EXTENSIONS OF SYLVESTER'S THEOREM

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1. Introduction. Sylvester [7] proposed the following question in 1893. If a finite set of points in a plane is such that on the line determined by any two points of the set there is always a third point of the set, is the set collinear? Equivalently, given a finite planar set of non-collinear points, does there exist a line containing exactly two of the points?

Interest in the question was revived by Erdős [3] and others in 1933 and was answered in the affirmative by Gallai (Grünwald) [4], Steinberg [4], Steenrod [4], A. Robinson [6], Motzkin [6], L. M. Kelly [5] and others in the 1930's and 1940's.

Motzkin showed that the analogous statement in three space is invalid. That is, the statement, "Given a finite set of non-coplanar points in a 3-space, there is a plane spanned by three points of the set which contains only these three points of the set," is false. Motzkin noted this in [6, p. 452] by observing that a set of six points in 3-space, 3 on each of two skew lines, is a counterexample.

Motzkin did conjecture a generalization of Sylvester's Theorem to R^n, an n-dimensional real affine space:
$U_n(n \geq 1)$: Given a finite subset K of R^n which is not contained in any hyperplane, then there is a hyperplane H spanned by points of K such that all but one of the points of $H \cap K$ are in one (n-2)-flat. (The empty set will be considered a flat of dimension -1.)

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Motzkin proved U_3. In this paper U_n is proved for $1 \leq n \leq 5$ and a new proof is given for $n = 3$.

The validity of U_n implies V_{n+1}:

$V_n (n \geq 2)$: If K is a finite subset of R^n lying in no hyperplane, then through each $p \in K$ there are an $(n-2)$-flat F and a line L each determined by points of K and spanning a hyperplane P such that $K \cap P \subset L \cup F$.

The statements U_n and V_n are discussed in section 3.

In section 2 another generalization of Sylvester's Theorem is proved for spaces of arbitrary finite dimension:

$W_n (n \geq 2)$: Let K be a finite subset of R^n such that (a) every subset of K having at most n points is affinely independent, (b) any hyperplane spanned by points of K contains at least $n + 1$ points of K. Then K is contained in a hyperplane.[2]

A set of points $p_0, p_1, \ldots, p_k$ is affinely independent if and only if $p_1 - p_0, p_2 - p_0, \ldots, p_k - p_0$ is a linearly independent set.[3] Since a set of 2 distinct points is affinely independent, W_n generalizes Sylvester's Theorem. The statement W_n is also an extension of Dirac's 3-dimensional version [2, p. 227] of Sylvester's Theorem.

2. The proof of W_n. For the case $n = 2$, W_2 is Sylvester's Theorem. Assume W_{n-1} is true and let p_0 be a point of K. Since K is a finite set of points, choose a hyperplane H not containing p_0 such that every line through p_0

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[2] Following the notation and terminology in [5], W_n could be stated in language motivated by Kelly and Moser's definition of "ordinary line".

[3] The definition is independent of the order in which points are enumerated.

and any other point of K intersects H. For each $p \neq p_o$ in K, let p' denote the point of intersection with H of the line through p_o and p, and let K' denote the collection of points p' for p in K. By the assumption of affine independence this projection determines a 1 - 1 correspondence between points of $K \sim \{p_o\}$ and K'. The proof will be completed by showing that K' satisfies the hypotheses of W_{n-1}. That is, (a') every subset of K' having at most (n-1) points is affinely independent, and (b') on the (n-2)-dimensional flat in H determined by any (n-1) points of K' there is an n-th point of K'.

To prove (a'), suppose there is a subset of $p_1', p_2', \ldots, p_m'$ consisting of $m \leq n-1$ points of K' which are affinely dependent. For each $j = 1, 2, \ldots, m$, let p_j be a point in K which projects from p_o onto p_j'. Then it is readily verified that the set $p_o, p_1, \ldots, p_m$ is an affinely dependent subset of K having at most n points, contradicting (a) of W_n.

