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OSCILLATIONS OF HIGHER ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract

Some new oscillation criteria for higher order neutral functional differential equations of the form

$$\frac{d^n}{dt^n}(x(t) + cx(t-h) + \bar{c}x(t+\bar{h})) + qx(t-g) + px(t+\bar{g}) = 0,$$

n is even, are established.

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1. Introduction

This paper deals with the even order neutral functional differential equation

$$(E)\ (x(t) + cx(t-h) + \bar{c}x(t+\bar{h}))^{(n)} + qx(x-g) + px(t+\bar{g}) = 0, \quad n \text{ even},$$

where c, $\bar{c}$, h, $\bar{h}$, p and q are real numbers, g and $\bar{g}$ are positive constants.

Recently, the oscillatory behavior of solutions of (E) has been studied by Grace [2]. The purpose of this paper is to obtain some new easily verifiable sufficient conditions, involving only the coefficients and the arguments, under which all solutions of (E) are oscillatory. Our technique, differing greatly from that in [2], is based on the study of the characteristic equation

$$(E^*) \qquad \lambda^n(1 + ce^{-\lambda h} + \bar{c}e^{\lambda\bar{h}}) + qe^{-\lambda g} + pe^{\lambda\bar{g}} = 0.$$

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Necessary and sufficient conditions (in terms of the characteristic equations) for the oscillations of all solutions of higher order neutral differential equations have been established by Bilchev, Grammatikopoulos and Stavroulakis [1], Ladas, Partheniadis and Sficas [3] and Wang [5]. The oscillation criteria obtained in this paper improve noticeably the results in [2] and [4].

Let $t_0 \geq 0$. By a solution of (E) we mean a continuous function x defined on the interval $[t_0 - \tau, \infty)$, where $\tau = \max\{h, g\}$ such that $x(t) + cx(t-h) + \bar{c}x(t+\bar{h})$ is n–times continuously differentiable for all $t \geq t_0$. As is customary, a solution is called *oscillatory* if it has arbitrarily large zeros. Otherwise, the solution is called non-oscillatory. Equation (E) is called *oscillatory* if all of its solutions are oscillatory.

2. Main results

The oscillation criteria in this paper are based on the following lemma which is extracted from results in [1] and [5].

LEMMA. *A necessary and sufficient condition for the oscillation of* (E) *is that its characteristic equation* (E^*) *has no real roots.*

Like Grace [2], we study the oscillatory character of the following equations, as some special cases of (E),

$$(E_1) \qquad \frac{d^n}{dt^n}(x(t) + cx(t-h) - \bar{c}x(t+\bar{h})) = qx(t-g) + px(t+\bar{g}),$$

$$(E_2) \qquad \frac{d^n}{dt^n}(x(t) + cx(t+h) - \bar{c}x(t+\bar{h})) = qx(t-g) + px(t+\bar{g}),$$

$$(E_3) \qquad \frac{d^n}{dt^n}(x(t) + cx(t-h) + \bar{c}x(t+\bar{h})) = qx(t-g) + px(t+\bar{g}),$$

and

$$(E_4) \qquad \frac{d^n}{dt^n}(x(t) - cx(t-h) - \bar{c}x(t+\bar{h})) + qx(t-g) + px(t+\bar{g}) = 0,$$

where n is even, $c, \bar{c}, g, h$ and $\bar{h}$ are non-negative, $g, \bar{g}, p$ and q are positive real numbers. The characteristic equations of (E_1)–(E_4) are respectively

$$(E_1^*) \qquad F_1(\lambda) := \lambda^n(1 + ce^{-\lambda h} - \bar{c}e^{\lambda\bar{h}}) - qe^{-\lambda g} - pe^{\lambda\bar{g}} = 0,$$

$$(E_2^*) \qquad F_2(\lambda) := \lambda^n(1 + ce^{\lambda h} - \bar{c}e^{\lambda\bar{h}}) - qe^{-\lambda g} - pe^{\lambda\bar{g}} = 0,$$

$$(E_3^*) \qquad F_3(\lambda) := \lambda^n(1 + ce^{-\lambda h} + \bar{c}e^{\lambda\bar{h}}) - qe^{-\lambda g} - pe^{\lambda\bar{g}} = 0$$

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and

$$(E_4^*) \qquad F_4(\lambda) := \lambda^n(1 - ce^{-\lambda h} - \bar{c}e^{\lambda\bar{h}}) + qe^{-\lambda g} + pe^{\lambda\bar{g}} = 0.$$

First of all, we consider (E_1) and establish the following result for its oscillatory character.

