Canad. Math. Bull. Vol. **53** (1), 2010 pp. 187–191
doi:10.4153/CMB-2010-033-x

On the Local Unipotent Fundamental Group Scheme

Sinan Ünver

Abstract. We prove a local, unipotent, analog of Kedlaya's theorem for the pro-p part of the fundamental group of integral affine schemes in characteristic p.

Introduction

Kedlaya proved that if $f\colon X \to Y$ is a morphism between two connected affine schemes $X := \operatorname{Spec}(A)$ and $Y := \operatorname{Spec}(B)$ over a field of characteristic p, then f induces an isomorphism between $\pi_1^p(X, \bar{x})$ and $\pi_1^p(Y, f(\bar{x}))$, the pro-p quotients of the algebraic fundamental group, if and only if f induces an isomorphism between $B/(F - \mathrm{id})(B)$ and $A/(F - \mathrm{id})(A)$ where F is the frobenius map [3, Corollary 2.6.10]. Theorem 8 is an analog of this result for the local, unipotent part of the fundamental group scheme.

1 Local Unipotent Fundamental Group Scheme

Let k be a perfect field of characteristic p. Let $\mathrm{G_a}$ denote the standard additive algebraic group structure on the affine line. For a scheme X/k, let $F\colon X \to X$ denote the absolute p-power frobenius map and $F_X\colon X \to X^{(p)}$, the relative frobenius map. Let α_p be the kernel of $F_{\mathrm{G_a}}\colon \mathrm{G_a} \to \mathrm{G_a^{(p)}}$.

A group scheme G/k is called unipotent if $G_{\bar{k}}$ has a composition series whose factor groups are $1, \mathrm{G_a}, \alpha_{\mathrm{p}}$, or $\underline{\mathbb{Z}}/p \cdot \underline{\mathbb{Z}}$. This is equivalent to requiring that G be affine and that every representation of G on a finite dimensional k-vector space V have a filtration $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_m = V$ by sub-representations whose successive quotients are trivial representations as shown in [1, Exposé XVII, Théorème 3.5, p. 547]. Note that such groups are called nilpotent in [5, p. 97].

We are interested in finite, local, unipotent group schemes over k. These are exactly the unipotent group schemes which have a finite composition series with factors α_p's by [1, Théorème 3.5].

Definition 1 Let X/k be a scheme and G/k a finite, local, unipotent group scheme. A G-torsor T over X is a finite, flat and surjective map $\pi\colon T \to X$ and an action $\varphi\colon G \times_k T \to T$ of G on T relative to X such that the map $G \times_k T \to T \times_X T$ defined by $(g, t) \to (\varphi(g, t), t)$ is an isomorphism.

Received by the editors February 21, 2007; revised July 25, 2007.
Published electronically December 4, 2009.
AMS subject classification: **11G25**.

Lemma 2 *Let G and X be as above. Let $\underline{G}$ denote the sheaf of groups represented by G on Sch_X, the fpqc site of schemes over X. Then there is an equivalence of categories between G-torsors on X and $\underline{G}$-torsors on Sch_X.*

Proof The proof follows from faithfully flat descent. ■

We will not distinguish between G and $\underline{G}$-torsors from now on.

Let $\mathcal{U}(X)$ denote the category whose objects are pairs (T, G) where G is a finite, local, unipotent group and T is a G-torsor over X. A morphism from (T, G) to (S, H) is a pair (α, ι), where $\iota\colon G \to H$ is a homomorphism and $\alpha\colon T \to S$ is a morphism of schemes over X that is equivariant with respect to G, where G acts on S through ι.

Lemma 3 *Let G be a finite, local, unipotent group. Any G-torsor T on $\mathrm{Spec}(k)$ is canonically trivial.*

Proof Note that the trivializations of T over k are in one-to-one correspondence with $T(k)$. Therefore the statement is equivalent to saying that $T(k)$ is a singleton.

