Bull. Aust. Math. Soc. **106** (2022), 264–268
doi:10.1017/S0004972722000065

ON REPUNIT CULLEN NUMBERS

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(Received 13 October 2021; accepted 20 December 2021; first published online 8 February 2022)

Abstract

We prove that if $s \geq 2$ is a fixed integer, then the equation $ns^n + 1 = (b^m - 1)/(b - 1)$ has only finitely many positive integer solutions (n, b, m) with $b \geq 2$ and $m \geq 3$. When $s = 2$, it has no solution.

2020 *Mathematics subject classification*: primary 11D61.

Keywords and phrases: Diophantine equations, applications of Baker's method.

1. Introduction

A Cullen number is a positive integer of the form $n2^n + 1$. Arithmetic properties of these numbers have been investigated in various papers. In 1976, Hooley [2] showed that almost all Cullen numbers are composite. Luca and Stănică [5] showed that there are only finitely many Fibonacci numbers among the Cullen numbers. More recently, Luca and Noubissie [4] showed that the largest prime factor of $n2^n + 1 \pm m!$ tends to infinity with $\max\{m, n\}$ and found all pairs (m, n) such that this number is of the form $\pm 3^a \cdot 5^b \cdot 7^c$ for some nonnegative integers a, b, c.

An s-Cullen number is a number of the form $ns^n + 1$ where $s \geq 2$ is a fixed integer. Grantham and Graves [1] studied the Diophantine equation

$$ns^n + 1 = \frac{b^m - 1}{b - 1}, \tag{1.1}$$

and showed that under the *abc* conjecture, it has only finitely many positive integer solutions (n, s, b, m) with $s \geq 2$, $b \geq 2$ and $m \geq 3$. In this note, we assume that $s \geq 2$ is fixed and prove that the equation (1.1) has only finitely many positive solutions in the remaining variables (n, b, m) again with $b \geq 2$ and $m \geq 3$. More precisely, we have the following result.

THEOREM 1.1.

(i) *For a fixed integer $s \geq 2$, the Diophantine equation (1.1) has only finitely many positive integer solutions (n, b, m) with $b \geq 2$ and $m \geq 3$.*
(ii) *When $s = 2$, it has no solution.*

2. The proof

We consider (i) and (ii) together. Let $s := q_1^{\alpha_1} \cdots q_k^{\alpha_k}$, where $q_1, \ldots, q_k$ are distinct primes and $\alpha_1, \ldots, \alpha_k$ are positive integers. We expand the right-hand side and obtain

$$ns^n = b^{m-1} + \cdots + b. \tag{2.1}$$

The proof proceeds in two cases.

Case 1. $\gcd(b, s) > 1$.

Assume that q is prime and $q \mid \gcd(s, b)$. Letting $\nu_q(\ell)$ denote the exponent of q in the factorisation of the nonzero integer ℓ, we see that $\nu_q(ns^n) \geq n\nu_q(s)$ in the left-hand side of (2.1) whereas in the right-hand side of (2.1), we have $\nu_q(b^{m-1} + \cdots + b) = \nu_q(b)$. Thus, if $q^\alpha \| s$, then $q^{n\alpha} \mid b$. Let i be such that $q_1, \ldots, q_i$ all divide b. If $i = k$, we then get that $s^n \mid b$, so the right-hand side of (2.1) is at least $b^2 > s^{2n} > ns^n$, which is a contradiction. In particular, when $s = 2$, Case 1 cannot occur. Next, take $i < k$. Write $s_1 := q_1^{\alpha_1} \cdots q_i^{\alpha_i}$ and put $s_2 := s/s_1$ and $b_1 := b/s_1^n$. Then (2.1) can be rewritten as

$$n(s_1 s_2)^n = b(b^{m-2} + \cdots + 1) = b\Big(\frac{b^{m-1} - 1}{b - 1}\Big) = b_1 s_1^n \Big(\frac{b^{m-1} - 1}{b - 1}\Big).$$

Cancelling s_1^n, we get

$$ns_2^n = b_1\Big(\frac{b^{m-1} - 1}{b - 1}\Big),$$

and since b_1 and s_2 are coprime, it follows that $b_1 \mid n$. Thus,

$$(n/b_1)s_2^n = b^{m-2} + \cdots + 1,$$

or

$$(n/b_1)s_2^n - 1 = b(b^{m-3} + \cdots + 1).$$

The right-hand side is nonzero, since $m \geq 3$, and $q_1^n \mid b$. Applying a linear form in q_1-adic logarithms to the left-hand side (which is nonzero since $s_2 \geq 2$), we get

$$n \leq \nu_{q_1}(b) \leq \nu_{q_1}((n/b_1)s_2^n - 1) \ll (\log n)^2,$$

where the constant implied by the $\ll$ symbol depends on s (in fact, it is of size $O(q_1 \log s_2) = O(s \log s)$, where the constant implied by O is absolute). This gives a bound on n in this case.

Case 2. $\gcd(b, s) = 1$.