To prove (b'), let J be an (n-2)-dimensional-flat determined by (n-1) points $q_1', q_2', \ldots, q_{n-1}'$ of K' and let $q_1, q_2, \ldots, q_{n-1}$ be the corresponding pre-images in K. Then by (b) applied to $p_o, q_1, \ldots, q_{n-1}$ there is an (n+1)-st point r of K on the hyperplane spanned by these points and since r' is distinct from each of $q_1', q_2', \ldots, q_{n-1}'$, it follows that r' is the desired n-th point of K' in J.

Thus, by the inductive hypothesis, all the points of K' lie in an (n-2)-flat F in H. Hence K lies in the hyperplane spanned by F and p_o. Statement W_n is proved.

A finite set B in R^n is an <u>affine basis</u> of R^n if and only if B is affinely independent and affinely spans R^n. (B <u>affinely spans</u> R^n if and only if every element of R^n is a linear combination of elements in B where the sum of the coefficients is equal to one.)

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Then an algebraic formulation of W_n is:

Let K be a nonempty finite set in R^n $(n \geq 2)$ such that (A) Every subset of at most n points of K is affinely independent and (B) K is not contained in a hyperplane. Then, there exists a subset A of K with n points such that for every $x \in K \sim A$, $A \cup \{x\}$ is an affine basis for R^n.

3. Motzkin's Conjecture. The generalization of Sylvester's Theorem to dimensions greater than 2 conjectured by Th. Motzkin [6] and proved by Motzkin in spaces of dimension 3 is proved here to be valid up to and including 5 dimensions. A new proof is presented for Motzkin's conjecture for spaces of dimension 3.

The proof of the generalization will follow by induction on the dimension n. In order to make clear the difficulties encountered when the dimension is greater than 5, the argument will be presented for arbitrary dimension to the place where our argument requires that the dimension be less than 6. The question of the validity of Motzkin's conjecture in spaces of dimension more than 5 is still open.

Definition. Let K be a subset of R^n, an n-dimensional affine space over the real number field. A j-dimensional flat M spanned by points of K is called a (K, j)-motzkin if and only if there is a $(j-1)$-flat $G \subset M$ and a point $p \in K \cap M$ so that p is the only point of $K \cap M$ not in G. In this situation M will be denoted by $p{:}G$. If $j = n-1$, M will be called a K-motzkin.

With this terminology, Motzkin's conjecture for a space of dimension n, becomes:

U_n $(n \geq 1)$: If K is a finite subset of R^n contained in no hyperplane, then there is a K-motzkin in R^n. (The empty set will be considered to be a flat of dimension -1.)

THEOREM 3.1. Statement U_n is true for $1 \leq n \leq 5$.

For A and B subsets of R^n, the flat spanned by A and B will be denoted by AB. The singleton $\{a\}$ will be denoted without brackets, a.

The following lemma is clear. It is basic in the proof of Theorem 3.1 and in Steinberg's [4] and Motzkin's [6, p. 452] elegant proof of Sylvester's Theorem.

LEMMA 3.2. Let x, a', b', c' be 4 distinct points on a line H' such that a' separates x and b' but c' does not. Let p be a point off H' and r' be a point on the line pb' such that $r' \neq p$, $r' \neq b'$. Then either $r'a'$ or $r'c'$ intersects the open segment (p, x).

Let K be a set of points of R^n satisfying the hypotheses of U_n. Choose a point $p \in K$. An easy argument shows that there is a line X through p so that X intersects every hyperplane spanned by points of K each in a single point.

By the finiteness of K there is a hyperplane H spanned by points of K which intersects X in $x \neq p$ so that no hyperplane spanned by points of K intersects the open segment (p, x). Choose a line H' in H and through the point x so that plane XH' intersects every (n-2)-flat spanned by points of $(K \cap H)$ each in a single point. It can be proved that H' exists.

