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FREE COMMUTATIVE SEMIFIELDS

B.J. Gardner

In memory of Ottó Steinfeld

A description is obtained of the free semifields with both fundamental operations commutative.

A semiring $(A, \#, *)$, where $*$ is distributive over $\#$, is called a *semifield* if either $(A, *)$ is a group or $\#$ has an identity (zero) e and $(A \setminus \{e\}, {}^*)$ is a group. For many purposes there is no great loss of generality in restricting attention to the case where $(A, *)$ is a group (see [4, 5]) and that is what we shall do here. Another reason for imposing this restriction is that the class of algebras so obtained (called *proper semifields* in [2]) is equational. Accordingly, we shall call an algebra $(A, +, \cdot)$ a semifield if $(A, +)$ is a group, $(A, \cdot)$ is a semigroup and $+$ distributes over $\cdot$. (It is perhaps more common to assign names to the operations in the opposite way; we prefer to have the group operation called $+$ because of ring-theoretic usage and because this is the natural notation in lattice ordered groups which constitute an important class of semifields.) We shall call a semifield *commutative* if both its operations are commutative. We shall obtain a description of the free commutative semifields and indicate some connections between these and free lattice ordered abelian groups.

Proposition 1. *Let X be a non-empty set, F the free abelian group on X (operation: $+$), S the free commutative semigroup on F (operation: $\cdot$). Extend $+$ is S by defining*

$$a_1a_2\cdots a_m + b_1b_2\cdots b_n = \prod_{i=1}^{m}\prod_{j=1}^{n}(a_i+b_j).$$

Then $(S, \cdot, +)$ is a commutative semiring.

Proof: We have

$$\begin{aligned}(a_1a_2\cdots a_m + b_1b_2\cdots b_n) + c_1c_2\cdots c_k \\ &= \Pi(a_i+b_j) + c_1\cdots c_k = \Pi((a_i+b_j)+c_r) \\ &= \Pi(a_i+(b_j+c_r)) = a_1a_2\cdots a_n + \Pi(b_j+c_r) \\ &= a_1a_2\cdots a_m + (b_1b_2\cdots b_n + c_1c_2\cdots c_k),\end{aligned}$$

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so $+$ is associative, while

$$\begin{aligned} a_1 a_2 \cdots a_m + b_1 b_2 \cdots b_n &= \Pi(a_i + b_j) = \Pi(b_j + a_i) \\ &= b_1 b_2 \cdots b_n + a_1 a_2 \cdots a_m, \end{aligned}$$

so $+$ is commutative. Finally,

$$\begin{aligned} a_1 a_2 \cdots a_m + (b_1 b_2 \cdots b_n c_1 c_2 \cdots c_k) &= \Pi(a_i + b_j)\Pi(a_i + c_r) \\ &(a_1 a_2 \cdots a_m + b_1 b_2 \cdots b_n)(a_1 a_2 \cdots a_m + c_1 c_2 \cdots c_k), \end{aligned}$$

so $+$ distributes over $\cdot$. □

Proposition 2. *Addition in S is cancellative.*

Proof: Let $a_1 \cdots a_m + b_1 \cdots b_n = a_1 \cdots a_m + c_1 \cdots c_k$, where all factors are in F. Then

(*) $$\prod_{i,j} (a_i + b_j) = \prod_{i,r} (a_i + c_r).$$

As F is torsion-free abelian, it carries a linear order, $<$, and we can assume that $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_m$, $b_1 \leqslant b_2 \leqslant \ldots \leqslant b_n$ and $c_1 \leqslant c_2 \leqslant \ldots \leqslant c_k$. Since we have free generators, the same factors occur on each side of (*). Since clearly $a_1 + b_1 \leqslant a_i + b_j$ for all i, j and $a_1 + c_1 \leqslant a_i + c_r$ for all i, r, we have $a_1 + b_1 = a_1 + c_1$, so $b_1 = c_1$. Thus we can re-write (*) as

$$\prod_i (a_i + b_1) \prod_{j>1;i} (a_i + b_j) = \prod_i (a_i + b_1) \prod_{r>1;i} (a_i + c_r),$$

and on cancelling, we get

$$a_1 \ldots a_m + b_2 \ldots b_n = \prod_{j>1;i} (a_i + b_j) = \prod_{r>1;i} (a_i + c_r) = a_1 \ldots a_m + c_2 \ldots c_k.$$

Repeating this argument, we see that the products $b_1 \ldots b_n$ and $c_1 \ldots c_k$ have the same factors with the same multiplicities. □

As the semiring $(S, \cdot, +)$ has cancellative addition, we can form its *semifield of differences* $D(S)$. (We use this term rather than "semifield of quotients" as our distributive operation is called addition.) For details of this construction see Rédei [**3**, Theorem 93, p.160]. This is our free semifield.