THEOREM 1. *If*

$$(1) \qquad p\bar{g}\left(\frac{e}{n}\right)^n + \bar{c}\exp\left(\left[\frac{p}{1+c}\right]^{1/n}\bar{h}\right) > 1 + c,$$

and

$$(2) \qquad q(g-h)^n\left(\frac{e}{n}\right)^n > 1 + c \quad \text{and} \quad g > h,$$

then (E_1) *is oscillatory.*

PROOF. For $\lambda \neq 0$, from (E_1^*), we have that

$$(3) \qquad -F_1(\lambda)/\lambda^n = (qe^{-\lambda g} + pe^{\lambda\bar{g}})/\lambda^n - (1 + ce^{-\lambda h} - \bar{c}e^{\lambda\bar{h}})$$

and

$$(4) \qquad -F_1(\lambda)e^{\lambda h}/\lambda^n = (qe^{-\lambda(g-h)} + pe^{\lambda(\bar{g}+h)})/\lambda^n - (e^{\lambda h} + c - \bar{c}e^{\lambda(\bar{h}+h)}).$$

Now, our strategy is to prove that $F_1(\lambda) < 0$ for all $\lambda \in (-\infty, \infty)$ in each of the following four cases.

Case 1. $0 < \lambda \leq (p/(1+c))^{1/n}$. From (3), in this case we have

$$-F_1(\lambda)/\lambda^n > (qe^{-\lambda g} + pe^{\lambda\bar{g}})/\lambda^n - (1+c) > p\Big/\left(\frac{p}{1+c}\right) - (1+c) = 0.$$

Thus $F_1(\lambda) < 0$ for $0 < \lambda \leq (p/(1+c))^{1/n}$.

Case 2. $\lambda > (p/(1+c))^{1/n}$. In view of (1) and the inequality $e^x \geq ex$ for all $x \geq 0$, (3) yields

$$-F_1(\lambda)/\lambda^n \geq qe^{-\lambda q}/\lambda^n + p\bar{g}^n\left(e^{\frac{\lambda}{n}\bar{g}}\Big/\frac{\lambda}{n}\bar{g}\right)^n\frac{1}{n^n} - (1+c) + \bar{c}\exp\left(\left[\frac{p}{1+c}\right]^{1/n}\bar{h}\right)$$

$$> p\bar{g}^n\left(\frac{e}{n}\right)^n - (1+c) + \bar{c}\exp\left(\left[\frac{p}{1+c}\right]^{1/n}\bar{h}\right) > 0,$$

which implies that $F_1(\lambda) < 0$ for $\lambda > (p/(1+c))^{1/n}$.

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Case 3. $\lambda < 0$. By using (2) and the inequality $e^x \geq ex$ for $x \geq 0$, from (4) we obtain

$$\begin{aligned} -F_1(\lambda)e^{\lambda h}/\lambda^n &\geq q(g-h)^n\left[e^{-\lambda(g-h)/n}/\lambda(g-h)/n\right]^n n^{-n} + pe^{\lambda(\bar{g}+h)}/\lambda^n - (e^{\lambda h}+c) \\ &\geq q(g-h)^n(e/n)^n - (1+c) > 0. \end{aligned}$$

Therefore, in this case $F_1(\lambda) < 0$ for $\lambda < 0$.

Cases 1–3 and $F_1(0) < 0$ imply that $F_1(\lambda) < 0$ for $\lambda \in (-\infty, \infty)$, that is , ($E_1^*$) has no real roots. By using the lemma, we conclude that (E_1) is oscillatory. This completes the proof.

REMARK 1. Theorem 1 improves Theorem 1 of Grace [2].

