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A NOTE ON THE DENSITY THEOREM

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In this note we prove:

THEOREM. *Let R be a right primitive ring with pair-wise non-isomorphic faithful irreducible modules $M_1, M_2, \ldots, M_k$. Let $D_i = \mathrm{End}_R M_i$. For each i, let $\{v_{ij}\}_{j=1}^{n_i}$ be elements of M_i linearly independent over D_i. For each i, let $\{u_{ij}\}_{j=1}^{n_i}$ be a set of elements of M_i. Then there exists an element r of R such that $u_{ij} = v_{ij}r$, for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n_i$.*

Thus the statement of the density theorem generalizes from the case of a single faithful irreducible module to the case where we have a finite collection of pairwise nonisomorphic faithful irreducible modules.

I would like to thank Professors Alperin and Herstein for suggesting the above theorem.

Proof. It is enough to show that for given $(a, b) \in N \times N$ there exists an element $r_{ab} \in R$ such that $v_{ij}r_{ab} = 0$ if $(i, j) \neq (a, b)$ and $v_{ij}r_{ab} \neq 0$ if $(i, j) = (a, b)$. Without loss of generality, we consider only the case where $(a, b) = (k, n_k)$. By the Jacoson Density Theorem [1], we can choose $t \in R$ such that $v_{kj}t = 0$, $j = 1, 2, \ldots, n_k - 1$, and $v_{kn_k}t \neq 0$. Consider the external direct sum of modules

$$\sum_{i=1}^{k-1} l_i(M_i),$$

where $l_i(M_i)$ stands for the direct sum of n_i copies of M_i. Let $\alpha = v_{11}t + v_{12}t + \cdots + v_{1n_1}t + v_{21}t + \cdots + v_{k-1n_{k-1}}t$. The relation f: $\alpha R \to M_k$ defined by $\alpha a \mapsto v_{kn_k}ta$, where $a \in R$, is a nonzero module homomorphism if it is well-defined as a function. This is impossible by the Jordan–Hölder Theorem, since M_k is not isomorphic to any other M_i. Thus there is an $s \in R$ such that $\alpha s = 0$ and $v_{kn_k}ts \neq 0$. Let $r_{kn_k} = ts$. This completes the proof.

REFERENCE

1. Herstein, Noncommutative rings, Carus Monograph, Math. Assoc. of America, (1968).

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