Theorem. *Let X be a non-empty set, F the free abelian group on X, S the free commutative semigroup on F. Define $\prod_{i=1}^{m} a_i + \prod_{j=1}^{n} b_j = \prod_{i,j} (a_i + b_j)$ for all $a_i, b_j \in F$.*

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This makes $(S, \cdot, +)$ a commutative semiring with cancellative addition and $D(S)$ is a free commutative semifield on X.

PROOF: Let A be a commutative semifield, $f\colon X \to A$ a function. Then there is a group homomorphism $f_1\colon F \to A$ defined by

$$f_1\left(\sum n_i x_i\right) = \sum n_i f(x_i).$$

But then as S is free on F, there is a semigroup homomorphism $f_2\colon S \to A$ given by

$$f_2\left(a_1^{m_1} a_2^{m_2} \cdots a_k^{m_k}\right) = f_1(a_1)^{m_1} f_1(a_2)^{m_2} \cdots f_1(a_k)^{m_k},$$

where $a_1, a_2, \cdots, a_k \in F$. Now consider f_2 in relation to $+$ on S. If $a_1, a_2, \cdots, a_m$, $b_1, b_2, \cdots, b_n \in F$, then

$$\begin{aligned} &f_2(a_1 a_2 \cdots a_m + b_1 b_2 \cdots b_n) \\ &\quad = f_2\left(\prod_{i,j}(a_i + b_j)\right) = \prod_{i,j} f_1(a_i + b_j) = \prod_{i,j}(f_1(a_i) + f_1(b_j)) \\ &\quad = \prod_j \left(\prod_i f_1(a_i) + f_1(b_j)\right) = \prod_i f_1(a_i) + \prod_j f_1(b_j) \\ &\quad = f_2(a_1 a_2 \cdots a_m) + f_2(b_1 b_2 \cdots b_n). \end{aligned}$$

Thus f_2 preserves $+$. Finally, we define $f_3\colon D(S) \to A$ by setting

$$f_3(u - v) = f_2(u) - f_2(v)$$

for all $u, v \in S$. It is clear that f_3 is well-defined. For $u, v, w, z \in S$, we have

$$\begin{aligned} f_3((u - v) + (w - z)) &= f_3((u + w) - (v + z)) = f_2(u + w) - f_2(v + z) \\ &= f_2(u) + f_2(w) - f_2(v) - f_2(z) = f_2(u) - f_2(v) + f_2(w) - f_2(z) \\ &= f_3(u - v) + f_3(w - z) \end{aligned}$$

and

$$\begin{aligned} f_3((u - v)(w - z)) &= f_3((u + z)(v + w) - (v + z)) \\ &= f_2((u + z)(v + w)) - f_2(v + z) = (f_2(u) + f_2(z))(f_2(v) + f_2(w)) - f_2(v + z) \\ &= (f_2(u) + f_2(z) - f_2(v + z))(f_2(v) + f_2(w) - f_2(v + z)) \\ &= (f_2(u) - f_2(v))(f_2(w) - f_2(z)) \\ &= f_3(u - v) f_3(w - z). \end{aligned}$$

Thus f_3 is a semifield homomorphism. For $x \in X$, we have

$$\begin{aligned} f_3(x+x-x) &= f_2(x+x) - f_2(x) \\ &= f_2(x) + f_2(x) - f_2(x) \\ &= f_2(x) = f_1(x) = f(x). \end{aligned}$$

But $x + x - x$ is the copy of x in $D(S)$. This completes the proof. ∎

Let η denote the minimum semilattice congruence on $(D(S), \cdot)$. Then $u\eta v$ if and only if each of u, v divides or equals a power of the other [1, p.131]. Since, for example, $uw = v^n$ implies $(t+u)(t+w) = t + uw = t + v^n = (t+v)^n$, η is in fact a semifield (semiring) congruence on $D(S)$, the minimum congruence for the variety defined by the identity $x^2 = x$, that is the class of lattice–ordered abelian groups (see [4, Section 3, Satz 1]). Thus $D(S)/\eta$ is a free lattice-ordered abelian group. Let us examine a bit more closely the relationship between the two free objects $D(S)$ and $D(S)/\eta$.

The original zero, 0, of F is an additive identity for S: if $a_1, \dots a_m \in F$, then

$$a_1 \dots a_m + 0 = (a_1 + 0) \dots (a_m + 0) = a_1 \dots a_m.$$

We can thus identify each $u \in S$ with the corresponding $u - 0$ in $D(S)$. Thus everything in $D(S)$ has the form $u - w$, where $u, w \in S$, so everything in $D(S)/\eta$ has the form $u\eta - w\eta$, where $u, w \in S$. If $u, v \in S$ and $u\eta v$, then either $u(r-s) = v^n$ or $u = v^n$ for some $r, x \in S$, $n \in \mathbb{Z}^+$. In the former case we have $(v+s)^n = v^n + s = u(r-s) + s = (u+s)(r-s+s) = (u+s)r$ and in the latter, $(v+0)^n = u + 0$. Thus there exists $s \in S$ such that $u+s$ divides or equals a power of $v+s$ in S. Similarly there exists $t \in S$ such that $v+t$ divides or equals a power of $u+t$ in S. But if $(v+s)^n = (u+s)r$ and $(u+t)^m = (v+t)q$, then $(v+s+t)^n = (v+s)^n + t = (u+s)r + t = (u+s+t)(r+t)$ and $(u+s+t)^m = (u+t)^m + s = (v+t)q + s = (v+s+t)(q+s)$ while if $(v+s)^n = (u+s)r$ and $(u+t)^m = v+t$, then $(u+s+t)^m = (u+t)^m + s = v+s+t$, and so on. Thus in the above we can replace s, t by $s+t$, that is assume $s = t$. We define the relation η^* on S as follows:

$u\eta^* v$ and only if there exists $s \in S$ such that each of $u+s$, $v+s$ divides or equals a power of the other in S.

This is the congruence induced on S by η, so $D(S)/\eta = D(S/\eta^*)$. Suppose now $a_1 \dots a_m = u\eta^* v = b_1 \dots b_n$, where the factors are in F. Then there exist $c_1, \dots, c_k, e_1, \dots, e_p \in F$ and $\ell \in \mathbb{Z}^+$ such that

$$(u + c_1 \dots c_k)(e_1 \dots e_p) = (v + c_1 \dots c_k)^\ell$$

or

$$(u + c_1 \dots c_k) = (v + c_1 \dots c_k)^\ell,$$

that is $\Pi(b_g + c_h)^\ell = \Pi(a_i + c_j)c_1 \dots e_p$ or $\Pi(a_i + c_j)$. Thus every $a_i + c_j$ is equal to some $b_g + c_h$. The converse holds also. On the other hand, if $\{a_1 + c_1, \dots, a_1 + c_k, \dots, a_m + c_k\} = \{b_1 + c_1, \dots, b_1 + c_k, \dots, b_n + c_k\}$ then $u + c_1 \dots c_k$ $(= \Pi(a_i + c_j))$ divides a power of $v + c_1 \dots c_k$ $(= \Pi(b_g + c_h))$. This gives us another description of S/η^*.

Let $P_f(F)$ denote the set of finite non-empty subsets of F. Let $+$ denote complex addition in $P_f(F)$ $(A + B = \{a + b \colon a \in A,\, b \in B\})$. Then $(P_f(F), +, \cup)$ is a semiring. Let ρ be defined on P_f as follows:

$$A\rho B \textit{ if and only if } A + C = B + C \textit{ for some } C.$$

Then $S/\eta^* \cong P_f(F)/\rho$; $D(S)/\eta$ is then the semifield of quotients of this. Note that $P_f(F)$ is the maximum semilattice (band) homomorphic image of $(F, \cdot)$ and ρ is the minimum cancellative congruence on $(P_f(F), +)$. Each of these congruences is compatible with the "other" operation.

The congruence ρ can be simply described when $|X| = 1$, that is $F = \mathbb{Z}$. If $a, n \in \mathbb{Z}$, $n \geqslant 0$, then

$$\begin{aligned} \{a,\, a+1,\, \dots,\, a+n\} &+ \{a,\, a+1,\, \dots,\, a+n\} \\ &= \{2a,\, 2a+1,\, \dots,\, 2a+n,\, 2a+n+1,\, \dots,\, 2a+2n\} \\ &= \{a,\, a+1,\, \dots,\, a+n\} + \{a,\, a+n\}, \end{aligned}$$

so $\{a,\, a+1,\, \dots,\, a+n\}\rho\{a,\, a+n\}$. It now follows that $\{a,\, \dots,\, a+i,\, \dots,\, a+n\}\rho\{a,\, a+n\}$ for any set of $a + i \in \{a+1, \dots, a+n-1\}$ and thus that $A\rho\{\min(A), \max(A)\}$ for all $A \in P_f(\mathbb{Z})$. If $a, b, c, d \in \mathbb{Z}$, $a \leqslant b$, $c \leqslant d$ and $\{a,\, b\} + C = \{c,\, d\} + C$, then $a + \min(C) = \min(\{a,\, b\} + C) = \min(\{c,\, d\} + C) = c + \min(C)$ so $a = c$ and similarly $b = d$. Thus $\{a,\, b\}\rho\{c,\, d\}$ if and only if $\{a,\, b\} = \{c,\, d\}$ and we have

$$P_f(\mathbb{Z})/\rho \cong \{(a,\, b) \colon a, b \in \mathbb{Z},\, a \leqslant b\}$$

where addition is componentwise and $(a,\, b)(c,\, d) = (\min\{a,\, c\},\, \max\{b,\, d\})$.

The semifield of differences of this, that is the free lattice ordered group on one generator, has an additive group which is free of rank two - an old result of Birkhoff.

The congruence ρ appears to be more difficult to describe for larger X.

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Department of Mathematics
The University of Tasmania
GPO Box 252C
Hobart, Tas. 7001
Australia