Next, we present a theorem which describes the oscillatory character of (E_2).

THEOREM 2. *If*

$$p(\bar{g}-h)^n(e/n)^n + \bar{c} > 1 + c \quad \text{and} \quad \bar{g} > h,\ \bar{h} > h, \tag{5}$$

and

$$qg^n\left(\frac{e}{n}\right)^n + \bar{c}\exp\left(-\left[\frac{q}{1+c}\right]^{1/n}\bar{h}\right) > 1 + c, \tag{6}$$

then (E_2) *is oscillatory.*

PROOF. From the characteristic equation (E_2^*) of (E_2), for $\lambda \neq 0$, we get

$$-F_2(\lambda)/\lambda^n = \left(qe^{-\lambda g} + pe^{\lambda\bar{g}}\right)/\lambda^n - \left(1 + ce^{\lambda h} - \bar{c}e^{\lambda\bar{h}}\right), \tag{7}$$

and

$$-F_2(\lambda)e^{-\lambda h}/\lambda^n = \left(qe^{-\lambda(g+h)} + pe^{\lambda(\bar{g}-h)}\right)/\lambda^n - \left(e^{-\lambda h} + c - \bar{c}e^{\lambda(\bar{h}-h)}\right). \tag{8}$$

As in the proof of Theorem 1, we consider the following four cases.

Case 1. $0 < \lambda \leq (p/(1+c))^{1/n}$. From (8) it follows that

$$-F_2(\lambda)e^{-\lambda h}/\lambda^n > pe^{\lambda(\bar{g}-h)}/(p/(1+c)) - (1+c) > 0.$$

which implies $F_2(\lambda) < 0$ for $0 < \lambda \leq (p/(1+c))^{1/n}$.

Case 2. $\lambda > (p/(1+c))^{1/n}$. Then from (8), (5) and the fact that $e^x \geq ex$ for $x \geq 0$, we have

$$-F_2(\lambda)e^{-\lambda h}/\lambda^n \geq p(\bar{g}-h)^n \left(e^{\lambda(\bar{g}-h)/n}/[\lambda(\bar{g}-h)/n]\right)^n n^{-n} - (1+c-\bar{c})$$
$$\geq p(\bar{g}-h)^n(e/n)^n - (1+c-\bar{c}) > 0,$$

Hence, in this case, $F_2(\lambda) < 0$.
Case 3. $-(q/(1+c))^{1/n} \leq \lambda < 0$. From (7) and (6) we get

$$-F_2(\lambda)/\lambda^n \geq qg^n(e^{-\lambda g/n}/(-\lambda g/n)^n n^{-n} - (1+c-\bar{c}\exp[-(q/(1+c))^{1/n}\bar{h}]) > 0,$$

that is, $F_2(\lambda) > 0$ for $-(q/(1+c))^{1/n} \leq \lambda < 0$.
Case 4. $\lambda < -(q/(1+c))^{1/n}$. In this case, using (7) we observe

$$-F_2(\lambda)/\lambda^n \geq qe^{-\lambda g} \Big/ \frac{q}{1+c} - (1+c) > 0$$

which implies $F_2(\lambda) < 0$ for $\lambda < -(q/(1+c))^{1/n}$.

From cases 1–4 and $F_2(0) < 0$, we see that for all $\lambda \in (-\infty, \infty)$, $F_2(\lambda) < 0$. By the Lemma, we conclude that (E_2) is oscillatory, and the proof of the theorem is complete.

REMARK 2. Our Theorem 2 improves Theorem 2 in [2].

The following result is concerned with the oscillatory behavior of (E_3).