LEMMA 3.3. Consider p, H, H', X and x as before. Assume there is no K-motzkin in R^n. Then, (a) there is an (n-3)-flat H_{n-3} in H spanned by points of K and there are 3 points a, b, and c of $K \cap H$ not in H_{n-3} so that $a' = H' \cap aH_{n-3}$, $b' = H' \cap bH_{n-3}$, $c' = H' \cap cH_{n-3}$ are situated as in Lemma 3.2 with respect to x. Also, (b) there is an $r \in K \cap pbH_{n-3}$, $r \neq p$, $r \notin bH_{n-3}$, (c) all the r's as in (b) are in pH_{n-3}, and (d) bH_{n-3} is not of the form $b{:}H_{n-3}$.

<u>Proof of 3.3.</u> (a) Assume there is no (n-3)-flat H_{n-3} with associated points a, b, and c so that aH_{n-3}, bH_{n-3} and cH_{n-3} are distinct. There must be at least 2 distinct flats, e.g., $aH_{n-3} \neq bH_{n-3}$, for otherwise H would have dimension n-2, a contradiction. Thus, since H is not a K-motzkin, there must be a $c \in K \cap aH_{n-3}$ or $c \in K \cap bH_{n-3}$,

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say $c \in aH_{n-3}$, and $c \neq a$, $c \notin H_{n-3}$. Let $B = \{k_o, \ldots, k_{n-3}\}$ be an affine basis of H_{n-3}, $B \subset K$. Let $k_o \in B$ be such that c depends on a and k_o, i.e.,

$$c = \lambda a + \sum_{i=0}^{n-3} \lambda_i k_i, \text{ where } \lambda + \sum_{i=0}^{n-3} \lambda_i = 1$$

$$\text{and } \lambda_o \neq 0 , \quad \lambda \neq 0 .$$

Consider $\bar{H}_{n-3}$, the affine hull of $\{b, k_1, \ldots, k_{n-3}\}$. Then, $a\bar{H}_{n-3}$, $k_o\bar{H}_{n-3}$, $c\bar{H}_{n-3}$ are distinct. Since $k_o\bar{H}_{n-3} = bH_{n-3}$, it follows that $c\bar{H}_{n-3} \neq k_o\bar{H}_{n-3}$ and $a\bar{H}_{n-3} \neq k_o\bar{H}_{n-3}$. If $c\bar{H}_{n-3} = a\bar{H}_{n-3}$, then

$$(\alpha)\ c = \lambda a + \lambda' b + \sum_{i=0}^{n-3} \lambda_i k_i,$$

$$\text{where } \lambda + \lambda' + \sum_{i=0}^{n-3} \lambda_i = 1 ,$$

and $\lambda_o = 0$.

Also, since $c \in aH_{n-3}$,

$$(\beta)\ c = \mu a + \sum_{i=0}^{n-3} \mu_i k_i , \text{ where } \mu + \sum_{i=0}^{n-3} \mu_i = 1 .$$

Subtracting (β) from (α),

$$(\gamma)\ 0 = (\lambda - \mu)a + \lambda' b + \sum_{i=0}^{n-3} (\lambda_i - \mu_i)k_i .$$

Since c depends on k_o, from (α) it follows that $\lambda' \neq 0$.

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Solving for b in (γ), it follows that $b \in aH_{n-3}$, a contradiction.

(b) If there is no $r \in pbH_{n-3}$, $r \notin bH_{n-3}$, $r \neq p$, then $p{:}bH_{n-3}$ is a K-motzkin.

(c) If $r \in pbH_{n-3}$, $r \neq p$, $r \notin bH_{n-3}$, $r \notin pH_{n-3}$, consider $r' = rH_{n-3} \cap XH'$. The point r' is on the line pb' and $r' \neq p$, $r' \neq b'$. Thus, by Lemma 3.2, either $r'a'$ or $r'c'$ intersects the open segment (p, x). Hence, either the hyperplane arH_{n-3} or crH_{n-3} intersects (p, x), contradicting the choice of H.

Statement (d) follows directly from (c) since if $bH_{n-3} = b{:}H_{n-3}$, then $bpH_{n-3} = b{:}pH_{n-3}$, a K-motzkin.

