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A RELATION BETWEEN S^1 AND S^3-INVARIANT HOMOTOPY IN THE STABLE RANGE

SHIRLEY M. F. GILBERT AND PETER ZVENGROWSKI

ABSTRACT. For any X and any $q > 0$, one has natural inclusions $\pi^{S^3}_{4q-1}(X) \subset \pi^{S^1}_{4q-1}(X) \subset \pi_{4q-1}(X)$, where the groups S^1 and S^3 act on S^{4q-1} in the standard way and $\pi^G_{4q-1}(X)$ are the G-invariant homotopy subsets, $G = S^1$ or $G = S^3$. It is proved here that for any space X of the homotopy type of a CW-complex and for $\pi_{4q-1}(X)$ in the stable range, the inclusion $\pi^{S^3}_{4q-1}(X) \subset \pi^{S^1}_{4q-1}(X)$ is in fact an equality when localized away from the prime 2.

1. **Introduction.** We recall that for a topological group G acting on a sphere S^n so that the orbit space $Y = S^n/G$ is a CW-complex and $\gamma: S^n \twoheadrightarrow Y$ the quotient map, the G-invariant homotopy subset ${\pi_n}^G(X)$ for any space X is defined as

$$\pi_n{}^G(X) = \mathrm{Im}\left[\gamma^\# : [Y,X] \longrightarrow [S^n,X] = \pi_n(X)\right] \subset \pi_n(X).$$

The first example seems to go back to J. H. C. Whitehead [14] for $G = Z_2$, and since the late 1960's G-invariant homotopy has been studied by numerous authors, especially for $G = Z_2$. The cases $G = S^1$ acting on S^{2n-1} or $G = S^3$ acting on S^{4n-1} (by the standard actions via complex or quaternionic multiplication, respectively), with which we shall be concerned, have been studied by Gilbert [4], Gilbert and Zvengrowski [5], Mukai [6], Ōshima [7,8,9], Randall [10], and Rees [11]. The case $G = Z_p$ acting on $S^{2n-1} (p > 2)$ was also studied in [4]. Terms such as symmetric or projective, complex projective, and quaternionic projective have also been used for G-invariant homotopy when $G = Z_2, S^1, S^3$ respectively.

For an introduction to the basic properties of G-invariant homotopy, including the fact that it is a subgroup of $\pi_n(X)$ in the stable range, or when X is an H-space, cf. § 1 of [5]. Another basic property is that if a subgroup H of G acts on S^n by restriction of the G-action then there is clearly a natural inclusion $\pi_n{}^G(X) \subset \pi_n{}^H(X)$ induced by the map $S^n/H \twoheadrightarrow S^n/G$.

In particular $S^1 \subset S^3$ as a subgroup, so we obtain a natural inclusion

$$\pi^{S^3}_{4q-1}(X) \subset \pi^{S^1}_{4q-1}(X). \tag{1}$$

Our main result (Theorem 3.2) in this note is that in the stable range and for X a space of the homotopy type of a CW-complex the inclusion (1) is in fact an equality

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when localized away from the prime 2. An example (3.3) is also given showing that the stability hypothesis is in fact necessary. This work is based on a portion of [4], and the key tool in the proof is the result of Harris [2] on the homotopy of the unitary and symplectic groups away from $p = 2$.

2. **Preliminaries on $\Sigma\mathbb{C}P^n_m$ and Q^s_r.** Here $\Sigma\mathbb{C}P^n_m = \Sigma(\mathbb{C}P^n/\mathbb{C}P^{m-1})$ is a CW-complex with a single cell in dimensions $2m+1, 2m+3, \ldots, 2n+1$ while $Q^s_r = Q^s/Q^{r-1}$ is a CW-complex with a single cell in dimensions $4r-1, 4r+3, \ldots, 4s-1$ (cf. Steenrod-Epstein [13], Ch. 4). The corresponding cohomology generators for any ring Λ with unit will be denoted

$$\Sigma\eta_t \in H^{2t+1}(\Sigma\mathbb{C}P^n_m; \Lambda) \approx \Lambda, \quad m \le t \le n,$$
$$\zeta_t \in H^{4t-1}(Q^s_r; \Lambda) \approx \Lambda, \quad r \le t \le s.$$

Finally denote the generators of $H^2(\mathbb{C}P^n; \Lambda)$ and $H^4(\mathbb{H}P^s; \Lambda)$ by y, z respectively.

LEMMA 2.1. *There exists a map $Dg\colon Q^{M-n}_{M-q} \longrightarrow \Sigma\mathbb{C}P^{2M-2n-1}_{2m-2q-2}$ such that*

$$(Dg)^*(\Sigma\eta_{2t}) = 0$$
$$(Dg)^*(\Sigma\eta_{2t-1}) = \zeta_t, \quad M-q \le t \le M-n,$$

where $1 \le n \le q$ and M is divisible by some suitably large integer.