Since (2.1) can be rewritten as

$$ns^n = b\Big(\frac{b^{m-1} - 1}{b - 1}\Big),$$

and s and b are coprime, we get $b \mid n$. Thus,

$$b^{m-1} - (b - 1)(n/b)s^n = 1. \tag{2.2}$$

https://doi.org/10.1017/S0004972722000065 Published online by Cambridge University Press

In particular,

$$b^{m-1} = (b-1)(n/b)s^n + 1.$$

Since $b \mid n$, it follows that $n^{m-1} \geq b^{m-1} > s^n \geq 2^n$, and so $m \gg n/\log n$. However, $b^{m-1} < n^2 s^n + 1 < s^{4n}$, so $m \ll n$, where the constant implied by the last $\ll$ symbol depends on s (it is in fact of size $O(\log s)$ where the constant implied by the O symbol is absolute). Next, we rewrite (2.2) as

$$1 - ((b-1)(n/b)b^{1-m}s^n = \frac{1}{b^{m-1}}.$$

On the left-hand side (which is nonzero since the right-hand side is nonzero), we apply a linear form in logarithms á la Baker to get

$$\begin{aligned} -(m-1)\log 2 \geq -(m-1)\log b &= \log|((b-1)(n/b))b^{-m}s^n - 1| \\ &\geq 0.5|n\log s - m\log b + \log((b-1)(n/b)| \\ &\gg -(\log n)^2 \log(\max\{m,n\}) \\ &\gg -(\log n)^3, \end{aligned}$$

where the constant implied by the $\ll$ symbol can be taken to be

$$O(\log s \log\log(s+1))$$

and the constant implied by O is absolute. Hence,

$$\frac{n}{\log n} \ll m \ll (\log n)^3,$$

giving $n \ll (\log n)^4$, so n is bounded. This finishes the proof of (i).

For (ii), when $s = 2$, only Case 2 is possible so b is odd and the left-hand side of (2.1) is even. Hence, $(b^{m-1}-1)/(b-1)$ is even and b is odd showing that $m-1$ is even. Thus, (2.2) gives

$$(b^{(m-1)/2})^2 - \delta(b-1)(n/b)(2^{\lfloor n/2 \rfloor})^2 = 1, \quad \text{for some } \delta \in \{1,2\}.$$

Thus, $(X, Y) := (b^{(m-1)/2}, 2^{\lfloor n/2 \rfloor})$ satisfy

$$X^2 - dY^2 = 1, \quad \text{with } d \in \{(b-1)(n/b), 2(b-1)(n/b)\}. \tag{2.3}$$

Thus, $(X, Y) = (X_k, Y_k)$ is the kth solution of the Pell equation (2.3). Since Y is a power of 2, by the existence of primitive divisors for Lucas sequences, the only possibilities are $k \in \{1, 2\}$. Thus,

$$2^{n/2} \leq \sqrt{d}Y_k \leq (X_1 + \sqrt{d}Y_1)^2 < e^{2\cdot 3\sqrt{d}\log d},$$

where the last inequality follows from Lemma 1 in [3]. Since $d \leq 2(b-1)(n/b) < 2n$, we get

$$(n/2)\log 2 < 6\sqrt{2n}\log(2n),$$

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giving $n < 10^5$. To reduce it, we played around with Mathematica. If $m \le 8$, then since $b \mid n$, we get

$$n2^n + 1 \le \frac{n^8 - 1}{n - 1},$$

which gives $n \le 29$. If $m \ge 9$, then $n2^n \equiv b + b^2 + b^3 + \cdots + b^7 \pmod{b^8}$. For each $n < 10^5$, we generated all odd divisors $b > 1$ of n and checked whether the above congruence held, and if it did, we recorded the value of b. The only b found was $b = 3$. Since $n2^n = (3^m - 1)/2 \ge (3^9 - 1)/2$, we get $n \ge 10$. If $m - 1 \ge n$, we then get $n2^n = 3^{m-1} + \cdots + 1 > 3^n$, which is a contradiction for $n \ge 10$. Thus,

$$n2^n = 3\Big(\frac{3^{m-1} - 1}{2}\Big),$$

so $2^{n+1} \mid 3^{m-1} - 1$. This implies that $m - 1$ is even and further, calculating the exponent of 2 in $3^{m-1} - 1$, we get

$$\begin{aligned} n + 1 \le \nu_2(3^{m-1} - 1) &= 3 + \nu_2((m - 1)/2) \\ &< 3 + (\log(n/2))/\log 2 = 2 + (\log n)/\log 2 \\ &< 1.5 \log n + 2, \end{aligned}$$

which is false for $n \ge 10$. This shows that $m \ge 9$ is not possible, and so only $m \le 8$ is possible for which we already saw that $n \le 29$. Finally, we took all $n \in [2, 29]$, found all odd divisors $b > 1$ of n (if any), calculated the potential m using the equation $m - 1 := \lfloor \log(n2^n + 1)/\log b \rfloor$ and checked whether (1.1) holds with $s = 2$. No solution was found.

This finishes the proof.

3. Comments

Closely related to Cullen numbers are the Woodall numbers of the form $n2^n - 1$ or, more generally, $ns^n - 1$ with some $s \ge 2$. Our argument does not extend to Woodall numbers so we leave the topic of exploring the analogous Diophantine equation (1.1) with Cullen numbers replaced by Woodall numbers to the interested reader.

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https://doi.org/10.1017/S0004972722000065 Published online by Cambridge University Press

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