LEMMA 3.4. Assume that there is no K-motzkin in R^n and U_k is valid, $k < n$. Let a, b, c, H_{n-3} be as in Lemma 3.3. Let $U \cup V \subset K$ be an affine basis of H_{n-3} so that U and V are disjoint and

(*) if $r \in K \cap pbH_{n-3}$, $r \neq p$, and $r \notin bH_{n-3}$ then $r \in pV$.

If H_{n-3}, a, b, c, U and V are such that V has minimal cardinality with respect to property (*) and if $U \cup V$ is nonempty, then (a) both U and V are nonempty and (b) if $\bar{a} \in K \cap H$ and $\bar{a} \notin H_{n-3}$, then $\bar{a} \in abU$.

<u>Proof.</u> (a) Observe that $V \neq \emptyset$, for otherwise $pbH_{n-3} = p{:}bH_{n-3}$, a K-motzkin. By assumption, $K \cap H$ satisfies U_{n-1} in H. Thus, there is a $(K \cap H, (n-2))$-motzkin $\tau{:}H_{n-3}(0)$ in H. Since, by assumption, H is not a K-motzkin, by Lemma 3.3, it may be assumed that there are three points a_o, b_o, $c_o \in K \cap H$ with a_o or $c_o = \tau$, so that $a'_o = H' \cap a_oH_{n-3}(0)$, $b'_o = H' \cap b_oH_{n-3}(0)$, $c'_o = H \cap c_oH_{n-3}(0)$

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are distinct points situated as in Lemma 3.2. For if a'_o, b'_o, c'_o were not distinct, then, for some $d \in K \cap H$, $H = \tau : dH_{n-3}(0)$ is a K-motzkin and, by 3.3(d), $b_o \neq \tau$. Let $k_o, k_1, \ldots, k_{n-3}$ be an affine basis for $H_{n-3}(0)$. Then $b_o, k_o, \ldots, k_{n-3}$ is an affine basis of $b_o H_{n-3}(0)$. By Lemma 3.3(d), there is a $\bar{b}_o \in K \cap b_o H_{n-3}(0)$, $\bar{b}_o \notin H_{n-3}(0)$, $\bar{b}_o \neq b_o$. Thus, $\bar{b}_o = \lambda b_o + \sum_{i=0}^{n-3} \lambda_i k_i$, where

$\lambda + \sum_{i=0}^{n-3} \lambda_i = 1$, $\lambda \neq 0$, and some $\lambda_i \neq 0$, say λ_o.

Consider the (n-3)-flats $H_{n-3}(1) = b_o k_1 \ldots k_{n-3}$ and $\bar{H}_{n-3}(1) = \bar{b}_o k_1 \ldots k_{n-3}$. Then either $a_o H_{n-3}(1)$, $k_o H_{n-3}(1)$, $c_o H_{n-3}(1)$ are distinct (n-2)-flats in H or $a_o \bar{H}_{n-3}(1)$, $k_o \bar{H}_{n-3}(1)$, $c_o \bar{H}_{n-3}(1)$ are distinct. For assume not. (It may be assumed without restriction that $a_o = \tau$.) Then,

$$(1) \quad c_o = \mu\tau + \mu' b_o + \sum_{i=1}^{n-3} \mu_i k_i, \quad \text{where } \mu + \mu' + \sum_{i=1}^{n-3} \mu_i = 1$$

and

$$(2) \quad c_o = \bar{\mu}\tau + \bar{\mu}' \bar{b}_o + \sum_{i=1}^{n-3} \bar{\mu}_i k_i, \quad \text{where } \bar{\mu} + \bar{\mu}' + \sum_{i=1}^{n-3} \bar{\mu}_i = 1 .$$

Subtracting (2) from (1),

$$(3) \quad 0 = (\mu - \bar{\mu})\tau + \mu' b_o - \bar{\mu}' \bar{b}_o + \sum_{i=1}^{n-3} (\mu_i - \bar{\mu}_i) k_i .$$

Since $\tau \notin b_o H_{n-3}(0)$, $\mu = \bar{\mu}$. Further, from (1) and (2), since $\tau : H_{n-3}(0)$ is (K, (n-2))-motzkin, μ' and $\bar{\mu}'$ are not zero. Thus, solving (3) for $\bar{b}_o$, we have $\bar{b}_o \in b_o k_1 \ldots k_{n-3}$, which contradicts $\lambda_o \neq 0$.