PROOF. Consider the map $g\colon \mathbb{C}P^{2q+1}_{2n} \longrightarrow \mathbb{H}P^q_n$ induced by the fibre maps f_{n-1} and f_q, where f_i is constructed from the diagram of fibrations

$$\begin{array}{ccccc} & & S^1 & \longrightarrow & S^3 & \longrightarrow S^2 \\ & & \downarrow & & \downarrow & \\ & & S^{4i+3} & \xrightarrow{=} & S^{4i+3} & \\ & & \downarrow & & \downarrow & \\ S^2 & \longrightarrow & \mathbb{C}P^{2i+1} & \xrightarrow{f_i} & \mathbb{H}P^i. & \end{array}$$

From the cohomology spectral sequence of f_i one easily sees that $f_i^*(z^t) = y^{2t}$, $t \le i$. Thus $g^*(z^t) = y^{2t}$, $n \le t \le q$. Dualizing g, for M divisible by a suitably large integer and using work of Atiyah [1] and James [3], we find

$$Q^{M-n}_{M-q} = D_{4M-1}(\mathbb{H}P^q_n) \xrightarrow{Dg} D_{4M-1}(\mathbb{C}P^{2q+1}_{2n}) = \Sigma\mathbb{C}P^{2M-2n-1}_{2M-2q-2}.$$

The stated behaviour of $(Dg)^*$ follows from Spanier [12], Theorem 6.1

Next let $p\colon \mathbb{C}P^{2M-2n-1}_{2M-2q-2} \longrightarrow \mathbb{C}P^{2M-2n-1}_{2M-2q-1}$ be the map killing the bottom cell, i.e. pinching $S^{4M-4q-4} = \mathbb{C}P^{2M-2q-2}_{2M-2q-2} \subset \mathbb{C}P^{2M-2n-1}_{2M-2q-2}$ to a point. Since $p^*(\eta_s) = \eta_s$, $2M-2q-1 \le s \le 2M-2n-1$, we form the composition $\psi = (\Sigma p) \circ (Dg)$, and have at once from Lemma 2.1 the following result.

PROPOSITION 2.2. *For $1 \leq n \leq q$ and M divisible by a suitably large power of 2, the map*

$$\psi = (\Sigma p) \circ (Dg) \colon Q_{M-q}^{M-n} \longrightarrow \Sigma \mathbb{C}P_{2M-2q-1}^{2M-2n-1}$$

satisfies

$$\psi^*(\Sigma\eta_{2t}) = 0$$
$$\psi^*(\Sigma\eta_{2t-1}) = \zeta, \quad M - q \leq t \leq M - n.$$

PROPOSITION 2.3. *Letting $J \colon S^{4M-4q-1} = Q_{M-q}^{M-q} \longrightarrow Q_{M-q}^{M-n}$ and $J' \colon S^{4M-4q-1} = \Sigma S^{4M-4q-2} = \Sigma \mathbb{C}P_{2M-2q-1}^{2M-2q-1} \longrightarrow \Sigma \mathbb{C}P_{2M-2q-1}^{2M-2n-1}$ be the respective inclusions into the bottom cells, $\psi \circ J \simeq J'$.*

PROOF. By 2.2 and the definitions of J, J', both $(\psi J)^*(\Sigma\eta_{2M-2q-1}) = J^*\psi^*(\Sigma\eta_{2M-2q-1}) = J^*(\zeta_{M-q})$ and $(J')^*(\Sigma\eta_{2M-2q-1})$ will equal the generator of $H^{4M-4q-1}(S^{4M-4q-1}; \Lambda) = \Lambda$. Taking $\Lambda = \pi_{4M-4q-1}(\Sigma \mathbb{C}P_{2M-2q-1}^{2M-2n-1}) \approx Z$ and using the Hopf classification theorem, this suffices to establish $\psi \circ J \simeq J'$.

For convenience in what follows we now take $n = 1$.

COROLLARY 2.4. *For any $i \geq 0$ there is a commutative diagram*

$$\begin{array}{ccc} \pi_i(S^{4M-4q-1}) & \xrightarrow{J_*} & \pi_i(Q_{M-q}^{M-1}) \\ \| & & \downarrow \psi_* \\ \pi_i(S^{4M-4q-1}) & \xrightarrow{J'_*} & \pi_i(\Sigma \mathbb{C}P_{2M-2q-1}^{2M-3}). \end{array}$$

REMARK 2.5. In §3 we shall take $i = 4M - r - 2$ with $2 \leq 2q < r$. A simple check then shows $\pi_{4M-r-2}(Q_{M-q}^{M-1}) = \pi_{4M-r-2}(Q_{M-q}^{M-1+j})$ and $\pi_{4M-r-2}(\Sigma \mathbb{C}P_{2M-2q-1}^{2M-3}) = \pi_{4M-r-2}(\Sigma \mathbb{C}P_{2M-2q-1}^{2M-3+j})$, $\quad 0 \leq j \leq \infty$, since in either case the extra cells attached are in dimensions greater than $4M - r - 1$.