THEOREM 3. *If*

(9)
$$p(\bar{g}-\bar{h})^n\left(\frac{e}{n}\right)^n > 1+\bar{c}+c\exp\left[-\left(\frac{p}{1+c+\bar{c}}\right)^{1/n}(h+\bar{h})\right], \quad \textit{and} \quad \bar{g} > \bar{h},$$

and

(10)
$$q(g-h)^n\left(\frac{e}{n}\right)^n > 1+c+\bar{c}\exp\left[-\left(\frac{q}{1+c+\bar{c}}\right)^{1/n}(\bar{h}+h)\right], \quad \textit{and} \quad g > h,$$

then (E_3) *is oscillatory.*

PROOF. From (E_3^*), for $\lambda \neq 0$ we obtain

(11) $$-F_3(\lambda)e^{-\lambda\bar{h}}/\lambda^n = (qe^{-\lambda(g+\bar{h})} + pe^{\lambda(\bar{g}+\bar{h})})/\lambda^n - (e^{-\lambda\bar{h}} + ce^{-\lambda(h+\bar{h})} + \bar{c})$$

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and

$$(12) \qquad -F_3(\lambda)e^{\lambda h}/\lambda^n = (qe^{-\lambda(g-h)} + pe^{\lambda(\bar{g}+h)})/\lambda^n - (e^{\lambda h} + c + \bar{c}e^{\lambda(h+\bar{h})}).$$

Now, consider the following four cases.

Case 1. $0 < \lambda \le (p/(1+c+\bar{c}))^{1/n}$. From (11), we get

$$-F_3(\lambda)e^{-\lambda\bar{h}}/\lambda^n > p/(p/(1+c+\bar{c})) - (1+c+\bar{c}) = 0.$$

Case 2. $\lambda > (p/(1+c+\bar{c}))^{1/n}$. In view of (11) and (9) we obtain

$$-F_3(\lambda)e^{-\lambda\bar{h}}/\lambda^n > p(\bar{g}-\bar{h})^n\left(\frac{e}{n}\right)^n - \left[1 + c\exp\left(-\left[\frac{p}{1+c+\bar{c}}\right]^{1/n}(h+\bar{h})\right) + \bar{c}\right] > 0.$$

Case 3. $-(q/(1+c+\bar{c}))^{1/n} \le \lambda < 0$. From (12) it follows that

$$-F_3(\lambda)e^{\lambda h}/\lambda^n > q\Big/\frac{q}{1+c+\bar{c}} - (1+c+\bar{c}) = 0.$$

Case 4. $\lambda < -(q/(1+c+\bar{c}))^{1/n}$. Then, from (12) and (10) we have

$$-F_3(\lambda)e^{\lambda h}/\lambda^n \ge q(g-h)^n\left(\frac{e}{n}\right)^n - \left(1 + c + \bar{c}\exp\left[-\left(\frac{q}{1+c+\bar{c}}\right)^{1/n}(\bar{h}+h)\right]\right) > 0.$$

From cases 1–4 and $F_3(0) < 0$, we observe $F_3(\lambda) < 0$ for all $\lambda \in (-\infty, \infty)$. By the lemma, (E_3) is oscillatory, and the proof of the theorem is complete.

REMARK 3. Theorem 3 improves Theorem 3 in [2].

Finally, we present an oscillation criterion for (E_4).

THEOREM 4. *If* $c + \bar{c} > 0$,

$$(13) \qquad p(\bar{g}-\bar{h})^n\left(\frac{e}{n}\right)^n > c\exp\left[-\left(\frac{p}{c+\bar{c}}\right)^{1/n}(h+\bar{h})\right] + \bar{c} \quad \textit{and} \quad \bar{g} > \bar{h}$$

and

$$(14) \qquad q(g-h)^n\left(\frac{e}{n}\right)^n > c + \bar{c}\exp\left[-\left(\frac{q}{c+\bar{c}}\right)^{1/n}(\bar{h}+h)\right] \quad \textit{and} \quad g > h,$$

then (E_4) *is oscillatory.*

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PROOF. By the characteristic equation (E_4^*) of (E_4) we obtain

$$(15)\qquad F_4(\lambda)e^{-\lambda\bar{h}}/\lambda^n = [qe^{-\lambda(g+\bar{h})} + pe^{\lambda(\bar{g}-\bar{h})}]/\lambda^n + e^{-\lambda\bar{h}} - ce^{-\lambda(h+\bar{h})} - \bar{c}$$

and

$$(16)\qquad F_4(\lambda)e^{\lambda h}/\lambda^n = [qe^{-\lambda(g-h)} + pe^{\lambda(\bar{g}+h)}]/\lambda^n + e^{\lambda h} - c - \bar{c}e^{\lambda(\bar{h}+h)}.$$

Consider the following four cases.