Thus, it may be assumed that the line H' intersects $a_oH_{n-3}(1)$, $k_oH_{n-3}(1)$, $c_oH_{n-3}(1)$ in 3 distinct points a'_1, b'_1, c'_1 where a_1, b_1, c_1 are a_o, k_o, c_o renamed so that a'_1 is in the open interval (x, b'_1) but c'_1 is not. By Lemma 3.3, every $r_1 \in K \cap pb_1H_{n-3}(1)$, $r_1 \neq p$, $r_1 \notin b_1H_{n-3}(1)$ is in $pH_{n-3}(1)$ and there is such an r_1. But, if $r_1 \in pH_{n-3}(1)$, then, $r_1 \in pb_oH_{n-3}(0)$, and, again by Lemma 3.3, $r_1 \in pH_{n-3}(0)$. Thus, every r_1 is in $pH_{n-3}(0) \cap pH_{n-3}(1) = pk_1 \ldots k_{n-3}$. Hence, U is nonempty.

(b) Let $U = \{b_o, b_1, \ldots, b_j\}$ and $V = \{k_{j+1}, k_{j+2}, \ldots, k_{n-3}\}$. It may be assumed from 3.3 (d) that there is a $\bar{b} \in K \cap bH_{n-3}$,

$$\bar{b} = \lambda b + \sum_{i=0}^{j} \lambda_i b_i + \sum_{i=j+1}^{n-3} \lambda_i k_i \,, \quad \text{where } \lambda + \sum_{i=0}^{n-3} \lambda_i = 1 \,, \quad \lambda \neq 0$$

and some $\lambda_i \neq 0$. It will be shown that all such $\bar{b}$'s are in bU.

Suppose that $\lambda_{j+1} \neq 0$.

Let $G_{n-3} = b_o \ldots b_j b k_{j+2} \ldots k_{n-3}$ and $\bar{G}_{n-3} = b_o \ldots b_j \bar{b} k_{j+2} \ldots k_{n-3}$. Then, either the flats $a\bar{G}_{n-3}$, $k_{j+1}\bar{G}_{n-3}$, $c\bar{G}_{n-3}$ are distinct or aG_{n-3}, $k_{j+1}G_{n-3}$, cG_{n-3} are distinct. For if not, $aG_{n-3} = cG_{n-3}$ and $a\bar{G}_{n-3} = c\bar{G}_{n-3}$, so c is represented by the following affine combinations,

$$(1) \quad c = \mu a + \sum_{i=0}^{j} \mu_i b_i + \mu' b + \sum_{i=j+2}^{n-3} \mu_i k_i \,, \quad \text{and}$$

$$(2) \quad c = \bar{\mu} a + \sum_{i=0}^{j} \bar{\mu}_i b_i + \bar{\mu}' \bar{b} + \sum_{i=j+2}^{n-3} \bar{\mu}_i k_i \,.$$

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Subtracting (2) from (1),

$$(3)\quad 0 = (\mu - \bar{\mu})a + \sum_{i=0}^{j} (\mu_i - \bar{\mu}_i)b_i + \mu' b - \bar{\mu}' \bar{b}$$

$$+ \sum_{i=j+2}^{n-3} (\mu_i - \bar{\mu}_i)k_i .$$

Thus, $\mu = \bar{\mu}$, since $a \notin bH_{n-3}$. If $\bar{\mu}' \neq 0$, solving (3) for $\bar{b}$, $\bar{b} \in b_o \ldots b_j bk_{j+2} \ldots k_{n-3}$, contradicting that $\lambda_{j+1} \neq 0$.

