A CHARACTERIZATION OF THE FINITE SIMPLE GROUP $L_4(3)$

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In this paper we aim to give a characterization of the finite simple group $L_4(3)$ (i.e. $PSL(4, 3)$) by the structure of the centralizer of an involution contained in the centre of its Sylow 2-subgroup. More precisely, we shall prove the following result.

THEOREM. *Let t_0 be an involution contained in the centre of a Sylow 2-subgroup of $L_4(3)$. Denote by H_0 the centralizer of t_0 in $L_4(3)$.*

Let G be a finite group of even order with the following properties:

(a) *G has no subgroup of index 2, and*

(b) *G has an involution t such that the centralizer $C_G(t) = H$ of t in G is isomorphic to H_0.*

Then G is isomorphic to $L_4(3)$.

The following notations are used.

$N_X(Y)$:	the normalizer of Y in the group X.
$C_X(Y)$:	the centralizer of Y in the group X.
$\{\cdots \mid \cdots\}$:	the set of elements $\cdots$ such that $\cdots$.
$\langle \cdots \mid \cdots \rangle$:	the group generated by $\cdots$ such that $\cdots$.
$[x, y]$:	$x^{-1}y^{-1}xy$
Y^x:	$x^{-1}Yx$
$[X : Y]$:	the index of a subgroup Y in X.
$\|X\|$:	the order of X.
$0(X)$:	the maximal odd-order normal subgroup of X.
$x \sim y(X)$:	x is conjugate to y in the group X.
Y char X:	Y is a characteristic subgroup of X.

1. Some properties of H_0

Let F_3 be the finite field of 3 elements and V be a 4-dimensional vector space over F_3. Take

$$t'_0 = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

which is an involution in $SL(4, 3)$. (Here we identify the linear transformations in $SL(4, 3)$ with the corresponding matrices in term of a fixed basis.) The centre of $SL(4, 3)$ is generated by

$$c = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

and is of order 2. Then a matrix (α_{ij}) in $SL(4, 3)$ satisfies $(\alpha_{ij})t'_0 = t'_0(\alpha_{ij}) \cdot c^r$ $(r = 0, 1)$ if and only if (α_{ij}) has the form

$$(\alpha_{ij}) = \begin{pmatrix} A & \\ & B \end{pmatrix} \quad \text{or} \quad (\alpha_{ij}) = \begin{pmatrix} & A \\ B & \end{pmatrix}$$

where (A) and (B) are 2×2 matrices over F_3 such that $\det(A) = \det(B) \neq 0$.

Denote by H'_0, the group of matrices in $SL(4, 3)$ which commute projectively with t'_0 i.e. which satisfy the relation $(\alpha_{ij})t'_0 = t'_0(\alpha_{ij})c^r$ $(r = 0, 1)$. We have

$$u' = \begin{pmatrix} & & 1 & 0 \\ & & 0 & 1 \\ 1 & 0 & & \\ 0 & 1 & & \end{pmatrix}, \qquad v' = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

belong to H'_0 and generate a four-group F'_0. Moreover, we get

$$u' \cdot \begin{pmatrix} A & \\ & B \end{pmatrix} = \begin{pmatrix} & B \\ A & \end{pmatrix}.$$

Denote by L'_0, the group of all matrices in $SL(4, 3)$ of the form

$$\begin{pmatrix} A & \\ & B \end{pmatrix}$$

where (A) and (B) belong to $SL(2, 3)$. Clearly then $H'_0 = F'_0 \cdot L'_0$ and $F'_0 \cap L'_0 = 1$. Let L'_1 be the subgroup of L'_0 of the form

$$\begin{pmatrix} A & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

where $(A) \in SL(2, 3)$. Hence $L'_1 \cong SL(2, 3)$. Put $L'_2 = u' L'_1 u'$. Therefore $L'_0 = L'_1 \times L'_2$.

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Now L_1' is generated by the following elements

$$a_1' = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix}; \quad b_1' = \begin{pmatrix} 1 & 1 & & \\ 1 & -1 & & \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix} \text{ and } \sigma_1' = \begin{pmatrix} -1 & 1 & & \\ 0 & -1 & & \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix}.$$

Put $a_2' = u'a_1'u'$, $b_2' = u'b_1'u'$, $\sigma_2' = u'\sigma_1'u'$. Let $H_0 = H_0'/\langle c\rangle$, and in the natural homomorphism of H_0' onto H_0, let the images of t_0', u', v', F_0', L_0', a_i', b_i', σ_i', L_i' be t_0, u, v, F_0, L_0, a_i, b_i, σ_i, L_i respectively ($i = 1, 2$). Then we get the following relations:

$H_0 \;= F_0 \cdot L_0$

$F_0 \;= \langle u, v\rangle$ is a four-group

$L_0 \;= L_1 \cdot L_2$ where $L_1 \cap L_2 = \langle t\rangle$, and $[L_1, L_2] = 1$
(i.e. L_1, L_2 commute elementwise).

$L_i \;= \langle a_i, b_i, \sigma_i | a_i^2 = b_i^2 = t_0, b_i^{-1}a_ib_i = a_i^{-1}, \sigma_i^{-1}a_i\sigma_i = b_i, \sigma_i^{-1}b_i\sigma_i = a_ib_i\rangle$
$va_iv = a_i^{-1}$, $vb_iv = b_ia_i$, $v\sigma_iv = \sigma_i^{-1}$.

The structure of H_0 is completely determined and it is now easy to compute the following results of H_0.

(1.1) Every element of H_0 can be written uniquely in the form $a_1^ib_1^j\sigma_1^kt_1^lt_2^m\sigma^nu^pv^q$ where $t_1 = a_1a_2$; $t_2 = b_1b_2$; $\sigma = \sigma_1\sigma_2$; $i = 0, 1, 2, 3$; $j = 0, 1$; $k = 0, 1, 2$; $l = 0, 1$; $m = 0, 1$; $n = 0, 1, 2$; $p = 0, 1$; $q = 0, 1$.

$$|H_0| = 2^7 \cdot 3^2$$

(1.2) The group $Q = \langle a_1, b_1, a_2, b_2\rangle\ F_0$ is a Sylow 2-subgroup of $L_4(3)$ and of H_0. $Z(Q) = \langle t_0\rangle$.

(1.3) The group $T = \langle \sigma_1, \sigma_2\rangle$ is a Sylow 3-subgroup of H_0 and is elementary abelian of order 9. We have $C_{H_0}(T) = \langle t_0\rangle \times T$, and $N_{H_0}\langle T\rangle = \langle t, u, v\rangle \cdot T$.

(1.4) There are seven classes of involutions in H_0 with representatives t_0, t_1, u, t_0u, uv, t_0uv and v.

(1.5) The centralizer of t_1 in H_0, $C_{H_0}(t_1) = A = \langle a_1, a_2, b_1b_2, u, v\rangle$ is a non-abelian group of order 64 with $Z(A) = A' = \langle t_0, t_1\rangle$ where A' denotes the commutator group of A. The group A contains precisely four elementary abelian groups of order 16, namely $E_1 = \langle t_0, t_1, t_2, u\rangle$, $E_2 = \langle t_0, t_1, t_3, uv\rangle$ ($t_3 = a_1t_2$); $K_1 = \langle t_0, t_1, u, v\rangle$ and $K_2 = \langle t_0, t_1, a_1v, t_2u\rangle$.

(1.6) The centralizer of u in H_0,

$$C_{H_0}(u) = U = \langle t_0, t_1, t_2, u, v\rangle \cdot \langle \sigma\rangle.$$

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We have $C_{H_0}(u) = C_{H_0}(t_0u)$. A Sylow 2-subgroup of U is

$$\langle t_0, t_1, t_2, u, v\rangle = E_1 \cdot K_1$$

and has as its centre the group $\langle t_0, t_1, u\rangle$.

(1.7) The centralizer of uv in H_0,

$$C_{H_0}(uv) = W = \langle t_0, t_1, t_3, u, v\rangle \cdot \langle \rho\rangle \qquad (\rho = \sigma_1^{-1}\sigma_2).$$

We have $C_{H_0}(uv) = C_{H_0}(t_0uv)$. A Sylow 2-subgroup of W is $\langle t_0, t_1, t_3, u, v\rangle$ with its centre equals to $\langle t_0, t_1, uv\rangle$.

(1.8) The centralizer of v in H_0 is $K_1 = \langle t_0, t_1, u, v\rangle$.

(1.9) We have $C_G(E_i) = E_i$ and $N_{H_0}(E_i)/E_i \cong S_4$, the symmetric group in 4 letters. So a Sylow 2-subgroup of $N_{H_0}(E_i)/E_i$ is dihedral of order 8. $(i = 1, 2)$.

(1.10) There are precisely two normal elementary abelian groups of order 16 in Q, namely E_1 and E_2. There is one and only one normal subgroup of order 32 in Q containing E_i. These are $\langle a_1, a_2, t_2, u\rangle \geqq E_1$ with its centre equals to $\langle t_0, t_1\rangle$ and $\langle a_1, a_2, t_3, uv\rangle \geqq E_2$ with its centre equals to $\langle t_0, t_1\rangle$.

2. Conjugacy of involutions

Throughout the rest of this paper, we shall suppose that G is a finite group of even order with properties (a) and (b). Since $C_G(t) = H$ is isomorphic to H_0, we identify H with H_0. Then $t = t_0$.

First we note the obvious fact that the group Q is a Sylow 2-subgroup of G, since by (1.2) $Z(Q) = \langle t\rangle$, a cyclic group of order 2.

(2.1) LEMMA. *The involution t_1 is not conjugate to t in G.*

PROOF. By way of contradiction, suppose that t_1 is conjugate to t in G. We have $A = C_H(t_1)$. Let T be a Sylow 2-subgroup of $C_G(t_1)$ containing A. By our assumption $[T : A] = 2$ and so $A \triangleleft T$. Let x be an element in $T-A$. Consider $x^{-1}E_1x \subseteqq A$. We know that there are precisely four distinct elementary abelian groups of order 16 in A namely E_1, E_2, K_1, K_2 where $K_2 = K_1^{b_2}$. Now if $E_1^x = E_1$, we get $E_1 \triangleleft \langle A, x\rangle = T$. If x does not normalize E_1, $x^{-1}E_1x \neq E_2$ since otherwise we would have two normal subgroups E_1 and E_2 of Q conjugate in G but not in $N_G(Q) \subseteqq H$, a contradiction to a theorem of Burnside [4, p. 203]. So $x^{-1}E_1x = K_1$ or K_2. Therefore $x^{-1}E_2x = E_2$, in which case we get $E_2 \triangleleft T$. Hence we have either E_1 or E_2 normal in T.

Suppose that $E_1 \triangleleft T$. Since $N_G(E_1) \supseteqq \langle Q, T\rangle$, we get $N_G(E_1) \nsubseteq H$. We have by (1.9) $C_G(E_1) = E_1$, and so $\mathscr{S} = N_G(E_1)/E_1$ is isomorphic to a subgroup of $GL(4, 2) \cong A_8$. A Sylow 2-subgroup $\bar{Q} = Q/E_1$ of $\mathscr{S}$ is dihedral of order 8. Consider $C_{\mathscr{S}}(a_1 E_1) \supseteq \bar{Q}$. By way of contradiction, suppose $Z(T/E_1) = Z(\bar{T}) = \langle vE_1 \rangle$ or $\langle a_1 v E_1 \rangle$. Then either $\langle E_1, v \rangle$ or $\langle E_1, a_1 v \rangle$ is normal in T. Since $Z(\langle E_1, v \rangle) = \langle t, t_1, u \rangle$ and

$$Z(\langle E_1, a_1 v \rangle) = \langle t, t_1, t_2 u \rangle$$

both of order 8, hence a contradiction to (1.10). Therefore

$$\langle \bar{Q}, \bar{T} \rangle \subseteq C_{\mathscr{S}}(a_1 E_1).$$

From the structure of A_8, the centralizer of any involution in A_8 has order $2^6 \cdot 3$ or $2^5 \cdot 3$, we get $|C_{\mathscr{S}}(a_1 E_1)| = 2^3 \cdot 3$ and hence $C_{\mathscr{S}}(a_1 E_1)$ has an abelian 2-complement. The conditions of Gorenstein-Walter's theorem [3] are satisfied by the group $\mathscr{S}$ and so we get the following possibilities for $\mathscr{S}$.