Let k' be the residue field of a closed point of T. Note that then k'/k is a finite extension, $T(k') \neq \varnothing$, and $T_{k'} \simeq G_{k'}$. That $T_{k'}$ is a local scheme implies that T is a local scheme. Hence k' is the residue field of the unique closed point. The closed imbedding $\mathrm{Spec}(k') \to T$ induces the closed imbedding $\mathrm{Spec}(k' \otimes_k k') \to T_{k'}$. In particular, $\mathrm{Spec}(k' \otimes_k k')$ is connected. Together with the assumption that k, is perfect this implies that $k' = k$. Hence $T(k) \neq \varnothing$ which implies that $T \simeq G$. Finally $T(k) \simeq G(k)$ is a singleton. ■

As in [5, Definition 2, p. 85], we say that $\mathcal{U}(X)$ satisfies property $\mathcal{P}$ if whenever $(f_i, \rho_i)\colon (T_i, G_i) \to (T, G)$ for $1 \leq i \leq 2$ are morphisms in $\mathcal{U}(X)$, then $(T_1 \times_T T_2, G_1 \times_G G_2)$ is an object of $\mathcal{U}(X)$.

Lemma 4 *The category $\mathcal{U}(X)$ satisfies property $\mathcal{P}$ if X is an integral scheme.*

Proof This follows from [5, Propositions 1, 2], except that there one needs to choose a basepoint on X. In our case, where we permit only local G as the structure group for the torsors, the choice of a basepoint will not be necessary.

Choose a finite extension k'/k such that $X(k') \neq \varnothing$ and fix $x_0 \in X(k')$. Then by [5] we know that the category of triples (T, G, ν), where $\nu \in T_{x_0}(k')$, satisfies property $\mathcal{P}$. But because of Lemma 3 this category is canonically equivalent to $\mathcal{U}(X)$ through the functor that forgets the point ν. This shows the property $\mathcal{P}$ for $\mathcal{U}(X)$. ■

From now on we assume that X is integral. Therefore by the construction in [5, pp. 86–87], there is a pro-finite, unipotent, local, fundamental group scheme $\pi_1^\circ(X)$ together with a torsor P/X under it such that, for any (T, G) as above there is a unique map $(P, \pi_1^\circ(X)) \to (T, G)$.

We can give a Tannakian description of this fundamental group if X/k is complete. Following [4], call a vector bundle V on X, F-trivial if there exists an $n \geq 1$ such that $F^n(V)$ is trivial. Then consider the category of F-trivial bundles which are also unipotent as vector bundles. This is a Tannakian category and it follows from the discussion in [4, pp. 144–145] that the fundamental group of this category (at a basepoint, but this is independent of the basepoint because of the above) is $\pi_1^\circ(X)$.

https://doi.org/10.4153/CMB-2010-033-x Published online by Cambridge University Press

Lemma 5 *Let G and X be as above. Then $\Gamma(X, G) = 0$.*

Proof Since X is an integral scheme, $\Gamma(X, \alpha_p) = 0$. Let $1 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G$ be a composition series whose factors are α_p's. The exact sequences $0 \to \Gamma(X, G_{m-1}) \to \Gamma(X, G_m) \to \Gamma(X, G_m/G_{m-1}) = 0$ and induction imply the statement. ■

Let G, H, T, S, and $\iota\colon G \to H$ be as above. Let $\mathrm{Hom}_{G-H}(T, S)$ denote morphisms between (T, G) and (S, H) in $\mathcal{U}$ where the map from G to H is ι.