Thus, it may be assumed that $\mu' = 0$ and $\bar{\mu}' = 0$. From (2), $c = \bar{\mu}a + \sum_{i=0}^{j} \bar{\mu}_i b_i + \sum_{i=j+2}^{n-3} \bar{\mu}_i k_i$. This implies that $cH_{n-3} = aH_{n-3}$, a contradiction.

Hence, 3 distinct (n-2)-flats exist and will be denoted by a_1G_{n-3}, b_1G_{n-3}, c_1G_{n-3} where $a_1' = H' \cap a_1G_{n-3}$, separates x and $b_1' = H' \cap b_1G_{n-3}$, but $c_1' = H' \cap c_1G_{n-3}$ does not.

Since it is assumed that there is no K-motzkin, by 3.3 there is an $r_1 \in K \cap pb_1G_{n-3}$, $r_1 \neq p$, $r_1 \notin b_1G_{n-3}$, and all such r_1's are in $pG_{n-3} \subset pbH_{n-3}$. Thus, by 3.3, all such r_1's are in pH_{n-3} and by (*) are in pV. Hence all of the r_1's are in $pV \cap pG_{n-3} = pk_{j+2} \ldots k_{n-3}$. This contradicts the minimality of V. Thus $\bar{b} \in bU$.

Now assume there is an $\bar{a} \in K \cap H$, $\bar{a} \notin bH_{n-3}$, $\bar{a} \neq a$, which, with respect to the affine basis $\{a, b\} \cup U \cup V$ for H, depends on an element in V, say k_{j+1}. Thus,

$$\bar{a} = \nu a + \nu' b + \sum_{i=0}^{j} \nu_i b_i + \sum_{i=j+1}^{n-3} \nu_i k_i \quad \text{where } \nu + \nu' + \sum_{i=0}^{n-3} \nu_i = 1,\ \nu \neq 0,$$
$$\nu_{j+1} \neq 0 .$$

As above let $G_{n-3} = b_o \dots b_j b k_{j+2} \dots k_{n-3}$. Then, the flats aG_{n-3}, $k_{j+1}G_{n-3}$, $\bar{a}G_{n-3}$ are distinct. For otherwise, $aG_{n-3} = \bar{a}G_{n-3}$ and hence $\bar{a}$ is represented as the affine combination $\bar{a} = \mu a + \mu' b + \sum_{i=0}^{j} \mu_i b_i + \sum_{i=j+2}^{n-3} \mu_i k_i$, which contradicts the dependency of $\bar{a}$ on k_{j+1}. Repeating the argument as above, it may be assumed that there is an r_1 in $K \cap pG_{n-3}$, with $r_1 \neq p$, $r_1 \notin b_1 G_{n-3}$ and all such r_1's are in $pV \cap pG_{n-3} = pk_{j+2} \dots k_{n-3}$, contradicting the minimality of V.

Thus, every $\bar{a} \in K \cap H$, $\bar{a} \notin bH_{n-3}$, is in $ab_o \dots b_j b = abU$ and part (b) of 3.4 is proved.

The proof of Theorem 3.1 will now be presented. The notation of Lemma 3.4 is used.

For $n = 1$, U_n is trivially true. For $n = 2$, U_n is Sylvester's Theorem. For $n = 3$, H_{n-3} is 0-dimensional, so in Lemma 3.4 either U is empty or V is empty. Hence, either $H = b{:}aH_{n-3}$ or $pbH_{n-3} = b{:}pH_{n-3}$.

Case: $n = 4$. For $n = 4$, H_{n-3} is 1-dimensional. Assuming there is no K-motzkin in R^4, by Lemma 3.4(a), U and V must each have exactly one element, say $U = \{u_o\}$ and $V = \{v_1\}$. It will be shown that $abu_o v_1 = v_1{:}abu_o$, a K-motzkin, or that $bpH_{n-3} = b{:}pH_{n-3}$, a K-motzkin. If $abu_o v_1$ does not have the form $v_1{:}abu_o$ then there is a $w \in K \cap abu_o v_1$ which, with respect to $\{a, b\} \cup U \cup V$, is affinely dependent on v_1 and abu_o. By Lemma 3.4(b), w is in H_{n-3} so w depends only on v_1 and u_o. Letting $U_o = \{w\}$, $U_o \cup V$ is an affine basis for H_{n-3} which satisfies (*) of Lemma 3.4 and moreover V is minimal with respect to property (*). Thus, all

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the $\bar{a}$'s as in Lemma 3.4(b) are in $abu_o \cap abw = ab$. So, bH_{n-3} has the form $b:H_{n-3}$, and it follows that $bpH_{n-3} = b:pH_{n-3}$, a K-motzkin.