Case 1. $0 < \lambda \le (p/(c+\bar{c}))^{1/n}$. From (15) we have

$$F_4(\lambda)e^{-\lambda\bar{h}}/\lambda^n \ge p\Big/\frac{p}{c+\bar{c}} + e^{-\lambda\bar{h}} - c - \bar{c} > 0.$$

Case 2. $\lambda > (p/(c+\bar{c}))^{1/n}$. In this case, from (15) and (13), it follows that

$$F_4(\lambda)e^{-\lambda\bar{h}}/\lambda^n \ge p(\bar{g}-\bar{h})^n\left(\frac{e}{n}\right)^n - c\exp\left[-\left(\frac{p}{c+\bar{c}}\right)^{1/n}(h+\bar{h})\right] - \bar{c} > 0.$$

Case 3. $-(q/(c+\bar{c}))^{1/n} \le \lambda < 0$. From (16) we get

$$F_4(\lambda)e^{\lambda h}/\lambda^n \ge q/\frac{q}{c+\bar{c}} + e^{\lambda h} - c - \bar{c} > 0.$$

Case 4. $\lambda < -(q/(c+\bar{c}))^{1/n}$. By using (16) and (14) we obtain

$$F_4(\lambda)e^{\lambda h}/\lambda^n \ge q(g-h)^n\left(\frac{e}{n}\right)^n - c - \bar{c}\exp\left[-\left(\frac{q}{c+\bar{c}}\right)^{1/n}(h+\bar{h})\right] > 0.$$

From cases 1–4 and $F_4(0) > 0$, it follows that $F_4(\lambda) > 0$ for all $\lambda \in (-\infty, \infty)$ which means that (E_4) is oscillatory, and the proof of the theorem is complete.

REMARK 4. Theorem 4 improves Theorem 4 in [2] and the theorem in [4].

The following examples are illustrative.

EXAMPLE 1. Consider the differential equation

$$(17)\qquad [x(t) + x(t-2\pi) - 2x(t+2\pi)]^{(12)} = 2\sin(t-8\pi) + 2\sin(t+\pi).$$

It is easy to check that conditions (1) and (2) are satisfied, and hence (17) is oscillatory and $x(t) = \sin t$ and $x(t) = \cos t$ are two solutions of (17).

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EXAMPLE 2. Consider the differential equation

$$\left[x(t) - e^4 x\left(t - \frac{\pi}{2}\right) - e^4 x\left(t + \frac{\pi}{2}\right)\right]^{(4)} + 3x\left(t - \frac{3\pi}{2}\right) + 2x\left(t + \frac{3}{2}\pi\right) = 0. \tag{18}$$

The conditions (13) and (14) are satisfied and hence (18) is oscillatory. One such solution is $x(t) = \cos t$.

REMARK 5. In the above examples we observe that conditions (1) and (22) of Theorems 1 and 4 in [2], respectively, are violated. Hence Theorems 1 and 4 in [2] fail to apply to (17) and (18) and we believe that the oscillatory behavior of (17) and (18) is not deducible from known oscillation criteria.

From the proofs of Theorems 1–4 and the lemma, it is easy to prove the following results.

COROLLARY 1. *If condition* (1) (*respectively* (5), (9) *or* (13)) *is satisfied, then every solution* $x(t)$ *of* (E_1) (*respectively* (E_2), (E_3) *or* (E_4)) *is either oscillatory or else* $x^{(j)}(t) \to 0$ *as* $t \to \infty$, $j = 0, 1, \ldots, n-1$.

COROLLARY 2. *If only condition* (2) (*respectively* (6), (10) *or* (14)) *is satisfied, then every solution* $x(t)$ *of* (E_1) (*respectively* (E_2), (E_3) *or* (E_4)) *is either oscillatory or else* $x^{(j)}(t) \to \infty$ *as* $t \to \infty$, $j = 0, 1, \ldots, n-1$.

REMARK 6. Our technique can be extended to odd order neutral functional differential equations.

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