Lemma 6 $\mathrm{Hom}_{G-H}(T, S)$ *has at most one element.*

Proof First note that if $\iota_* T$ denotes the push-forward of the G-torsor T to an H-torsor, then $\mathrm{Hom}_{G-H}(T, S) = \mathrm{Hom}_H(\iota_* T, S)$. Hence it suffices to prove the claim in the case when $G = H$ and ι is the identity map. Note that any morphism between G-torsors T and S is necessarily an isomorphism. Therefore we may assume that $T = S$. Let $\alpha \in \mathrm{Hom}_G(T, T)$. Let U be an fpqc covering of X that trivializes T, *e.g.*, let $U := T$. Choosing a trivialization $G_U \to T_U$ of T on U and using the isomorphism $p_1^* T_U \simeq p_2^* T_U$ on $U' := U \times_X U$ we obtain an isomorphism $G_{U'} \to G_{U'}$ of G-torsors on U'. Let $g' \in \Gamma(U', G_{U'})$ be the image of 1 under this map. Then in this description α is given by $h \in \Gamma(U, G_U)$ such that $p_1^* h = g' \cdot p_2^* h \cdot (g')^{-1}$. Since G being unipotent is nilpotent [1, Exposé XVII, Corollaire 3.7, p. 548,], there is a finite filtration $1 = G_0 \to G_1 \to \cdots \to G_n =: G$ such that G_k/G_{k-1} is the center of G/G_{k-1}. We take the push-forward of the torsor via the quotient map $G \to G/G_{n-1}$, which has an abelian target. The above identity gives $p_1^* h = p_2^* h$, and hence by faithfully flat descent gives an element in $\Gamma(X, G/G_{n-1})$ which vanishes by Lemma 5. Therefore $h \in \Gamma(U, G_{n-1})$. Taking the push-forward of T via the map $G \to G/G_{n-2}$ and proceeding as above we see that $h \in \Gamma(U, G_{n-2})$ and proceeding by induction $h = 1$. ■

The following lemma is an explicit characterization of α_p-torsors on an affine scheme.

Lemma 7 *Let $X := \mathrm{Spec}(A)$ be an affine scheme over k. The natural map*

$$A/A^p \to H^1_{\mathrm{fpqc}}(X, \alpha_p) \simeq \mathrm{Tors}(X, \alpha_p)$$

is an isomorphism.

Proof We have an exact sequence $0 \to \alpha_p \to \mathrm{G_a} \to \mathrm{G_a^{(p)}} \to 0$ of sheaves of groups on Sch_X. Considering the long exact sequence for cohomology and noting that the cohomology groups of coherent sheaves are the same on the Zariski and the fpqc site, and that on an affine scheme cohomology of a coherent sheaf vanishes in positive degrees proves the statement. The last isomorphism is a consequence of Lemma 2. ■

Theorem 8 *Let $X := \mathrm{Spec}(A)$ and $Y := \mathrm{Spec}(B)$ be two integral affine schemes over k. Then a map $f\colon X \to Y$ induces an isomorphism from $\pi_1^\circ(X)$ to $\pi_1^\circ(Y)$ if and only if it induces an isomorphism from B/B^p to A/A^p.*

Proof By the universal property of the fundamental group, the natural map from $\mathrm{Hom}(\pi^\circ(X), \alpha_p)$ to $\mathrm{Tors}(X, \alpha_p)$ is an isomorphism. Together with Lemma 7 this implies necessity.

Conversely assume that the induced map $B/B^p \to A/A^p$ is an isomorphism. This implies that the map $H^1_{\mathrm{fpqc}}(Y, \alpha_p) \to H^1_{\mathrm{fpqc}}(X, \alpha_p)$ is an isomorphism. We will show that f induces an isomorphism on the set of isomorphism classes of torsors.

Lemma 9 *Let G be a finite, local, unipotent group. The map*

$$H^1_{\mathrm{fpqc}}(Y, G) \to H^1_{\mathrm{fpqc}}(X, G)$$

is an isomorphism.

Proof In the following $H^i_{\mathrm{fpqc}}(X, G)$, for $0 \le i \le 2$, denotes the non-abelian cohomology of the associated sheaves of groups on Sch_X. This coincides with the ordinary cohomology groups when G is abelian, [2, pp. 168–170, 259–264].