(i) $\mathscr{S}/\mathscr{M} \cong PSL(2, q)$; $q \pm 1 = |C_{\mathscr{S}}(a_1 E_1)\mathscr{M}/\mathscr{M}|$

(ii) $\mathscr{S}/\mathscr{M} \cong PGL(2, q)$; $q \pm 1 = \frac{1}{2}|C_{\mathscr{S}}(a_1 E_1)\mathscr{M}/\mathscr{M}|$

(iii) $\mathscr{S}/\mathscr{M} \cong \bar{Q}$ or

(iv) $\mathscr{S}/\mathscr{M} \cong A_7$

where in all cases $\mathscr{M} = 0(\mathscr{S})$.

Suppose that $|\mathscr{M}| \neq 1$. Consider the action of the four group $\mathscr{V} = \langle a_1 E_1, b_1 E_1 \rangle$ on $\mathscr{M}$. Since $a_1 E_1$, $b_1 E_1$, $a_1 b_1 E_1$ are conjugate in $\mathscr{S}$, we get that $|\mathscr{M}| = 3^3$ or 3. Since $|A_8| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$, we must have $|\mathscr{M}| = 3$ therefore $\mathscr{V} \cdot \mathscr{M} = \mathscr{V} \times \mathscr{M}$. Now we look at

$$N_{\mathscr{S}}(\mathscr{V}) = N_G\langle a_1, b_1, E_1 \rangle \cap N_G(E_1)/E_1.$$

Since $\langle t \rangle = Z\langle a_1, b_1, E_1 \rangle$, we have $N_G\langle a_1, b_1, E_1 \rangle \subseteq H$. Thus

$$N_G\langle E_1, a_1, b_1 \rangle / E_1 \cong A_4,$$

a contradiction to $\mathscr{V} \cdot \mathscr{M} = \mathscr{V} \times \mathscr{M}$. Hence $\mathscr{M} = 1$. Clearly then (i), (ii) and (iii) cannot arise.

Thus we are in case (iv). The non-trivial elements of E_1 separate into 4 sets of involutions namely $\{t\}$; $\{u, tt_1 u, tt_2 u, tt_1 t_2 u\}$; $\{tu, t_1 u, t_2 u, t_1 t_2 u\}$ and $\{t_1, t_2, t_1 t_2, tt_1, tt_2, tt_1 t_2\}$, each of these sets lie in a different conjugate class of H. Let $\mu \in N_G(E_1)$ be an element of order 5. Since $C_G(E_1) = E_1$, we get that μ acts fixed-point-free on E_1. Together with the fact that $t_1 \sim t(G)$, we conclude that all involutions are conjugate in G. Now let $\lambda \in N_G(E_1)$ be an element of order 7 in G. Since all involutions of E_1 are conjugate to t and because $7 \nmid |H|$, we get that λ acts fixed-point-free on E_1, a contradiction since $7 \nmid (|E_1|-1)$. Thus we have shown that $E_1 \ntriangleleft T$.

By exactly the same reasoning, we get a contradiction if $E_2 \triangleleft T$. Hence t_1 is not conjugate to t in G. The proof is complete.

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(2.2) LEMMA. *The elementary abelian groups E_1, E_2, K_1 are not conjugate to one another in G.*

PROOF. We have shown that E_1 is not conjugate to E_2 in G. Suppose, by way of contradiction, E_1 is conjugate to K_1 in G. Since 2^7 divides the order of $N_G(E_1)$, and by our assumption, we get a Sylow 2-subgroup of $N_G(K_1)$ is of order 2^7. There exists a 2-group in $N_G(K_1)$ containing A such that $[T : A] = 2$. Now $Z(A) = \langle t, t_1 \rangle$ is characteristic in A and so normal in T. Since $N_G(Z(A)) \cap H = Q$ and $K_1 \ntriangleleft Q$, therefore we obtain $T \nsubseteq H$. Let $x \in T - A$. Then $x^{-1}tx \in \{t_1, tt_1\}$, a contradiction to (2.1). Similarly we can show that E_2 is not conjugate to K_1 in G. The proof is finished.

(2.3) LEMMA. *If 64 divides the order of $C_G(u)$ then u and tu do not lie in the same conjugate class in G.*

PROOF. Let $T \subseteqq C_G(u)$ be a group of order 64 containing

$$U = \langle t, t_1, t_2, u, v \rangle$$

and let $x \in T - U$. Then x normalizes $Z(U) = \langle t, t_1, u \rangle$. By (2.1),

$$x^{-1}tx \in \{tt_1u, t_1u, tu\}.$$

We shall consider each possibility in turn. If $x^{-1}tx = tt_1u$, then $x^{-1}tux = tt_1$. The proof is finished since $tt_1u \sim u(H)$ and $tt_1 \sim t_1(H)$. Next if $x^{-1}tx = t_1u$, then we get $x^{-1}tux = t_1$ and so $t \sim t_1(G)$ since $t_1u \sim tu(H)$, a contradiction to (2.1). Lastly if $x^{-1}tx = tu$, then we have $x^{-1}t_1x \in \{t_1, tt_1, tt_1u\}$. Now if $x^{-1}t_1x = t_1$, then $x^{-1}tt_1x = tt_1u \sim u(H)$ and so lemma is proved. The case $x^{-1}t_1x = tt_1$ is not possible, since this would imply

$$x^{-1}tx = x^{-1}t_1 \cdot tt_1x = tt_1 \cdot t_1 = t$$

(Here we use the fact $x^2 \in U$). Finally if $x^{-1}t_1x = tt_1u$, there is nothing to prove. The proof of this lemma is complete.

(2.4) LEMMA. *If 64 divides the order of $C_G(uv)$, then uv, tuv do not lie in the same conjugate class in G.*

PROOF. As in (2.3).

(2.5) LEMMA. *If u is conjugate to t in G, then tu is conjugate to t_1 in G. Moreover, we have $N_G(E_1)/E_1 \cong S_5$, the symmetric group in 5 letters.*

PROOF. The first part of this lemma is obvious from (2.3).

Consider $N_G(E_1)$. We have $N_H(E_1) = Q \cdot \langle \sigma \rangle$ and $N_H(E_1)/E_1 \cong S_4$. Let T be a Sylow 2-subgroup of $C_G(u)$ containing $\underline{U} = \langle t, t_1, t_2, u, v \rangle$. There exists $x \in T - \underline{U}$ with $x \in N_G(\underline{U})$ and so $x^{-1}E_1x \subseteqq \underline{U}$. By (1.6) and (2.2), we get $x^{-1}E_1x = E_1$. Hence $N_G(E_1) \nsubseteq H$.

A Sylow 2-subgroup of $\mathscr{S} = N_G(E_1)/E_1$ is dihedral of order 8. Suppose by way of contradiction that, $\bar{Q}$ has one class of involution in $\mathscr{S}$. Then there exists an element $g \in N_G(E_1)$ such that $g^{-1}\langle E_1, a_1\rangle g = \langle E_1, v\rangle$, which is a contradiction since $Z(\langle E_1, a_1\rangle) = \langle t, t_1\rangle$ whereas

$$Z(\langle E_1, v\rangle) = \langle t, t_1, u\rangle.$$

Since we have $C_G(E_1) = E_1$, $\mathscr{S}$ is isomorphic to a subgroup of A_8. Suppose that $0(\mathscr{S}) = \mathscr{M} \neq 1$. Then consider the action of the four group $\vartheta = \langle a_1 E_1, b_1 E_1\rangle$ on $\mathscr{M}$. Using the facts that involutions of ϑ are conjugate in $\mathscr{S}$ and that the centralizer of any involution in A_8 has order $2^6 \cdot 3$ or $2^5 \cdot 3$, we get by Brauer-Wielandt [10], $|\mathscr{M}| = 27$ or 3. Since $27 \nmid |A_8|$, we must have $|\mathscr{M}| = 3$, and so $\vartheta \cdot \mathscr{M} = \vartheta \times \mathscr{M}$. We look at $N_{\mathscr{S}}(\vartheta) = N_G(\langle E_1, a_1, b_1\rangle) \cap N_G(E_1)/E_1$. Since $\langle t\rangle = Z(\langle E_1, a_1, b_1\rangle)$; we get $N_G(\langle E_1, a_1, b_1\rangle) \subseteqq H$. Hence $N_G(\langle E_1, a_1, b_1\rangle)/E_1 \cong A_4$, a contradiction to $\vartheta \cdot \mathscr{M} = \vartheta \times \mathscr{M}$. Thus we have shown $0(\mathscr{S}) = 1$.

By our earlier remark, we must have $|C_{\mathscr{S}}(a_1 E_1)| = 2^3 \cdot 3$ or 2^3. Hence we may now apply Gorenstein-Walter's theorem [3] to get $\mathscr{S} \cong PGL(2, 11)$; $PGL(2, 13)$; $PGL(2, 3)$ or $PGL(2, 5)$. The first two cases cannot arise since 11 and 13 do not divide $|A_8|$. $\mathscr{S} \cong PGL(2, 3) \cong S_4$ would contradict the fact that $N_G(E_1) \nsubseteq H$. Therefore we obtain $\mathscr{S} \cong PGL(2, 5) \cong S_5$. The proof is finished.

(2.6) Lemma. *If uv is conjugate to t in G, then tuv is conjugate to t_1 in G. Moreover we have $N_G(E_2)/E_2 \cong S_5$, the symmetric group in 5 letters.*

Proof. As in (2.5).

(2.7) Lemma. *If u is conjugate to t in G, the group $Y_1 = N_G(E_1) \cap C_G(tu)$ has the following structure. $Y_1 = \langle E_1, v, z\rangle \cdot \langle \sigma\rangle$ such that $z^2 = 1$; $ztz = u$; $zt_1z = t_1$; $zt_2z = t_2$; $z\sigma z = \sigma$; and $zvz = v$ or tuv.*

Proof. By (2.5), we see there exists an element $\mu \in N_G(E_1)$ of order 5 acting fixed-point-free on E_1 and so it follows that t_1 is conjugate to tu in $N_G(E_1)$. Now $A \subseteqq N_G(E_1) \cap C_G(t_1)$ and A is a Sylow 2-subgroup of $C_G(t_1)$, for otherwise, we would have t_1 in the centre of a group of order 2^7, a contradiction to (2.1). We get that 2^6 divides $|N_G(E_1) \cap C_G(tu)|$. We know that $\sigma \in N_G(E_1) \cap C_G(tu) = Y_1$ and $\mu \notin Y_1$. Hence $|Y_1| = 2^6 \cdot 3$ and therefore $Y_1 = \tilde{A} \cdot \langle \sigma\rangle$ with $\tilde{A} \cong A$.