3. **Application to S^1 and S^3-Invariant Homotopy.**

Theorem 2.1 of [5] gives a general method for calculating $\pi_{n+k}^G(X)$ where X is $(r-1)$-connected, in the stable range. We take $q \geq 1$ and wish to consider $\pi_{4q-1}^G(S^r)$, where $G = S^1, S^3$ and $2q < r$ for stability. In the case $G = S^1$, as shown in [5] Lemma 2.2 (a) (which we "suspend" one time here), this gives

$$\pi_{4q-1}^{S^1} \approx \mathrm{Ker}[J'_* \colon \pi_{4M-r-2}(S^{4M-4q-1}) \longrightarrow \pi_{4M-r-2}(\Sigma \mathbb{C}P_{2M-2q-1}^{\infty})].$$

Notice by Remark 2.5 above that there is no change if we replace $\Sigma \mathbb{C}P_{2M-2q-1}^{\infty}$ by $\Sigma \mathbb{C}P_{2M-2q-1}^{2M-3}$, so this is the same map J' that appears in Corollary 2.4. Similarly (cf. [4], Theorem 2.3.4),

$$\pi_{4q-1}^{S^3}(S^r) \approx \mathrm{Ker}[J_* \colon \pi_{4M-r-2}(S^{4M-4q-1}) \longrightarrow \pi_{4M-r-2}(Q_{M-q}^{M-1})].$$

Before stating the main theorem we introduce some notation. Let $\pi_*^s(X)$ denote the stable homotopy of X, and $\bar{\pi}_n(X) = \pi_n(X) \otimes Z[1/2]$ the homotopy group localized away from $p = 2$. Finally let $W_{n,k} = U(n)/U(n-k)$ and $X_{n,k} = Sp(n)/Sp(n-k)$ denote the respective complex and quaternionic Stiefel manifolds.

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THEOREM 3.1. *In the stable range $2q < r$ one has*

$$\bar{\pi}^{S^3}_{4q-1}(S^r) = \bar{\pi}^{S^1}_{4q-1}(S^r).$$

PROOF. By the remarks at the beginning of this section, $\pi^{S^3}_{4q-1}(S^r)$ and $\pi^{S^1}_{4q-1}(S^r)$, being in the stable range, are simply $\operatorname{Ker} J_*$ and $\operatorname{Ker} J'_*$ respectively in Corollary 2.4, with $i = 4M - r - 2$. To show equality of these kernels away from 2 it clearly suffices, again using Corollary 2.4, to show $\bar{\psi}_*$ is monic. We will do this by defining another map φ to which Harris' results [2] are applicable, and then comparing φ and ψ.

Consider the composite inclusion

$$\begin{array}{ccc} X_{2M-2q-1,M+q} & \rightarrowtail & W_{4M-4q-2,2M+2q} \\ & \searrow^{i} & \downarrow \\ & & W_{4M-4q-1,2M+2q} \end{array}$$

which we may assume without loss of generality is cellular. According to Chapter IV of [13] there are imbeddings

$$Q^{2M-2q-1}_{M-q} \subset X_{2M-2q-1,M+q}$$
$$\Sigma\mathbb{C}P^{4M-4q-2}_{2M-2q-1} \subset W_{4M-4q-1,2M+2q},$$

and in fact the left hand spaces are both the $8M-8q-2$ skeleta of the corresponding right hand spaces (a simple numerical check shows that there are no possible product cells in dimension$\leq 8M-8q-2$). Thus there is an induced map $Q^{2M-2q-1}_{M-q} \xrightarrow{\varphi} \Sigma\mathbb{C}P^{4M-4q-2}_{2M-2q-1}$, and for $m < 8M-8q-2$ we have

$$\begin{array}{ccc} \pi_m(X_{2M-2q-1,M+q}) & \xrightarrow{i_*} & \pi_m(W_{4M-4q-1,2M+2q}) \\ \uparrow\approx & & \uparrow\approx \\ \pi_m(Q^{2M-2q-1}_{M-q}) & \xrightarrow{\varphi_{1*}} & \pi_m(\Sigma\mathbb{C}P^{4M-4q-2}_{2M-2q-1}) \\ \uparrow\approx & & \uparrow\approx \\ \pi_m(Q^{M-1}_{M-q}) & \xrightarrow{\varphi*} & \pi_m(\Sigma\mathbb{C}P^{2M-3}_{2M-2q-1}). \end{array}$$

Localizing away from 2, we have from Harris [2], p. 412, that $\bar{i}_*$ is monic onto a direct summand (for all m). Hence so is $\bar{\varphi}_*$ (for $m < 8M-8q-2$). The theorem would now be immediate if one knew $\varphi \simeq \psi$. This seems plausible but probably difficult to prove. However, using Proposition 2.2 above and [13], p. 52, Theorem 5.4, we see that $\varphi^* = \psi^*$ in cohomology and it seems easiest to complete the proof at this stage by a fairly simple application of the Adams spectral sequence at any odd prime p.