Consider a central extension $0 \to \alpha_p \to G \to G' \to 0$. First note that $H^2_{\mathrm{fpqc}}(X, \alpha_p)$ vanishes. This follows from the long exact sequence corresponding to $0 \to \alpha_p \to G_a \to G_a^{(p)} \to 0$ and the assumption that X is affine. This central extension defines an exact sequence for cohomology, with the appropriate notion of exactness,

$$0 = H^0_{\mathrm{fpqc}}(X, G') \to H^1_{\mathrm{fpqc}}(X, \alpha_p) \to H^1_{\mathrm{fpqc}}(X, G) \to H^1_{\mathrm{fpqc}}(X, G') \to H^2_{\mathrm{fpqc}}(X, \alpha_p) = 0,$$

(see [2, p. 284]), and similarly on Y. Therefore if the induced map $H^1_{\mathrm{fpqc}}(Y, G') \to H^1_{\mathrm{fpqc}}(X, G')$ is an isomorphism, then so is $H^1_{\mathrm{fpqc}}(Y, G) \to H^1_{\mathrm{fpqc}}(X, G)$. And hence this is so by induction based on the fact that G is nilpotent.

This proves that f induces an equivalence between isomorphism classes of torsors. ■

By Lemma 6 we know that f also induces an isomorphism between morphisms of torsors. This proves that f induces an equivalence between $\mathcal{U}(Y)$ and $\mathcal{U}(X)$, and hence an isomorphism between the corresponding fundamental groups. ■

Remarks. (i) Let A be a ring and $A^{p^{-\infty}}$ be a perfection of A. Let B_n be the subring of $A^{p^{-\infty}}[x^{p^{-n}}]$ consisting of those $f(x^{p^{-n}})$ such that $f(0) \in A$. Then $B_n \subseteq B_{n+1}$ and let $B := \bigcup_{1 \le n} B_n$. Then the natural inclusion $A \to B$ induces an isomorphism $A/A^p \simeq B/B^p$.

(ii) One could ask whether the following statement is true: if $f\colon X \to Y$ is a morphism between integral schemes over k such that $f^{-1}(\mathcal{O}_Y)/f^{-1}(\mathcal{O}_Y)^p \to \mathcal{O}_X/\mathcal{O}_X^p$ is an isomorphism, then the induced map $f_*\colon \pi_1^\circ(X) \to \pi_1^\circ(Y)$ is an isomorphism. The following example shows that this need not be true. Let $X := \mathbb{A}^1$, $Y := \mathbb{P}^1$ and f be the open immersion. Note that since $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$, there are no non-trivial unipotent bundles on Y, and hence by the description of $\mathcal{U}(Y)$, when Y is complete, proceeding as in preceding Lemma 5 we see that $\pi_1^\circ(Y) = 1$, whereas $\pi_1^\circ(X) \ne 1$, since there are non-trivial α_p-torsors on X by Lemma 7.

Acknowledgement The author thanks James Borger for mathematical suggestions.

References

[1] M. Demazure and A. Grothendieck, eds., *Schémas en groupes. II.* (SGA 3) Lecture Notes in Mathematics 152. Springer-Verlag, Berlin, 1962–1964.
[2] J. Giraud, *Cohomologie non-abélienne.* Die Grundlehren der mathematischen Wissenschaften 179. Springer-Verlag, Berlin, 1971.
[3] K. Kedlaya, *On the geometry of p-typical covers in characteristic p.* Canad. J. Math. **60**(2008), no. 1, 140–163. doi:10.4153/CJM-2008-006-8
[4] V. B. Mehta and S. Subramanian, *On the fundamental group scheme*. Invent. Math **148**(2002), no. 1, 143–150. doi:10.1007/s002220100191
[5] M. Nori, *The fundamental group-scheme.* Proc. Indian Acad. Sci. Math. **91**(1982), no. 2, 73–122. doi:10.1007/BF02967978

Department of Mathematics, Koc University, Istanbul, Turkey
e-mail: sunver@ku.edu.tr