p-subregular case it is enough to completely determine the structure of $G_v^{\Gamma(v)}$ (see Lemma 2.4) and almost completely determine the graph! We first show some of the 'local' consequences of this condition in general.

Lemma 2.3. *Let L be a transitive permutation group, let (Γ, G) be a locally-L pair, let (u, v) be an arc of Γ and suppose that G contains an abelian normal subgroup intersecting G_{uv} nontrivially. Then, L contains an abelian normal subgroup which is not semiregular.*

Proof. Suppose, for a contradiction, that each abelian normal subgroup of L is semiregular and let N be an abelian normal subgroup of G with $N_{uv} \neq 1$. Since N is an abelian normal subgroup of G, N_v is an abelian normal subgroup of G_v and hence $N_v^{\Gamma(v)}$ is an abelian normal subgroup of $G_v^{\Gamma(v)} \cong L$. It follows that $N_v^{\Gamma(v)}$ is semiregular and hence N_{uv} fixes $\Gamma(v)$. It follows that fixing an arc in N fixes all adjacent arcs and, by connectedness, that $N_{uv} = 1$, which is a contradiction. □

In view of Theorem 2.2 and Lemma 2.3, it is natural to study weakly p-subregular permutation groups which admit an abelian normal subgroup which is not semiregular.

Lemma 2.4. *Let p be a prime and let L be a weakly p-subregular permutation group. If L contains an abelian normal subgroup which is not semiregular, then L is permutation isomorphic to $\mathbb{Z}_p \operatorname{wr} \mathbb{Z}_2$.*

Proof. Let Ω be the set on which L is a permutation group. By definition, there exist $x, y \in \Omega$ such that $|L_x| = p$ and $x^{\overline{L}} \cup y^{\overline{L}} = \Omega$, where $\overline{L} = \langle L_x, L_y\rangle$. Let N be an abelian normal subgroup of L which is not semiregular. Since N is normal and not semiregular, N_x and N_y must be nontrivial. Since $|L_x| = |L_y| = p$ is prime, it follows that $N_x = L_x$ and $N_y = L_y$ and therefore $\overline{L} \leq N$. Finally, since N is abelian and not semiregular, it is not transitive and hence $\overline{L} = N$.

Since N is abelian, and $L_x \neq L_y$, it follows that $N = L_x \times L_y \cong \mathbb{Z}_p^2$. Since L is transitive, there exists $h \in L$ such that $x^h = y$. Write $L_x = \langle a\rangle$ and $L_y = \langle b\rangle$. Since $(L_x)^h = L_y$, we have that $a^h = b$ and, since N has index 2 in L, it follows that h^2 is contained in N which is abelian and hence $y^h = x$ and $b^h = a$ and, in particular, h does not commute with either a or b.

We now show that there exists an involution in $L \setminus N$. If $h^2 = 1$, then we are done. Otherwise, h^2 has order p. If p is odd, then h^p is an involution in $L \setminus N$. If $p = 2$, then $h^2 = ab$ and it is easy to check that ha is an involution in $L \setminus N$.

Let h' be an involution in $L \setminus N$. Clearly, $a^{h'} = a^h = b$ and $b^{h'} = a$. This shows that, as an abstract group, L is isomorphic to $\mathbb{Z}_p \operatorname{wr} \mathbb{Z}_2$. Finally, note that $\Omega = x^N \cup y^N = x^{L_y} \cup y^{L_x}$, while h' interchanges x^{L_y} and y^{L_x}, which concludes the proof. □

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Lemma 2.4 will allow us to use the following theorem of Praeger and Xu.

Theorem 2.5 [4, Theorem 1]. *Let p be a prime and let Γ be a $2p$-valent G-arc-transitive graph such that G has an abelian normal p-subgroup which is not semiregular on the vertices of Γ. Then $\Gamma \cong \mathrm{C}(p, r, s)$ for some $r \geq 3$ and $1 \leq s \leq r - 1$.*

We are now ready to prove Theorem A, which we restate for convenience.

Theorem A. *Let (Γ, G) be a locally-L pair and let (u, v) be an arc of Γ. If L is weakly p-subregular for some prime p then either:*

(1) $|G_{uv}| \leq |\mathrm{A}(\Gamma)|^3$; *or*

(2) *L is permutation isomorphic to $\mathbb{Z}_p$ wr $\mathbb{Z}_2$ and $\Gamma \cong \mathrm{C}(p, r, s)$ for some $r \geq 3$ and $1 \leq s \leq r - 1$.*

Proof. By Theorem 2.2, we may assume that G contains an abelian normal subgroup N intersecting G_{uv} nontrivially. By Lemma 2.3, this implies that L contains an abelian normal subgroup which is not semiregular. By Lemma 2.4, it follows that L is permutation isomorphic to $\mathbb{Z}_p$ wr $\mathbb{Z}_2$. In particular, L has degree $2p$ and hence Γ is $2p$-valent.

Since L is weakly p-subregular, we have that $|G_{uv}^{\Gamma(v)}| = p$ and hence G_{uv} is a p-group. Let M be the group generated by the elements of order p in N. This is an elementary abelian p-group which is characteristic in N and hence normal in G. Since G_{uv} is a p-group, it follows that there is an element of order p in N_{uv} and hence $M_{uv} \neq 1$. Finally, by Theorem 2.5, it follows that $\Gamma \cong \mathrm{C}(p, r, s)$ for some $r \geq 3$ and $1 \leq s \leq r - 1$. □

3. Further work

The two main tools in the proof of Theorem A are Theorems 2.2 and 2.5 [4, Theorem 1]. It seems quite likely that the conclusion of Theorem 2.2 holds under a much weaker hypothesis, especially if we are willing to replace the constant 3 in the exponent by a different constant. For example, we make the following conjecture, which would generalise Theorem 2.2.

Conjecture 3.1. For any k, there exists a constant d_k such that, if Γ is k-valent and G-arc-transitive, and the arc-stabiliser G_{uv} is soluble, then either $|G_{uv}| \leq |\mathrm{A}(\Gamma)|^{d_k}$ or G contains an elementary abelian normal p-subgroup intersecting G_{uv} nontrivially.

Note that the solubility of G_{uv} can be checked locally. Indeed, it is not hard to see that G_{uv} is soluble if and only if $G_{uv}^{\Gamma(v)}$ is. A missing piece of the puzzle seems to be our lack of a generalisation of Theorem 2.5. We thus pose the following question.

Question 3.2. What can be said about a G-arc-transitive graph Γ such that G has an abelian normal p-subgroup which is not semiregular on the arcs of Γ?

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