We have the group $C_G(tu) \cap H = U$ a subgroup of index 2 in Y_1. The group $\langle t_1, t_2\rangle\langle \sigma\rangle$ is the smallest normal subgroup of $C_G(tu) \cap H$ with 2-factor group. Hence $\langle t_1, t_2\rangle\langle \sigma\rangle$ char $C_H(tu)$ and it follows that it is normal in Y_1. Let T be a Sylow 2-subgroup of Y_1 containing $\underline{U} = \langle E_1, v\rangle$ and let $z \in T - \underline{U}$. We know from the isomorphism of T and A, that $Z(T)$ is a four-group. Obviously $tu \in Z(T)$. Since $\langle t_1, t_2\rangle$ char $\langle t_1, t_2\rangle\langle \sigma\rangle$ and so

$\langle t_1, t_2\rangle \lhd T$. Hence $\langle t_1, t_2\rangle$ has non-trivial intersection with $Z(T)$. So $1 \neq \langle t_1, t_2\rangle \cap Z(T) \subseteqq \langle t_1, t_2\rangle \cap Z(\underline{U}) = \langle t_1\rangle$. Thus $Z(T) = \langle tu, t_1\rangle$.

From the fact $\langle t_1, t_2\rangle\langle\sigma\rangle$ is normal in Y_1, it follows that

$$z^{-1}\sigma z \in \langle t_1, t_2\rangle\langle\sigma\rangle.$$

Replacing z by zv if necessary, we can suppose that $z^{-1}\sigma z = \sigma \cdot x$, where $x \in \langle t_1, t_2\rangle$. Again replacing z by zt_1, zt_2 or zt_1t_2 if necessary, we get $z^{-1}\sigma z = \sigma$. We have $\langle t_1, t_2\rangle \lhd Y_1$ and so it follows $z^{-1}t_2z = t_2$ or t_1t_2. Comparing the action of $z^{-1}t_2z$ on σ by conjugation with those of t_2, t_1t_2, we conclude that $z^{-1}t_2z = t_2$.

Next we want to determine the action of z on $\langle t, u\rangle$. We have $Z(\underline{U}) = \langle t, t_1, u\rangle$ char $\underline{U}$ and therefore $\langle t, t_1, u\rangle \lhd T$. In $\langle t, t_1, u\rangle$ by (2.5), the only elements conjugate to t in Y_1 are tt_1u and u. It follows that $z^{-1}tz = u$; $z^{-1}uz = t$ $(z^2 \in H)$. Because $\langle E_1, v\rangle \lhd T$, we get $z^{-1}vz = vs$ for some $s \in E_1$. From the fact $(z^{-1}vz)\sigma(z^{-1}vz) = \sigma^{-1}$, we see that

$$s \in E_1 \cap C_G(\sigma) = \langle t, u\rangle.$$

If $z^{-1}vz = tv$, then $(z^2)^{-1}vz^2 = tuv$, a contradiction since v and tuv are not conjugate in H. Similarly, $z^{-1}vz = uv$ is impossible. Thus $z^{-1}vz = v$ or tuv.

From the structure of A, we know that z has order at most 4 and all elements of order 4 have their squares lying in $Z(A)$. So we have $z^2 \in Z(T)$ and from the fact $z \in C_G(\sigma)$, we obtain either $z^2 = 1$ or $z^2 = tu$, in which case replacing z by zu, we have $(zu)^2 = 1$. Hence all the statements of the lemma are completely proved.

We note also that each successive replacing of z does not affect the earlier conclusions. The proof of this lemma is finished.

(2.8) Lemma. *If uv is conjugate to t in G, then we have*

$$Y_2 = C_G(uv) \cap N_G(E_2) = \langle E_2, v, z'\rangle\langle\rho\rangle \qquad (\rho = \sigma_1^{-1}\sigma_2)$$

such that $(z')^2 = 1$; $z'tz' = uv$; $z'tt_1z' = tt_1$, $z'tt_3z' = tt_3$; $z'vz' = v$ *or* tu; $z'\rho z' = \rho$.

Proof. As in (2.7).

(2.9) Lemma. *The group G is not 2-normal.*

Proof. Suppose by way of contradiction, that G is 2-normal. Since $\langle t\rangle$ is the centre of a Sylow 2-subgroup Q of G. It follows from Hall-Grün's theorem [4, p. 216], the greatest 2-factor group of G is isomorphic to that of $N_G(Z(Q)) = H$ i.e. isomorphic to H/L which is a four-group. But this contradicts condition (a). Hence G is not 2-normal.

(2.10) Lemma. *The involution t is conjugate to an involution in $\{u, v, uv\}$.*

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PROOF. By (2.9), G is not 2-normal, hence there exists an element x in G such that $t \in Q \cap x^{-1}Qx$, but $\langle t \rangle$ is not the centre of $x^{-1}Qx$. The centre of $x^{-1}Qx$ is $\langle x^{-1}tx \rangle$ and thus $x^{-1}tx \neq t$. On the other hand, $t \in x^{-1}Qx$ and so t and $x^{-1}tx$ commute. Therefore $x^{-1}tx \in H$. Without loss of generality, we may assume that $x^{-1}tx \in \{u, tu, v, uv, tuv\}$ (since $x^{-1}tx \neq t_1$ by (2.1)). Interchanging u by tu; v by tv, if necessary, we may and shall suppose $x^{-1}tx$ is an element in $\{u, v, uv\}$.

To prove the next lemma, the following unpublished result of Thompson is indispensable.

LEMMA A (Thompson) [7]. *Suppose $\mathfrak{G}$ is a finite group of even order which has no subgroup of index 2. Let $\mathscr{S}_2$ be a Sylow 2-subgroup of $\mathfrak{G}$ and let $\mathscr{M}$ be a maximal subgroup of $\mathscr{S}_2$. Then for each involution I of $\mathfrak{G}$, there is an element B of $\mathfrak{G}$ such that $B^{-1}IB \in \mathscr{M}$.*

(2.11) LEMMA. *The group G has precisely two conjugate classes of involutions $\mathscr{K}_1$ and $\mathscr{K}_2$ with the representatives t and tu respectively: $\mathscr{K}_1 \cap H$ is the union of 4 conjugate classes of involutions of H with representatives t, u, v, uv; $\mathscr{K}_2 \cap H$ is the union of 3 conjugate classes of H with representatives t_1, tu, tuv.*

PROOF. By (2.10), there exists an element x in G, such that

$$x^{-1}tx \in \{u, uv, v\}.$$

Suppose that $x^{-1}tx = u$. We have $M = \langle a_1, a_2, b_1, b_2, u \rangle$ is a maximal subgroup of Q, a Sylow 2-subgroup of G. By (2.1); (2.4), the involutions of M lie in two conjugate classes in G with representatives t and tu. By lemma A, we see that involutions uv, uvt and v are conjugate to some involutions in M. By (2.4), uv, tuv lie in different conjugate classes of G. Hence, interchanging v by vt if necessary, we may suppose uv is conjugate to t in G and so tuv is conjugate to tu in G. To decide whether v is conjugate to t or tu, we use (2.7) and (2.8) and get the following possibilities.

(i) $zvz = v$ and $z'vz' = tu$. Then we have $ztvz = uv$, a contradiction, since tu and uv lie in two different conjugate classes of G.

(ii) $zvz = tuv$ and $z'vz = v$. Then we have $z'vtz' = u$, a contradiction as in (i).

(iii) $zvz = tuv$, and $z'vz' = tu$. Then by (1.8)

$$|C_G(v) \cap C_G(t)| = |C_G(tuv) \cap C(u)| = 2^4,$$

but when $z' \in C_G(u)$ and therefore $\langle z', t, u, v, t_1 \rangle \in C_G(u) \cap C_G(tuv)$, a contradiction.

Thus we are in the last case (iv) where $zvz = v$, and $z'vz' = v$. Then $zz'tz'z = tv$ proving all the statements of this lemma.

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Suppose $x^{-1}tx = uv$. We take as a maximal subgroup of Q, the group $\langle a_1, a_2, b_1, b_2, uv\rangle$ and apply the same proof as in previous cases.

Finally if $x^{-1}tx = v$. We have the group $\langle a_1, a_2, b_1, b_2, v\rangle$ is a maximal subgroup of Q. By lemma A again, interchanging u by tu and/or v by tv if necessary, we get the same conclusions.

Since by (2.10), one of these cases must happen, we have proved our lemma.

3. The centralizer of an involution in $\mathscr{K}_2$

We begin with a preliminary result. The notation in this proof is independent of the rest of the paper.

PROPOSITION 1. *Let G be a finite group of even order with the following properties*:

(1) *The centralizer $C(\alpha)$ in G of an involution α contained in the centre of a Sylow 2-subgroup of G is $\langle\alpha, \beta\rangle \times F$ where $\langle\alpha, \beta\rangle$ is a four group and F is isomorphic to S_4 (the symmetric group in 4 letters).*

(2) *If S is a Sylow 2-subgroup of G then $C(S') = S$ where S' denotes the commutator group of S.*

(3) *The involutions α, β, $\alpha\beta$ are not conjugate to each other in G.*

Then either $G = C(\alpha)$ or G is isomorphic to the direct product of a group of order 2 and D where $D \cong S_6$.

PROOF. Put $F = V \cdot \langle\rho\rangle \cdot \langle\tau\rangle$ where $V = \langle\tau_1, \tau_2\rangle$ is a four-group. We have $\rho^{-1}\tau_1\rho = \tau_2$; $\rho^{-1}\tau_2\rho = \tau_1\tau_2$, $\tau\tau_1\tau = \tau_1$; $\tau\tau_2\tau = \tau_1\tau_2$; $\tau\rho\tau = \rho^{-1}$ and $\tau^2 = \rho^3 = 1$. Obviously $S = \langle\alpha, \beta\rangle \times (V\langle\tau\rangle)$ is a Sylow 2-subgroup of G. $V\langle\tau\rangle$ is dihedral of order 8 and we have $S' = \langle\tau_1\rangle$. Hence by (2), $C(\tau_1) = S$. Finally $Z(S) = \langle\alpha, \beta, \tau_1\rangle$ is elementary of order 8.

(i) *Non-trivial elements of $Z(S)$ lie in 7 distinct conjugate classes of G.*

By way of contradiction, suppose there are 2 involutions in $Z(S)$ conjugate to each other in G. Then by a transfer theorem of Burnside [4], they are conjugate in $N(Z(S))$. We must have $N(Z(S)) > S$. Since $C(Z(S)) \subseteqq C(\tau_1) = S$, we get $N(Z(S))/S$ is isomorphic to a subgroup of $GL(3, 2)$. Clearly $7 \nmid |N(Z(S))|$, otherwise there exists an element of order 7 in $N(Z(S))$ which acts fixed-point-free on $Z(S)$. This requires, in particular, that α, β, $\alpha\beta$ lie in one conjugate class of G, contradicting condition (3). Therefore the order of $N(Z(S))$ is $2^5 \cdot 3$. Let $\lambda \in N(Z(S))$ and $O(\lambda) = 3$. We want to determine the orbits of λ on $Z(S)$. By condition (3), the elements α, β, $\alpha\beta$ lie in 3 distinct orbits, a contradiction to the fact $|Z(S)| = 8$.

(ii) *The focal group S^* of S in G contains V.*

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This is obvious since $\rho^{-1}\tau_1\rho = \tau_2$ and $\rho^{-1}\tau_2\rho = \tau_1\tau_2$.

(iii) *The case $S^* = S$ is not possible.*

By way of contradiction, suppose that $S = S^*$. This means that G has no subgroup of index 2. Consider the group $\langle\beta\rangle\times(V\langle\tau\rangle)$. It is maximal subgroup of S and has at most 5 conjugate classes of involutions with representatives τ_1, τ, β, $\beta\tau_1$ and $\beta\tau$. By lemma A, we get that G has at most 5 conjugate classes of involutions. This is a contradiction since by (i), we know that G has at least 7 classes of involutions.