In fact, at any such p, with $k = M-q$, we have that on the p-primary component of homotopy

(2) $$\varphi_*\colon {}_p\pi_m(Q^{M-1}_k) \longrightarrow {}_p\pi_m(\Sigma\mathbb{C}P^{2M-3}_{2k-1})$$

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is monomorphism onto a direct summand, by the above quoted result of Harris. Since this is in the stable range, φ_* is induced by the homorphism of Adams spectral sequences at the E_2 level

$$\varphi_{\#}\colon \mathrm{Ext}_{\mathcal{A}(p)}\big(H^*(Q_k^{M-1};Z_p),Z_p\big)\longrightarrow \mathrm{Ext}_{\mathcal{A}(p)}\big(H^*(\Sigma\mathbb{C}P_{2k-1}^{2M-3};Z_p),Z_p\big)$$

which in turn is induced by the map

$$\varphi^*\colon H^*(\Sigma\mathbb{C}P_{2k-1}^{2M-3};Z_p)\longrightarrow H^*(Q_k^{M-1};Z_p).$$

Finally, since $\psi^* = \varphi^*$, it will induce the same maps of Adams spectral sequences and hence of p-primary stable homotopy, so condition (2) also holds for ψ_*, completing the proof.

REMARK. Some extra light may be shed on the above proof by observing $H^*(\Sigma\mathbb{C}P_{2k-1}^{2M-3};Z_p) = \Sigma_{t=0}^{M-k-1}H^{4k+4t-1}(\mathbb{C}P_{2k-1}^{2M-3};Z_p)\oplus\Sigma_{t=0}^{M-k-2}H^{4k+4t+1}(\mathbb{C}P_{2k-1}^{2M-3};Z_p)$ is a splitting over the Steenrod algebra $\mathcal{A}(p)$ since deg $\mathcal{P}^i = 2i(p-1)$ is divisible by 4. Thus the Adams spectral sequence for $\Sigma\mathbb{C}P_{2k-1}^{2M-3}$ splits at the E_2 level, say $'E_2\oplus {}''E_2$, and Harris' result implies that this splitting passes to E_∞ (in the stable range), i.e. no Adams differentials connect $'E_2$ and $''E_2$ or vice-versa. Thus $'E_2$ and $''E_2$ both give rise to direct summands of the p-primary component ${}_p\pi_*^s(\Sigma\mathbb{C}P_k^{2M-3})$, at least in the stable range, and all φ_* or ψ_* are doing is identifying ${}_p\pi_*^s(Q_k^{M-1})$ with the direct summand arising from $'E_2$.

THEOREM 3.2. *Let X be an $(r-1)$-connected space having the homotopy type of a CW-complex. Then in the stable range $2q\le r$ one has*

$$\bar{\pi}_{4q-1}^{S^3}(X) = \bar{\pi}_{4q-1}^{S^1}(X).$$

PROOF. The theorem is already known for X a sphere (3.1). For X a finite $(r-1)$-connected CW-complex we may suppose without loss of generality that $X^{(r-1)} = *$. The theorem then extends to X by a routine application of the exact stable homotopy sequence of the cofibration sequence of the attaching map of each cell of X to a finite subcomplex, the fact that localization preserves exactness, and the 5-lemma. Since localization also commutes with direct limits the theorem then extends to arbitrary $(r-1)$-connected CW-complexes and hence to any space having this homotopy type.

To see that Theorem 3.1 fails outside the stable range, first recall that in the unstable range $\pi_n^G(X)$ is only a subset of $\pi_n^G(X)$, so by $\bar{\pi}_n^G(X)$ we mean the subset $\{\alpha\otimes 1 : \alpha\in\pi_n^G(X)\}\subset\pi_n(X)\otimes Z[1/2] = \bar{\pi}_n(X)$.

EXAMPLE 3.3. In [7] it is proved that $\pi_3^{S^1}(S^2) = \bar{\pi}_3^{S^1}(S^2)\approx\{k^2\nu : k\in Z\}$, where ν is the class of the Hopf map. On the other hand it is clear that $\pi_3^{S^3}(S^2) = 0$ since $Y = S^3/S^3 = *$ in this case. Thus Theorem 3.1 does not hold outside the stable range.

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The University of Calgary
Calgary, Alberta