Case: $n = 5$. For $n = 5$, H_{n-3} is 2-dimensional. Assuming that there is no K-motzkin in R^5, by Lemma 3.4(a), V has one or two elements.

Suppose that V has two elements v_1 and v_2 and that $U = \{u_o\}$. If there is a $w \in K \cap H$ which, with respect to $\{a, b\} \cup U \cup V$, is affinely dependent on both V and abU, then by Lemma 3.4(b), w is in H_{n-3}, so w depends only on V and U. Letting $U_o = \{w\}$, then, as in the case $n = 4$, $bpH_{n-3} = b:pH_{n-3}$, a K-motzkin.

So assume every w in $K \cap H$ is affinely dependent with respect to $\{a, b\} \cup U \cup V$ on only one of V or abU. If the flat aff V spanned by V is a (K, 1)-motzkin, then aff $V = v_1:v_2$, so $H = v_1:v_2\ abu_o$, a K-motzkin. If aff V is not a (K, 1)-motzkin, then there is a w in K which is affinely dependent on v_1 and v_2, so $w:abu_o$, $v_1:abu_o$, and $v_2:abu_o$ are distinct (K, n-2)-motzkins in H, contradicting Lemma 3.3(d) unless there is a K-motzkin.

There remains only the case where V has one element v_2 and U has two, u_o and u_1. If there is no w in $K \cap H$ which, with respect to $\{a, b\} \cup U \cup V$, is affinely dependent on both V and abU then $H = v_2:abU$, a K-motzkin. So suppose there is such a w in $K \cap H$. Then by Lemma 3.4(c) $w \in abU$ or $w \in H_{n-3}$. If there is no w in H_{n-3} then $H = v_2:abU$, a K-motzkin. If there is a w in H_{n-3}, w is an affine combination of v_2 and at least one of u_o and u_1. If w is affinely dependent on both u_o and u_1, let $U_o = \{w, u_1\}$

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and $U_1 = \{u_o, w\}$. Then $U_i \cup V$ $(i = 0, 1)$ is an affine basis for H_{n-3} which satisfies (*) of 3.4 and also V is minimal with respect to property (*). Hence by 3.4, all the $\bar{a}$'s as in 3.4(b) are in $abU \cap abU_o \cap abU_1 = ab$. Thus, $pbH_{n-3} = b{:}pH_{n-3}$, a K-motzkin. So assume that each w in $K \cap H_{n-3}$ is affinely dependent only on u_1. Thus, $v_2{:}abu_o$, $w{:}abu_o$, and $u_1{:}abu_o$ are distinct $(K, n-2)$-motzkins in H, contradicting Lemma 3.3(d) unless there is a K-motzkin.

The theorem is proved.

For dimension $n \geq 6$, the cases where V has more than 2 elements must be treated in order to verify or disprove U_n. The question remains open.

THEOREM 3.6. U_n implies V_{n+1}.

Consider K in R^{n+1}. Let p be a point of K. Let H be a hyperplane in R^{n+1} not containing p such that each line through p and any other point k of K intersects H in a point k' and consider $K' = \{k' \mid k \in K\}$.

If K' lies in an $(n-1)$-flat, then K lies in a hyperplane of R^{n+1} contradicting the hypothesis. Suppose K' does not lie in an $(n-1)$-dimensional subflat of H. Then, by U_n, there is a $(K', (n-1))$-motzkin $P' = k'_o : F'$ in H. Hence $P = pk'_o F'$, where $L = pk'_o$ and $F = pF'$ satisfies V_{n+1}.

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