(iv) *The case $|S^*| = 16$ is not possible.*

Suppose on the contrary, we have the order of S^*, the focal group of S in G, is 16. This means that G has a subgroup of index 2 but has no subgroup of index 4. Let M be a subgroup of G of index 2. By D. G. Higman [5], we have $S \cap M = S^*$ and S^* is a Sylow 2-subgroup of M. We have two cases to consider. If $\langle\alpha, \beta\rangle \subseteqq S^*$, then by (ii), we have $S^* = \langle\alpha, \beta\rangle\times V$. Then $\langle\alpha\rangle\times V$ is a maximal subgroup of S^* and has at most 3 classes of involutions with representative τ_1, α, $\alpha\tau_1$ (we use the fact $\rho \in M$). So by lemma A, M has at most 3 classes of involutions in contradiction to (i) since $Z(S) \subseteqq M$. Next suppose that $S^* \cap \langle\alpha, \beta\rangle$ is of order 2. We have $S^* = V(\langle\alpha, \beta, \tau\rangle \cap S^*)$. There exists an element $\tau' \in S^* \cap \langle\alpha, \beta, \tau\rangle$ such that $V\langle\tau'\rangle$ is a dihedral and has at most 2 conjugate classes in M. Also $V\langle\tau'\rangle$ is a maximal subgroup of S^* and hence by Thompson, we obtain that M has at most 2 conjugate classes of involution, a contradiction to the fact that $Z(S) \cap S^*$ is of order 4 and by (ii) its involutions lie in 3 distinct conjugate classes of G.

(v) *If $V = S^*$, then we have $G = C(\alpha) = \langle\alpha, \beta\rangle\times F$.*

We have in this case a normal subgroup M of index 8 in G such that $M \cap S = V$. Because $\rho \in M$ and $V\langle\rho\rangle \cong A_4$, all involutions of V are conjugate in M and a Sylow 2-subgroup of M is a four group. Also we have $C_M(\tau_1) = S \cap M = V$. By a result of Suzuki [9,] we have either $V \lhd M$ or $M \cong A_5$. If $V \lhd M$, then $M = V\langle\rho\rangle$ (since $C_M(V) = V$). Therefore $G = S \cdot M = C(\alpha) = \langle\alpha, \beta\rangle\times F$. If $M \cong A_5$, because the automorphism group of A_5 is S_5, it follows that $C(M) \neq 1$. Clearly $C(M) \cap M = 1$. From the fact $\tau \notin C(M)$, we obtain that $|C(M)| = 4$. Now

$$C(M) \subseteqq C(V) = \langle\alpha, \beta\rangle\times V,$$

so

$$C(M) \subseteqq \langle\alpha, \beta\rangle\times V - V \qquad (\because C(M) \cap M = 1).$$

Let $C(M) = \langle z_1, z_2\rangle$, a four group. It follows that $z_1 = \alpha v_1$; $z_2 = \beta v_2$ where $v_1, v_2 \in V$. Since α, z_1, β, z_2 centralize ρ; we get v_1, v_2 commute with ρ.

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By the structure of A_4, $v_1 = v_2 = 1$. Thus we get $C(M) = \langle \alpha, \beta \rangle$ and therefore contradicts condition (1).

(vi) *If the order S^* is 8, then G is a product of a group of order 2 with a subgroup of G isomorphic to S_6.*

Since $|S^*| = 8$, it means that G has a normal subgroup M of index 4 in G and G has no subgroup of index 8 (Here we use the fact $V \subsetneqq S^*$ and S/V is abelian). We have $S \cap M = S^*$ and V is a maximal subgroup of S^*. Since $\rho \in M$, involutions in V are conjugate in G and so by lemma A, M has only one class of involutions. By (i), we must have $S^* \cap \langle \alpha, \beta \rangle = 1$. Therefore $S^* \subsetneqq \langle \alpha, \beta \rangle \times (V\langle \tau \rangle) - \langle \alpha, \beta \rangle$ and so is dihedral of order 8. Now τ_1 is in the centre of S^* and we have $C_M(\tau_1) = S \cap M = S^*$.

Let $0(M)$ be the largest normal odd order subgroup of M. V acts on $C(M)$ and since all involutions of V are conjugate in M, we get

$$|C_{0(M)}(\tau_1)| = |C_{0(M)}(\tau_2)| = |C_{0(M)}(\tau_1\tau_2)| = 1$$

because $C(\tau_1)$ is a 2-group. By Brauer-Wielandt's result [10], $0(M) = 1$.

Application of Gorenstein-Walter's theorem [3], produces the result: $M \cong PSL(2, q)$ $q \pm 1 = |C_M(\tau_1)|$ or $M \cong A_7$. The second case cannot happen since the centralizer of an involution in A_7 is divisible by 3. Therefore $M \cong PSL(2, 7)$ or $PSL(2, 9)$. Since the automorphism group of $PSL(2, 7)$ is $PGL(2, 7)$, we get $C(M) \neq 1$. So we have either $G = \langle \alpha, \beta \rangle \times M$ or G contains a subgroup iscmorphic to $PGL(2, 7)$. The first possibility cannot arise since it contradicts condition (1). The second possibility is ruled out by the fact that a Sylow 2-subgroup of $PGL(2, 7)$ is dihedral of order 16 and so contains an element of order 8, in contradiction to the structure of S.

We are left with the case $M \cong PSL(2, 9) \cong A_6$. Since $PGL(2, 9)$ contains elements of order 8, we conclude that G does not contain a subgroup isomorphic to $PGL(2, 9)$. Also $C(M)$ cannot have order 4, because by similar argument as in (v), G would be equal to $\langle \alpha, \beta \rangle \times M$, a contradiction to condition (1). It follows that $C(M)$ is of order 2 and $C(M) \cap M = 1$ and $\alpha \notin C(M)$. Let $C(M) = \langle z \rangle$. From the fact $C(M) \subsetneqq C(V) = \langle \alpha, \beta \rangle \times V$ and $C(M) \cap M = 1$, we get $C(M) \subsetneqq \langle \alpha, \beta \rangle V - V$. Hence $C(M) = \langle h \cdot v \rangle$ where $h \in \langle \alpha, \beta \rangle$, $v \in V$. Since hv and h commute with ρ, we get $v = 1$. Therefore $h = \beta$ or $\alpha\beta$.

Now the automorphism group $\mathscr{A}$ of $PSL(2, 9)$ has the property $\mathscr{A}/A_6$ is a four-group. $\mathscr{A}$ is an extension of $PGL(2, 9)$ by the field automorphism f of order 2. Now $PGL(2, 9)$ is the group of all non-singular 2×2 matrices (α_{ij}) with $\alpha_{ij} \in GF(9)$ considered modulo the group of all 2×2 scalar matrices and we have $f(\alpha_{ij})f = (\alpha_{ij}^3)$. Let ζ be a generator of the multiplicative group of $GF(9)$. Then $\zeta^4 = -1$. Put

$$a = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}; \quad b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad c = \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix} f.$$

We verify that $a^4 = 1 = b^2$, $b^{-1}ab = a^{-1}$, $c^{-1}ac = a^{-1}$; $c^{-1}bc = a^{-1}b$; $c^2 = a^2$. Since $\langle a, b\rangle$ is a Sylow 2-subgroup of $PSL(2, 9)$, it follows $\langle a, b, c\rangle$ is a Sylow 2-subgroup of $\langle PSL(2, 9), c\rangle$. We shall produce an element of $\langle PSL(2, 9), c\rangle$ which is of order 8. We note that

$$\begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix} \in PGL(2, 9) - PSL(2, 9),$$

and $(cb)^2 = c^2c^{-1}bcb = a^2a^{-1}bb = a$, so is of order 8.

Now $\langle\alpha\rangle M$ is isomorphic to a subgroup of index 2 of $\mathscr{A}$ containing $PSL(2, 9)$. There are 3 such subgroups namely $PGL(2, 9)$; $\langle PSL(2, 9), c\rangle$ and $\langle PSL(2, 9), f\rangle$. We have shown that the first two cases cannot arise, so we have $\langle\alpha\rangle M \cong \langle PSL(2, 9), f\rangle$. It is well known that $PSL(2, 9)$ is isomorphic to A_6. Hence S_6 is isomorphic to a subgroup of index 2 in $\mathscr{A}$. We check that S_6 has no element of order 8. It follows

$$S_6 \cong \langle PSL(2, 9), f\rangle \cong \langle\alpha\rangle M.$$

Therefore $G = C(M) \times (\langle\alpha\rangle M)$.

The proof of this lemma is now complete.

We can now begin the determination of the structure of $C_G(tu)$. Consider the factor group $C_G(tu)/\langle tu\rangle = \bar{C}$. We note that $\langle t, u\rangle/\langle tu\rangle$ is in the centre of a Sylow 2-subgroup $\langle z, u, v, t, t_1, t_2\rangle/\langle tu\rangle$ of $\bar{C}$. Since tu is conjugate neither to t nor to u, we have $N_G\langle t, u\rangle \subseteq C_G(tu)$. Hence we obtain $N_G\langle t, u\rangle = \langle z, u, v, t, t_1, t_2\rangle\langle\sigma\rangle = Y_1$. Hence we get the centralizer of $\langle t, u\rangle/\langle tu\rangle$ in $\bar{C}$ is

$$(\langle z, t, u\rangle/\langle tu\rangle) \times (\langle v, t_1, t_2, \sigma\rangle\langle tu\rangle/\langle tu\rangle)$$

where $\langle z, t, u\rangle/\langle tu\rangle$ is a four-group and $\langle v, t_1, t_2, \sigma\rangle\langle tu\rangle/\langle tu\rangle$ is isomorphic to S_4 and so the group $\bar{C}$ satisfies condition (1) of the proposition.

To check that condition (2) of the proposition is alsc fulfilled by the group $\bar{C}$, we look at $C_G(t_1)/\langle t_1\rangle$. Now $\langle a_1, a_2, t_2, u, v\rangle/\langle t_1\rangle$ is a Sylow 2-subgroup of $C_G(t_1)/\langle t_1\rangle$ and $\langle t, t_1\rangle/\langle t_1\rangle$ is its commutator group. The group $N_G\langle t, t_1\rangle$ is contained in H, since t is not conjugate to t_1 or tt_1. It follows that $N_G\langle t, t_1\rangle \cap C_G(t_1) = A$. Since t_1 is conjugate to tu in G, it follows that the centralizer of the commutator group $\langle t_1, tu\rangle/\langle tu\rangle$ of $\langle z, u, v, t, t_1, t_2\rangle/\langle tu\rangle$ in $\bar{C}$ is $\langle z, u, v, t, t_1, t_2\rangle/\langle tu\rangle$.

Next we want to show that $\langle z, tu\rangle/\langle tu\rangle$; $\langle t, u\rangle/\langle tu\rangle$ and $\langle zt\rangle/\langle tu\rangle$ are not conjugate to each other in $\bar{C}$. It is clear that $\langle zt\rangle/\langle tu\rangle$ is not conjugate to $\langle z, tu\rangle/\langle tu\rangle$ or $\langle t, u\rangle/\langle tu\rangle$ since $\langle zt\rangle$ is cyclic whereas $\langle z, tu\rangle$ and $\langle t, u\rangle$ are four groups. Both $\langle t, u\rangle$ and $\langle z, tu\rangle$ are normal in $\langle z, u, v, t, t_1, t_2\rangle$.

If $\langle t, u\rangle/\langle tu\rangle$ were conjugate in $\bar{C}$ to $\langle z, tu\rangle/\langle tu\rangle$, by a transfer theorem of Burnside [4], they would be conjugate in

$$N_{\bar{C}}(\langle z, u, v, t, t_1, t_2\rangle/\langle tu\rangle) \subseteqq N_{\bar{C}}(\langle tu, t_1\rangle/\langle tu\rangle) = \langle z, u, v, t, t_1, t_2\rangle/\langle tu\rangle,$$

a contradiction.

Applying Proposition 1 on the group $\bar{C}$, we get either $C_G(tu) = Y_1$ or $\bar{C}$ is the direct product of a group of order 2 by a subgroup which is isomorphic to S_6. The case $C_G(tu) = Y_1$ is not possible since we have by (2.8) and (2.11), an element

$$z' \in C_G(tu) - Y_1.$$

We shall now take a close look at the remaining case. Let $\tilde{D}$ be the complete inverse image in $C_G(tu)$ of the subgroup of $\bar{C}$ which is isomorphic to S_6. From the proof of Proposition 1, we see that $\langle t, u, t_1, t_2\rangle \subseteqq \tilde{D}$. Since $\langle z, t, u\rangle \nsubseteq \tilde{D}$, we have either $\langle t, u, t_1, t_2, zv\rangle$ or $\langle t, u, t_1, t_2, v\rangle$ is a Sylow 2-subgroup of $\tilde{D}$. Suppose that $\langle t, u, t_1, t_2, zv\rangle \subseteqq \tilde{D}$. Let $\tilde{\boldsymbol{D}}$ be the subgroup of $\tilde{D}$ such that $\tilde{\boldsymbol{D}}/\langle tu\rangle \cong A_6$. We know that $t_1 \in \tilde{\boldsymbol{D}}$ and

$$\langle t, u\rangle/\langle tu\rangle \in \tilde{D}/\langle tu\rangle - \tilde{\boldsymbol{D}}/\langle tu\rangle.$$

Hence zvt^r ($r = 0$, or 1) is conjugate to t_1 modulo $\langle tu\rangle$. Hence there exists an element $g \in \tilde{\boldsymbol{D}}$ such that $g^{-1}zvt^r g = t_1 \cdot h$ where $h \in \langle tu\rangle$ and so $g^{-1}zvt^r \cdot tug = t_1 \cdot h \cdot tu$. From (2.7), zvt^r is conjugate to $zvt^r \cdot tu$. It follows then t_1 is conjugate to tt_1u, a contradiction to (2.11). Therefore

$$\langle t, u, t_1, t_2, v\rangle \subseteqq \tilde{D}.$$

We check that $\langle t, u, t_1, t_2, v\rangle$ splits over $\langle tu\rangle$. So by a theorem of Gaschütz, [4, p. 246], $\tilde{D}$ splits over $\langle tu\rangle$. Hence there is a subgroup D of $\tilde{D}$ isomorphic to S_6 such that $\tilde{D} = \langle tu\rangle \times D$, and we may suppose that $t \in D$. Let $\underset{\sim}{D}$ be the subgroup of D such that $\underset{\sim}{D} \cong A_6$. By the structure of A_6 all involutions in A_6 are conjugate in A_6. In $\langle t, u, t_1, t_2, v\rangle$, we observe that elements of order 4 have their squares equal to t_1. Therefore we conclude that all involutions in $\underset{\sim}{D}$ lie in $\mathscr{K}_2$ (in the notation of (2.11)). These facts imply that a Sylow 2-subgroup of $\underset{\sim}{D}$ is $\langle tt_2uv, tuv\rangle$.

We have by Proposition 1 that

$$C_G(tu) = (\langle zt\rangle \times \underset{\sim}{D})\langle t\rangle \quad \text{or} \quad (\langle z, tu\rangle \times \underset{\sim}{D})\langle t\rangle$$

where in both cases, we have $\underset{\sim}{D}\langle t\rangle = D \cong S_6$. Suppose that

$$C_G(tu) = (\langle z, tu\rangle \times \underset{\sim}{D})\langle t\rangle.$$

Clearly $z \in \mathscr{K}_2$ and $\langle z, tu\rangle \times \underset{\sim}{D}$ is a subgroup of index 2 in $C_G(z)$. We want to determine $C_G(z) \cap C_G(v)$. Suppose there is an element

$$g \in C_G(z) - (\langle z, tu\rangle \times \underset{\sim}{D})$$

and g centralizes v. Now

$$\langle z, tu\rangle = Z(\langle z, tu\rangle \times \underset{\sim}{D})$$

and therefore $\langle z, tu\rangle \lhd C_G(z)$. Thus $g^{-1}tug = ztu \quad (g \notin C_G(tu))$. So $g^{-1}tuvg = ztuv$. But we have $\underset{\sim}{D} = (\langle z, tu\rangle \times \underset{\sim}{D})'$ char $C_G(z)$. Therefore $g^{-1}\underset{\sim}{D}g = \underset{\sim}{D}$ giving $g^{-1}tuvg \subseteqq \underset{\sim}{D}$ a contradiction. Hence we have shown that

$$C_G(z) \cap C_G(v) \subseteqq \langle z, tu\rangle \times \underset{\sim}{D}.$$

Using the fact $tuv \in \underset{\sim}{D}$ and centralizer of an involution in A_6 has order 8, we conclude that $C_G(z) \cap C_G(v)$ has order 32, in contradiction to the fact that $C_G(x) \cap C(t)$ with $x \in \mathscr{K}_2 \cap C_G(t)$ has order 2^6 or $32 \cdot 3$. Thus we have finally proved that $C_G(tu) = (\langle zt\rangle \times \underset{\sim}{D})\langle t\rangle$.

(3.1) Lemma. *The centralizer $C_G(tu)$ of tu in G has the following structure*:

$$C_G(tu) = (\langle zt\rangle \times \underset{\sim}{D})\langle t\rangle \quad \textit{where} \quad \langle \underset{\sim}{D}\rangle\langle t\rangle = D \cong S_6.$$

(3.2) Lemma. *The group G is simple.*

Proof. Suppose that $0(G) \neq 1$. Act on $0(G)$ by the four group $\langle v, t\rangle$. We know that $C_G(x)$ has no odd-order normal subgroup by the structure of H, for all $x \in \mathscr{K}_1$, $\langle t, v\rangle$ acts fixed-point-free on $0(G)$, a contradiction to a theorem of Burnside. We have therefore proved that G has no non-trivial odd order normal subgroup.

Suppose that G has a proper normal subgroup N with odd factor-group G/N. Then Q being a Sylow 2-subgroup of G is contained in N. The Frattini argument gives $G = N \cdot N_G(Q)$. But $N_G(Q) = Q$ and hence $G = N \cdot Q = N$, a contradiction. Thus G has no proper normal subgroup with odd factor group.

Next suppose that G has a proper non-trivial normal subgroup M such that $|M|$ and $|G : M|$ are both even. Suppose that $\mathscr{K}_1 \cap M$ is not empty. Then $\mathscr{K}_1 \subseteqq M$ and in particular t and u are in M. Hence $tu \subseteqq M$. So $\mathscr{K}_2 \cap M \neq \phi$ giving $\mathscr{K}_2 \subseteqq M$. Thus all involutions of G are contained in M. It follows that Q, being generated by its involutions is in M, a contradiction. This gives $\mathscr{K}_1 \cap M = \phi$. Therefore $\mathscr{K}_2 \cap M \neq \phi$ and so $tt_1, t_1 \in M$. This implies that $t \in M$, a contradiction. Hence the proof is now complete.

4. Structures of a Sylow 3-subgroup of G and its normalizer in G

In § 3, we have $C_G(tu) = (\langle zt\rangle \times \underset{\sim}{D})\langle t\rangle$. A Sylow 3-subgroup of $\underset{\sim}{D}$ is elementary abelian of order 9, and is self-centralizing in $\underset{\sim}{D}$. Therefore Sylow 3-subgroups of $\underset{\sim}{D}$ are independent (i.e. two distinct Sylow 3-subgroups of $\underset{\sim}{D}$ intersect in the identity only). Let T_1 be the unique Sylow 3-subgroup of $\underset{\sim}{D}$

containing $\langle\sigma\rangle \subseteqq C_G(tu)$. Therefore $T_1 = C_G(\sigma) \cap C_G(tu)$. Since $\langle t, uv\rangle$ normalizes $\langle\sigma\rangle$, it normalizes $C_G(\sigma) \cap C_G(tu) = T_1$. Thus we have

$$\langle zt\rangle \times \langle t, uv\rangle \subseteqq N_G(T_1) \cap C_G(tu).$$

From the structure of S_6, we know that the normalizer in S_6 of a Sylow 3-subgroup of S_6 is a splitting extension of the Sylow 3-subgroup by a dihedral group of order 8 (e.g. $\langle(123), (456)\rangle$ is a Sylow 3-subgroup of S_6 and

$$N_{S_6}(\langle(123), (456)\rangle) = \langle(1524)(36), (12)\rangle \cdot \langle(123), (456)\rangle.$$

Therefore we get

$$N_G(T_1) \cap C(tu) = (\langle zt\rangle \cdot \langle a, t\rangle)T_1$$

where $a^2 = tuv$, $tat = a^{-1}$, $a^{-1}zta = zt$. Clearly $C_G(T_1) \cap C_G(tu) = \langle zt\rangle \times T_1$ and $C_G(T_1) \lhd N_G(T_1)$. Let $U \supseteqq \langle zt\rangle$ be a Sylow 2-subgroup of $C_G(T_1)$. If $U \supset \langle zt\rangle$, then $|C_G(T_1) \cap C_G(tu)|$ would be divisible by 8, which contradicts the structure of $C_G(tu)$. It follows a Sylow 2-subgroup of $C_G(T_1)$ is cyclic of order 4. By a result of Burnside, $C_G(T_1)$ has a normal 2-complement $M_1 \supseteqq T_1$. The Frattini argument gives

$$\begin{aligned} N_G(T_1) &= (N_G(zt) \cap N_G(T_1))C_G(T_1) \subseteqq (C_G(tu) \cap N_G(T_1))C_G(T_1) \\ &= \langle zt\rangle \cdot \langle a, t\rangle M_1. \end{aligned}$$

Thus

$$N_G(T_1) = (\langle zt\rangle \cdot \langle a, t\rangle)M_1.$$

Since M_1 char $C_G(T_1)$ we get $M_1 \lhd N_G(T_1)$, and so $\langle v, t\rangle \subseteqq N_G(T_1)$ acts on M_1. Because $\{v, vt, t\} \subseteqq \mathscr{K}_1$, by a result of Brauer-Wielandt [10], M_1 is a 3-group.

Now $\langle t, u\rangle$ also acts on M_1. We have $C_{M_1}(tu) = T_1$; $C_{M_1}\langle t, u\rangle = \langle\sigma\rangle$. Hence $|M_1| = |C_{M_1}(t)||C_{M_1}(u)|$. Because t and u are conjugate in $N_G(T_1)$, we get $|C_{M_1}(t)| = |C_{M_1}(u)|$. Hence $|M_1| = 3^2$ or 3^4. We shall show that $|M_1| = 3^2$ is not possible.

Let $T = \langle\sigma_1, \sigma_2\rangle \subseteqq H = C_G(t)$. Then

$$C_G(t) \cap \langle\sigma_1, \sigma_2\rangle = \langle t\rangle \times T \quad \text{and} \quad N_G(T) \cap H = \langle t, u, v\rangle \cdot T.$$

Now $\langle t\rangle$ is a Sylow 2-subgroup of $C_G(T)$ and therefore by a result of Burnside, $C_G(T)$ has a normal 2-complement M and $C_G(T) = \langle t\rangle \cdot M$. We have $C_G(T) \lhd N_G(T)$ and so by the Frattini argument,

$$N_G(T) = (C_G(t) \cap N_G(T)) \cdot C_G(T) = \langle t, u, v\rangle M.$$

Since M char $C_G(T)$, we have $M \lhd N_G(T)$. Therefore $\langle v, t\rangle$ acts on M and hence M is a 3-group. By way of contradiction, suppose $|M_1| = 3^2$, then T_1 is a Sylow 3-subgroup of G and so is T. But $C_G(T)$ has a different structure from that of $C_G(T_1)$, a contradiction to Sylow's theorem. Hence we must have $|M_1| = 81$.

We want to show that M_1 is abelian. We have

$$N_G(T_1) = (\langle zt\rangle \cdot \langle a, t\rangle)M_1.$$

By the structure of S_6, there exists an element $\lambda \in T_1$, inverted by t and a^2. Therefore $\lambda \in C_G(uv) \cap C_G(tu)$. Consider the action of the four-group $\langle uv, vt\rangle$ on M_1. We have $C_{M_1}(\langle uv, vt\rangle) = \langle\lambda\rangle$. Therefore

$$|C_{M_1}(uv)| = |C_{M_1}(vt)| = 3^2.$$

Next consider the action of $\langle v, t\rangle$ on M_1. We have $C_{M_1}\langle t, v\rangle = 1$. Therefore

$$|M_1| = |C_{M_1}(t)|\,|C_{M_1}(vt)|\,|C_{M_1}(v)|$$

giving $C_{M_1}(v) = 1$. Thus the involution v acts fixed-point-free on M_1. By a result of Zassenhaus, M_1 is abelian. By a result of Gorenstein-Walter [3], $M_1 = C_G(t)C_G(vt)$. Showing that M_1 is elementary abelian of order 81.

We shall next take a closer look at M_1. Since

$$az \in N_G(T_1) \quad \text{and} \quad (az)^{-1}t(az) = vt,$$

we get

$$C_{M_1}(vt) = (az)^{-1}C_{M_1}(t)(az).$$

Because $\sigma \in C_{M_1}(t)$, and there is an unique subgroup of order 9 in $C_G(\sigma) \cap H$ namely $T = \langle\sigma_1, \sigma_2\rangle$, we get $C_{M_1}(t) = T$. Let $(az)^{-1}\sigma_1(az) = \zeta_1$, $(az)^{-1}\sigma_2(az) = \zeta_2$. Then $C_{M_1}(vt) = \langle\zeta_1, \zeta_2\rangle$. We also observe that $u\zeta_1 u = \zeta_2^{-1}$ using the relation $(az)u(az)^{-1} = uv$. Collecting the results proved so far, we have the following lemma.

(4.1) LEMMA. *Let T_1 be the Sylow 3-subgroup of $C_G(tu)$ containing $\langle\sigma\rangle$. Then we have $C_G(T_1) = \langle zt\rangle M_1$ and $N_G(T_1) = (\langle zt\rangle \cdot \langle a, t\rangle) \cdot M_1$ where*

$$z^2 = tuv; \quad tat = a^{-1}; \quad M_1 = C_{M_1}(t)C_{M_1}(vt);$$
$$C_{M_1}(t) = \langle\sigma_1, \sigma_2\rangle; \quad C_{M_1}(vt) = \langle\zeta_1, \zeta_2\rangle$$

with $(az)^{-1}\sigma_i(az) = \zeta_i$ $(i = 1, 2)$ and $u\zeta_1 u = \zeta_2^{-1}$.

Next we shall investigate the structure of $C_G(\sigma_1)$. We have

$$C_G(\sigma_1) \cap H = T \cdot Q_2$$

where $Q_2 = \langle a_2, b_2\rangle$, a quaternion group containing the unique involution t. Clearly Q_2 is a Sylow 2-subgroup of $C_G(\sigma_1)$. We shall use the following result of Brauer-Suzuki [9]. If X is a finite group with a generalized quaternion Sylow 2-subgroup, then $X/0(X)$ has only one involution. In our case, denote $0(C_G(\sigma_1)) = V$. Then $\langle\sigma_1\rangle \subseteqq V$ and $C_G(\sigma_1)/V$ has only one involution tV. It follows that $\langle t\rangle V$ is normal in $C_G(\sigma_1)$ and so (by Frattini's argument,

$$C_G(\sigma_1) = (C_G(t) \cap C_G(\sigma_1))V = Q_2 \cdot T \cdot V.$$

Because $Q_2T \cong SL(2, 3)$ is not 3-closed, it follows that $T \not\subseteq V$ and so $T \cap V = \langle \sigma_1 \rangle$. We get $C_G(\sigma_1) = \langle Q_2, \sigma_2 \rangle V = S_2 \cdot V$ where $S_2 \cong SL(2, 3)$ and $S_2 \cap V = 1$. Since $C_G(t) \cap V = \langle \sigma_1 \rangle$, it follows that t acts fixed-point-free on $V/\langle \sigma_1 \rangle$ and so $V/\langle \sigma_1 \rangle$ is abelian $V' \subseteq \langle \sigma_1 \rangle \subseteq Z(V)$.

Now v inverts σ_1. Therefore $N_G\langle \sigma_1 \rangle = \langle v \rangle S_2 V$. Since V is characteristic in $C_G(\sigma_1)$, we have $V \lhd N_G\langle \sigma_1 \rangle$. Thus the four group $\langle v, t \rangle$ acts on V and so V is a 3-group. Using Brauer-Wielandt's result, we get

$$|V| = |C_V(t)|\, |C_V(v)| \cdot |C_V(vt)|.$$

We know that $C_V(t) = \langle \sigma_1 \rangle$ and from the fact $M_1 \subseteq C_G(\sigma_1)$, we get that $|C_V(vt)| = 9$. Now v is conjugate to vt in $C_G(\sigma_1)$ i.e. $v = a_2^{-1} vta_2$, we get $|C_V(v)| = |C_V(vt)|$. Thus $|V| = 3^5$. By Gorenstein-Walter [3],

$$V = C_V(t)C_V(v)C_V(vt).$$

Put $C_V(v) = \langle \zeta_3, \zeta_4 \rangle$ where $\zeta_3 = a_2^{-1}\zeta_1 a_2$, $\zeta_4 = a_2^{-1}\zeta_2 a_2$. We have

$$V = \langle \sigma_1, \zeta_1, \zeta_2, \zeta_3, \zeta_4 \rangle.$$

Since $V/\langle \sigma_1 \rangle$ is abelian and so elementary abelian of order 81, we may represent $\langle v \rangle S_2$ on the 'vector space' $V/\langle \sigma_1 \rangle$ over $GF(3)$. We get in terms of the basis $\zeta_1\langle \sigma_1 \rangle$, $\zeta_2\langle \sigma_1 \rangle$, $\zeta_3\langle \sigma_1 \rangle$, $\zeta_4\langle \sigma_1 \rangle$;

$$a_2 \to \begin{pmatrix} & -I \\ I & \end{pmatrix}; \quad t \to \begin{pmatrix} -I & \\ & -I \end{pmatrix}; \quad v = \begin{pmatrix} -I & \\ & I \end{pmatrix}; \quad \sigma_2 \to \begin{pmatrix} I & C \\ O & D \end{pmatrix}$$

where (I) is the 2×2 unit matrix, and C, D are 2×2 matrices over $GF(3)$. Let b_2 be represented by

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

where (A_i) $i = 1, 2, 3, 4)$ is 2×2 matrix over $GF(3)$. Using the relation $b_2^{-1}a_2b_2 = a_2^{-1}$, we get $A_3 = A_2$, $A_4 = -A_1$. Since $\sigma_2^{-1}v\sigma_2 = \sigma_2 v$, we get $D = I$. By the relations $\sigma_2^{-1}a_2\sigma_2 = b_2$; $\sigma_2^{-1}b_2\sigma_2 = a_2b_2$, we obtain $A_2 = I$, $A_1 = I$, $C = -I$. Therefore we have

$$\sigma_2 \to \begin{pmatrix} I & -I \\ O & I \end{pmatrix}; \quad b_2 \to \begin{pmatrix} I & I \\ I & -I \end{pmatrix}.$$

Hence we have $\sigma_2^{-1}\zeta_3\sigma_2 = \zeta_1^{-1}\zeta_3\sigma_1^{\varepsilon_1}$; $\sigma_2^{-1}\zeta_4\sigma_2 = \zeta_2^{-1}\zeta_4\sigma_1^{\varepsilon_2}$ where $\varepsilon_i = 0, 1$ or -1 and $i = 1, 2$.

Since $V/\langle \sigma_1 \rangle$, is abelian, we have $\zeta_3^{-1}\zeta_2\zeta_3 = \zeta_2\sigma_1^{\varepsilon}$ where $\varepsilon = 0, 1$ or -1. Conjugating both sides of the equation $\zeta_3^{-1}\zeta_2\zeta_3 = \zeta_2\sigma_1^{\varepsilon}$ by the element a_2, we get $\zeta_4^{-1}\zeta_1\zeta_4 = \zeta_1\sigma_1^{\varepsilon}$. Consider the group $C_G(\langle \sigma_1, \zeta_1 \rangle) \subseteq C_G(\sigma_1)$. We have $C_G(\langle \sigma_1, \zeta_1 \rangle) \subseteq P = \langle \sigma_1, \sigma_2, \zeta_1, \zeta_2, \zeta_3, \zeta_4 \rangle$. Suppose that $C_G(\langle \sigma_1, \zeta_1 \rangle) = P$, then $\varepsilon = 0$, and so $C_G(\langle\zeta_1, \zeta_2\rangle)$ is divisible by 3^5, a contradiction to the structure of $C_G(T)$ since T is conjugate to $\langle\zeta_1, \zeta_2\rangle$ in G. So $\varepsilon \neq 0$.

We observe that $\langle\sigma_1, \zeta_1\rangle \lhd P$. Since $N_G(\langle\sigma_1, \zeta_1\rangle)/C_G(\langle\sigma_1, \zeta_1\rangle)$ is isomorphic to a subgroup of $GL(2, 3)$, we get that $C_G(\langle\sigma_1, \zeta_1\rangle)$ is of order 3^5. So $C_G(\langle\sigma_1, \zeta_1\rangle) = \langle\sigma_1, \sigma_2, \zeta_1, \zeta_2, x\rangle$ where $x \in \langle\zeta_3, \zeta_4\rangle$. Then we have the commutator group $(C_G(\langle\sigma_1, z_1\rangle))'$ of $C_G(\langle\sigma_1, \zeta_1\rangle)$ is

$$\langle\sigma_1, x^{-1}\sigma_2^{-1}x\sigma_2\rangle \neq \langle\sigma_1, \zeta_1\rangle \qquad \text{if} \quad x \neq \zeta_3.$$

From the structure of $C_G(\sigma_1)/\langle\sigma_1\rangle$ and the fact that $C_G(\langle\zeta_1, \zeta_2\rangle)$ is not divisible by 3^5, we get $Z(C_G(\langle\sigma_1, \zeta_1\rangle)) = \langle\sigma_1, \zeta_1\rangle$. Therefore we have

$$Z(C_G(\langle\sigma_1, \zeta_1\rangle)) \cap (C_G(\langle\sigma_1, \zeta_1\rangle))' = \langle\sigma_1\rangle \text{ char } C_G(\langle\sigma_1, \zeta_1\rangle)$$

and so

$$\langle\sigma_1\rangle \lhd N_G(C_G(\langle\sigma_1, \zeta_1\rangle)) = N_G(\langle\sigma_1, \zeta_1\rangle)$$

(since $\langle\sigma_1, \zeta_1\rangle = Z(C_G(\langle\sigma_1, \zeta_1\rangle))$). By (4.1), there is an element $taz \in C_G(tu)$, such that $(taz) \in N_G(\langle\sigma_1, \zeta_1\rangle)$ but $taz \in N_G\langle\sigma_1\rangle$, a contradiction to $\langle\sigma_1\rangle \lhd N_G(\langle\sigma_1, \zeta_1\rangle)$. Therefore we have shown that

$$C_G(\langle\sigma_1, \zeta_1\rangle) = V_1 = \langle\sigma_1, \sigma_2, \zeta_1, \zeta_2, \zeta_3\rangle.$$

Similarly we can prove that

$$C_G(\langle\sigma_1, \zeta_2\rangle) = V_3 = \langle\sigma_1, \sigma_2, \zeta_1, \zeta_2, \zeta_3\rangle$$

with *uaz* playing the role of *taz*.

Now we are in a position to determine ε_i $(i = 1, 2)$. By conjugating the equations $\sigma_2^{-1}\zeta_3\sigma_2 = \zeta_1^{-1}\zeta_3\sigma_1^{\varepsilon_1}$ and $\sigma_2^{-1}\zeta_4\sigma_2 = \zeta_2^{-1}\zeta_4\sigma_1^{\varepsilon_2}$ by the element vt, we verify that $\varepsilon_1 = \varepsilon_2 = 0$ using the fact $\zeta_3 \in C_G(\zeta_1)$ and $\zeta_4 \in C_G(\zeta_2)$. Except for the unknown $\varepsilon \neq 0$, we have determined the structure of P completely. In particular we see that $Z(P) = \langle\sigma_1\rangle$. The fact implies that $N_G(P) \subseteqq N_G\langle\sigma_1\rangle$ and therefore P is a Sylow 3-subgroup of G. By the structure of $N_G\langle\sigma_1\rangle$, we see that $N_G(P) = V \cdot (N_G(\sigma_2) \cap S_2\langle v\rangle) = \langle v, t\rangle \cdot P$. Collecting the results found so far, we have proved the following lemma.

(4.2) LEMMA. *A Sylow 3-subgroup P of G and its normalizer $B = N_G(P)$ in G have the following structures.*

$$P = T \cdot T_2 \cdot T_3; \quad B = N_G(P) = \langle v, t\rangle \cdot P,$$

where

$$\begin{aligned} T &= C_p(t) &&= \langle\sigma_1, \sigma_2\rangle \\ T_2 &= C_p(vt) &&= \langle\zeta_1, \zeta_2\rangle \\ T_3 &= C_p(v) &&= \langle\zeta_3, \zeta_4\rangle \end{aligned}$$

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$M = T \cdot T_2$ is elementary abelian

$$[\zeta_3, \zeta_1] = 1 = [\zeta_4, \zeta_2]$$
$$[\zeta_4, \zeta_1] = \sigma_1^{\varepsilon} = [\zeta_3, \zeta_2]$$
$$[\sigma_2, \zeta_3] = \zeta_1$$
$$[\sigma_2, \zeta_4] = \zeta_2.$$

5. Final characterization

We shall now determine the structure of $N_G\langle v, t\rangle$. First we note by (4.1) that the element $taz \in C_G(tu)$ satisfies the following relations: $(taz)^2 = v$; $(taz)^{-1}t(taz) = vt$ and $(taz)^{-1}v(taz) = v$. Therefore $taz \in N_G\langle v, t\rangle$. Also using (4.1), we show that

$$taz \in N_G\langle \sigma_1, \zeta_1\rangle = N_G(C_G\langle \sigma_1, \zeta_1\rangle) = N_G(V_1).$$

Because $\langle \zeta_3\rangle = C_G(v) \cap V_1$, we get $(taz)^{-1}\zeta_3(taz) = \zeta_3^{\delta_1}$ where $\delta_1 = 1$ or -1. Next consider the element uaz in $C_G(tu)$. Again we verify that $(uaz)^2 = v$, $(uaz)^{-1}t(uaz) = vt$, and $(uaz)^{-1}v(uaz) = v$. So $(uaz) \in N_G\langle v, t\rangle$. Also we check that $uaz \in N_G\langle \sigma_1, \zeta_2\rangle = N_G(C_G\langle \sigma_1, \zeta_2\rangle) = N_G(V_3)$. So we get once more $(uaz)^{-1}\zeta_4(uaz) = \zeta_4^{\delta_2}$ $\delta_2 = 1$ or -1.

We can now construct the following table using (4.1) and the results just found to show the actions of the elements taz, a_2, uaz on V_1, $V_2 (= V)$, V_3 respectively by conjugation.

TABLE I

	taz	a_2	uaz	$(taza_2)^3$	$(a_2uaz)^3$
σ_1	ζ_1	σ_1	ζ_2	$\sigma_1^{\delta_1}$	$\sigma_1^{\delta_2}$
σ_2	ζ_2	—	ζ_1	—	—
ζ_1	σ_1^{-1}	ζ_3	σ_2^{-1}	$\zeta_1^{\delta_1}$	—
ζ_2	σ_2^{-1}	ζ_4	σ_1^{-1}	—	$\zeta_2^{\delta_2}$
ζ_3	$\zeta_3^{\delta_1}$	ζ_1^{-1}	—	$\zeta_3^{\delta_1}$	—
ζ_4	—	ζ_2^{-1}	$\zeta_4^{\delta_2}$	—	$\zeta_4^{\delta_2}$

If δ_1 is equal to (-1), then we have

$$(taza_2)^3 v \in N_G\langle v, t\rangle \cap C_G\langle \sigma_1, \zeta_1\rangle$$

and $(taza_2)^3 v$ inverts ζ_3, a contradiction since

$$N_G\langle v, t\rangle \cap C_G\langle \sigma_1, \zeta_1\rangle = 1.$$

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Hence we must have $\delta_1 = 1$ and consequently $(taza_2)^3 = 1$. Similarly, we obtain $\delta_2 = 1$ and $(a_2uaz)^3 = 1$. Since $uaz = tu \cdot taz$ and $taz \in C_G(tu)$, we have that taz and uaz commute. Thus we have shown that

$$\langle taz, a_2, uaz\rangle \subseteqq N_G\langle v, t\rangle$$

and the following relations hold for the group $\langle taz, az, uaz\rangle$;

$$(taz)^2 \equiv a_2^2 \equiv (uaz)^2 \equiv (taza_2)^3 \equiv (a_2uaz)^3 \equiv 1$$

(mod $\langle v, t\rangle$); $(taz)(uaz) = (uaz)(taz)$. By Moore's result, we get

$$\langle taz, a_2, uaz\rangle/\langle v, t\rangle \cong S_4$$

(the symmetric group in 4 letters). Since $C_G\langle v, t\rangle$ is of order 16, we have also proved that $\langle taz, a_2, uaz\rangle = N_G\langle v, t\rangle$. Therefore we have proved the following lemma.

(5.1) LEMMA. *We have*

$$N = N_G\langle v, t\rangle = \langle taz, a_2, uaz\rangle$$

where $N_G\langle v, t\rangle/\langle v, t\rangle \cong S_4$. Moreover, the actions of the elements taz, a_2, uaz on V_1, V_2, V_3 respectively are shown in Table I with $\delta_1 = \delta_2 = 1$.

We shall next show that the set of elements in BNB i.e. the set of elements of the double cosets BxB with $x \in N$, forms a subgroup of G. Moreover we shall compute the order of BNB. But first we want to define a few notations.

Put $W = N/\langle v, t\rangle$ and $taz\langle v, t\rangle = r_1$, $a_2\langle v, t\rangle = r_2$, $uaz\langle v, t\rangle = r_3$. Then elements of W are generated by the involutions r_1, r_2, r_3. For any $w \in W$, let $l(w) = l$ be the smallest non-negative integer such that $w = r_{i_1} \cdot r_{i_2} \cdots r_{i_l}$ where $r_{ij} \in \{r_1, r_2, r_3\}$. Let $\omega(r_1) = taz$, $\omega(r_2) = a_2$ and $\omega(r_3) = uaz$. For any $w \in W$ and $w = r_{i_1}r_{i_2}\cdots r_{i_s}$, let

$$\omega(w) = \omega(r)\omega_{i_1}(r_{i_2})\cdots\omega(r_{i_s}).$$

For notational convenience, we shall denote BwB ($\omega \in W$) to mean $B\omega(w)B$.

(5.2) LEMMA. *The set of elements in $G_i = B \cup Br_iB$ ($i = 1, 2, 3$) is a subgroup of G.*

PROOF. Representing the elements taz, ζ_4 on the 'vector space' $M = \langle \sigma_1, \sigma_2, \zeta_1, \zeta_2\rangle$ over $GF(3)$, we get

$$taz \to \begin{pmatrix} & & -1 & 0 \\ & & 0 & -1 \\ 1 & 0 & & \\ 0 & 1 & & \end{pmatrix}; \quad \zeta_4 \to \begin{pmatrix} 1 & 0 & \varepsilon & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

We note that since $C_G(M) = M$, the representation is faithful.

Consider the element $taz\zeta_4$. We have

$$(taz\zeta_4)^3 \to \begin{pmatrix} -\varepsilon & & & \\ & 1 & & \\ & & -\varepsilon & \\ & & & 1 \end{pmatrix}.$$

Suppose $\varepsilon = 1$, then we have $v(taz\zeta_4)^3 \in C_G(\sigma_1, \zeta_1) = V_1$, a contradiction. Therefore $\varepsilon = -1$. Then we get $(taz\zeta_4)^3 \in M \cap C_G(v) = 1$. So $(taz\zeta_4)^3 = 1$. Similarly we get $(uaz\zeta_3)^3 = 1$ and we know that $(a_2\sigma_2)^3 = 1$. Therefore, we have putting $\zeta_4 = X_1$, $\sigma_2 = X_2$, $\zeta_3 = X_3$,

$$(*) \qquad (\omega(r_i)X_i)^3 = 1 \qquad (i = 1, 2, 3)$$

Suppose that we have $g_i = b_i\omega(r_i)b'_i \in Br_iB$ with b_i, b'_i in B. Then the element $g'_i = (b'_i)^{-1} \cdot \omega(r_i)(\omega(r_i)^{-2} \cdot b_i^{-1}) \in Br_iB$ and we have $g_i \cdot g'_i = 1$.

Clearly to show that G_i is a subgroup of G, we need only to show that $\omega(r_i)X_i^\delta\omega(r_i) \in G_i$ ($\delta = 0, 1$, or -1), since for any $b \in B$, we can write $b = v_iX_i$ with $v_i \in \langle v, t\rangle V_i$ where v_i is normalized by $\omega(r_i)$. We have three cases to consider.

(a) $\delta = 0$. Then we have $\omega(r_i) \cdot \omega(r_i) \in \langle v, t\rangle \subseteqq B$.

(b) $\delta = 1$. Then $\omega(r_i)X_i\omega(r_i) = X_i^{-1}\omega(r_i) \cdot \omega(r_i)^{-2}X_i^{-1} \in Br_iB$ by $(*)$.

(c) $\delta = -1$. Then $\omega(r_i)X_i^{-1}\omega(r_i) = \omega(r_i)^2X_i\omega(r_i)X_i\omega(r_i)^2 \in Br_iB$ by $(*)$.

Therefore we have shown that G_i is closed under taking inverses and multiplication. Thus G_i is a subgroup of G.

(5.3) LEMMA. *For any i and $w \in W$, if $l(r_iw) \geqq l(w)$, then $r_iBw \subseteqq Br_iwB$.*

PROOF. Since $W \cong S_4$, and r_i satisfies the Moore's relation, we may identify r_1, r_2, r_3 with the transposition (12), (23), (34) in S_4 respectively. Let $C_0 = \{1\}$, $C_1 = \{r_1, r_2, r_3\}$. We shall give a method of constructing C_n for $n \geqq 2$. Suppose that the sets $C_0, \cdots, C_{n-1}$ have been constructed. Let $\bar{C}_n$ be the set of all 'words' of length n. Define $C_n = \bar{C}_n - \bigcup_{0 \leqq i \leqq n-1} C_i$. Then clearly elements w in C_n has $l(w) = n$.

To check that for those $w \in W$ with $l(r_iw) \geqq l(w)$, we have

$$r_iBw \subseteqq Br_iwB,$$

we need only to see that $r_iX_iw \subseteqq Br_iwB$. It is easily verified that for those $w \in W$ such that $l(r_1w) \geqq l(w)$, we can always write $r_iX_iw = r_iwY_i$ with $Y_i \in B$ using Table I.

The computations are summarized in Table II, which is self-explanatory.

TABLE II

w	$= r_{i_1} \ldots r_{i_s}$	$l(w)$	$l(r_1 w)$	$l(r_2 w)$	$l(r_3 w)$	Y_1	Y_2	Y_3
(12)	r_1	1	0	2	2		ζ_2	ζ_3
(23)	r_2	1	2	0	2	ζ_2^{-1}		ζ_1^{-1}
(34)	r_3	1	2	2	0	ζ_4	ζ_1	
(132)	$r_1 r_2$	2	1	3	3		ζ_4	ζ_1^{-1}
(123)	$r_2 r_1$	2	3	1	3	σ_2		σ_1
(12)(34)	$r_1 r_3$	2	1	3	1		σ_1^{-1}	
(243)	$r_2 r_3$	2	3	1	3	σ_1		σ_2
(234)	$r_3 r_2$	2	3	3	1	ζ_2^{-1}	ζ_3	
(13)	$r_1 r_2 r_1$	3	2	2	4			σ_1
(1432)	$r_1 r_2 r_3$	3	2	4	4		ζ_4	σ_2
(1342)	$r_3 r_1 r_2$	3	2	4	2		σ_1^{-1}	
(1243)	$r_2 r_1 r_3$	3	4	2	4	ζ_1		ζ_2
(1234)	$r_3 r_2 r_1$	3	4	4	2	σ_2	ζ_3	
(24)	$r_2 r_3 r_2$	3	4	2	2	σ_1		
(143)	$r_1 r_2 r_1 r_3$	4	3	3	5			ζ_2
(142)	$r_1 r_2 r_3 r_2$	4	3	5	3		ζ_2^{-1}	
(13)(24)	$r_2 r_3 r_1 r_2$	4	5	3	5	ζ_3		ζ_4
(134)	$r_3 r_2 r_1 r_2$	4	3	5	3		ζ_1^{-1}	
(124)	$r_2 r_3 r_2 r_1$	4	5	3	3	ζ_1		
(1423)	$r_1 r_2 r_1 r_3 r_2$	5	4	4	6			ζ_4
(14)	$r_1 r_2 r_3 r_2 r_1$	5	4	6	4		σ_2	
(1324)	$r_2 r_3 r_1 r_2 r_1$	5	6	4	4	ζ_3		
(14)(23)	$r_1 r_2 r_3 r_2 r_1 r_2$	6	5	5	5			

(5.4) LEMMA. *The set $BNB = G_0$ is a subgroup of G and the double coset Bw_1B is different from Bw_2B if $w_1 = w_2$.*

PROOF. It follows from (3.1), (5.2), (5.3) and Tits [8].

We shall next compute the order of G_0. We check that

$$w_0 = \omega(r_1 r_2 r_3 r_2 r_1 r_2) \in C_G\langle v, t\rangle$$

and so is an involution. The group $\langle v, t, w_0\rangle$ is elementary and different

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from $\langle v, t, u\rangle = \langle v, t, \omega(r_1)^{-1}\omega(r_3)\rangle$. Consider the group $I = P \cap w_0 P w_0$. It is acted on by $\langle v, t\rangle$. By Brauer-Wielandt [10], we get

$$I = C_I(t) C_I(vt) C_I(v).$$

Now we have $C_I(t) = T \cap w_0 T w_0$. Since $\langle t, v, w_0\rangle \neq \langle t, u, v\rangle$, we get either $\langle t, v, w_0\rangle = \langle t, v, t_1\rangle$ or $\langle t, v, ut_1\rangle$. In either case, by the structure of H, we get $T \cap w_0 T w_0 = 1$. Since $T_2 = \omega(r_1)^{-1} T \omega(r_1)$, we obtain

$$T_2 \cap w_0 T_2 w_0 = (T \cap T^{\omega(r_1) w_0 \omega(r_1)^{-1}})^{\omega(r_1)}.$$

We have again that

$$\omega(r_1) w_0 \omega(r_1)^{-1} \in C_G(v, t) \quad \text{and} \quad \langle t, u, v\rangle \neq \langle t, v, \omega(r_1) w_0 \omega(r_1)^{-1}\rangle.$$

So we get $T_2 \cap w_0 T_2 w_0 = C_I(vt) = 1$. Lastly, by exactly the same reason, we prove that $T_3 \cap w_0 T_3 w_0 = C_I(v) = 1$ showing that $I = 1$.

TABLE III

w	B_w	$(B_w)'$
1	1	P
(12)	$\langle\zeta_4\rangle$	V_1
(23)	$\langle\sigma_2\rangle$	V_2
(34)	$\langle\zeta_3\rangle$	V_3
(132)	$\langle\sigma_2, \zeta_2\rangle$	$\langle\sigma_1, \zeta_1, \zeta_3, \zeta_4\rangle$
(123)	$\langle\zeta_2, \zeta_4\rangle$	$\langle\sigma_1, \sigma_2, \zeta_1, \zeta_3\rangle$
(12)(34)	$\langle\zeta_3, \zeta_4\rangle$	$\langle\sigma_1, \sigma_2, \zeta_1, \zeta_2\rangle$
(243)	$\langle\zeta_1, \zeta_3\rangle$	$\langle\sigma_1, \sigma_2, \zeta_2, \zeta_4\rangle$
(234)	$\langle\sigma_2, \zeta_1\rangle$	$\langle\sigma_1, \zeta_2, \zeta_3, \zeta_4\rangle$
(13)	$\langle\sigma_2, \zeta_2, \zeta_4\rangle$	$\langle\sigma_1, \zeta_1, \zeta_3\rangle$
(1432)	$\langle\sigma_1, \zeta_1, \zeta_3\rangle$	$\langle\sigma_2, \zeta_2, \zeta_4\rangle$
(1342)	$\langle\sigma_2, \zeta_1, \zeta_2\rangle$	$\langle\sigma_1, \zeta_3, \zeta_4\rangle$
(1243)	$\langle\sigma_1, \zeta_3, \zeta_4\rangle$	$\langle\sigma_2, \zeta_1, \zeta_2\rangle$
(1234)	$\langle\sigma_1, \zeta_2, \zeta_4\rangle$	$\langle\sigma_2, \zeta_1, \zeta_3\rangle$
(24)	$\langle\sigma_2, \zeta_1, \zeta_3\rangle$	$\langle\sigma_1, \zeta_2, \zeta_4\rangle$
(143)	$\langle\sigma_1, \zeta_1, \zeta_3, \zeta_4\rangle$	$\langle\sigma_2, \zeta_2\rangle$
(142)	$\langle\sigma_1, \sigma_2, \zeta_1, \zeta_3\rangle$	$\langle\zeta_2, \zeta_4\rangle$
(13)(24)	$\langle\sigma_1, \sigma_2, \zeta_1, \zeta_2\rangle$	$\langle\zeta_3, \zeta_4\rangle$
(134)	$\langle\sigma_1, \sigma_2, \zeta_2, \zeta_4\rangle$	$\langle\zeta_1, \zeta_3\rangle$
(124)	$\langle\sigma_1, \zeta_2, \zeta_3, \zeta_4\rangle$	$\langle\sigma_2, \zeta_1\rangle$
(1423)	V_1	$\langle\zeta_4\rangle$
(14)	V_2	$\langle\sigma_2\rangle$
(1324)	V_3	$\langle\zeta_3\rangle$
(14)(23)	P	1

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Define for any $w \in W$, the group B_w generated by all elements x in P such that $\omega(w)x\omega(w)^{-1}$ is in $w_0 P w_0$. Using the informations obtained so far and taking advantages of the identification of W with S_4 in Table II, we can construct the group B_w for all $w \in W$, and these groups B_w are shown in Table III.

We observe that for every B_w, there exists the subgroups $(B_w)'$ such that $B_w(B_w)' = P$ and $B_w \cap (B_w)' = 1$.

(5.5) LEMMA. *Every element of G_0 can be written uniquely in the 'normal' form $h \cdot p\omega(w) \cdot p_w$ with $h \in \langle v, t\rangle$, $p \in P$ and $p_w \in B_w$. The order of G_0 is* $2^7 \cdot 3^6 \cdot 5 \cdot 13$.

PROOF. By (5.4), the group G_0 is the set of elements in BNB. Hence for any element $x \in G_0$, we get that $X = b_1\omega(w)b_2$, $b_i \in B$. We have that $P = Bw \cdot (Bw)'$. We may write $b_2 = hp_2'p_2$ with $h \in \langle v, t\rangle$, $p_2 \in B_w$ and $p_2' \in (B_w)'$. Since, we have $\omega(w)h\omega(w)^{-1} \in \langle v, t\rangle$ and $\omega(w)p_2'\omega(w)^{-1} \in P$, we get $x = b \cdot \omega(w) \cdot p_2$ showing the existence of the 'normal' form.

To prove uniqueness, suppose that we have $b\omega(w)b_w = b'\omega(w')(b_{w'})$. By Tits [8], we get $w = w'$ and so we have $b\omega(w)b_{w'} = b'\omega(w)(b_w)'$. Therefore we get $(b')^{-1}b = \omega(w)b_w(b_{w'})^{-1}\omega(w)^{-1}$. Since we have $(b')^{-1}b \in B$ and $\omega(w)b_w(b_{w'})^{-1}\omega(w)^{-1} \in P^{w_0}$, we obtain $(b')^{-1}b \in B \cap P^{w_0} \subseteqq P$. The uniqueness follows from the fact $P \cap P^{w_0} = 1$.

By Tits [8], the 24 double cosets BwB are distinct. Therefore we have

$$|G_0| = \sum_w |BwB| = |B| \sum_w |B_w| = 2^7 \cdot 3^6 \cdot 5 \cdot 13.$$

The proof of this lemma is now complete.

Before the final proof of the theorem we need the following result of Thompson [7].

LEMMA B (Thompson). *Let $\mathscr{M}$ be a subgroup of the group $\mathscr{X}$ such that*

(a) *$|\mathscr{M}|$ is even*

(b) *$\mathscr{M}$ contains the centralizer of each of its involutions.*

(c) *$\bigcup_{s \in \mathscr{X}} \mathscr{M}^s$ is of odd order.*

Then $i(\mathscr{X}) = 1$ where $i(\mathscr{X})$ is the number of conjugate classes of involutions of $\mathscr{X}$.

Conclusion of the proof of the theorem

Using the informations of our tables (I, II and III), we can multiply any two elements of G_0 in the 'normal' form to get the product *uniquely* in the 'normal' form. (Uniqueness of product since we have determined ε, δ_1, δ_2). Now if X is any finite group satisfying (a) and (b) of the theorem, then X has a subgroup X_0 of order $|L_4(3)|$ with uniquely determined multiplication table. Hence taking X to be $L_4(3)$, we see that $X_0 = L_4(3)$ and so $L_4(3) \cong G_0$. Consequently G_0 satisfies conditions (a) and (b) of lemma B. If condition (c) of this lemma were true for G_0, then we would get $i(G) = 1$, a contradiction to (2.11). So $\bigcap_{g \in G} G_0^g$ is even and normal in G. By (3.2) we get immediately that $G = G_0$. The proof of the theorem is now complete.

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