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p-ADIC QUOTIENT SETS: LINEAR RECURRENCE SEQUENCES

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Abstract

Let $(x_n)_{n\geq 0}$ be a linear recurrence of order $k \geq 2$ satisfying $x_n = a_1x_{n-1} + a_2x_{n-2} + \cdots + a_kx_{n-k}$ for all integers $n \geq k$, where $a_1, \ldots, a_k, x_0, \ldots, x_{k-1} \in \mathbb{Z}$, with $a_k \neq 0$. Sanna ['The quotient set of k-generalised Fibonacci numbers is dense in $\mathbb{Q}_p$', *Bull. Aust. Math. Soc.* **96**(1) (2017), 24–29] posed the question of classifying primes p for which the quotient set of $(x_n)_{n\geq 0}$ is dense in $\mathbb{Q}_p$. We find a sufficient condition for denseness of the quotient set of the kth-order linear recurrence $(x_n)_{n\geq 0}$ satisfying $x_n = a_1x_{n-1} + a_2x_{n-2} + \cdots + a_kx_{n-k}$ for all integers $n \geq k$ with initial values $x_0 = \cdots = x_{k-2} = 0, x_{k-1} = 1$, where $a_1, \ldots, a_k \in \mathbb{Z}$ and $a_k = 1$. We show that, given a prime p, there are infinitely many recurrence sequences of order $k \geq 2$ whose quotient sets are not dense in $\mathbb{Q}_p$. We also study the quotient sets of linear recurrence sequences with coefficients in certain arithmetic and geometric progressions.

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1. Introduction and statement of results

For a set of integers A, the set $R(A) = \{a/b : a, b \in A, b \neq 0\}$ is called the ratio set or quotient set of A. Several authors have studied the denseness of ratio sets of different subsets of $\mathbb{N}$ in the positive real numbers (see [3, 5–7, 15, 16–20, 24, 25, 29, 30]). An analogous study has also been done for algebraic number fields (see [12, 28]).

For a prime p, let $\mathbb{Q}_p$ denote the field of p-adic numbers. The denseness of ratio sets in $\mathbb{Q}_p$ has been studied by several authors (see [1, 2, 10, 13, 14, 21–23, 27]). Let $(F_n)_{n\geq 0}$ be the sequence of Fibonacci numbers, defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all integers $n \geq 2$. In [14], Garcia and Luca showed that the ratio set of Fibonacci numbers is dense in $\mathbb{Q}_p$ for all primes p. Later, Sanna [27, Theorem 1.2] showed that, for any $k \geq 2$ and any prime p, the ratio set of the k-generalised Fibonacci numbers is dense in $\mathbb{Q}_p$. Sanna remarked that his result could be extended to other linear recurrences over the integers. However, he used some specific properties of the k-generalised Fibonacci numbers in the proof. Therefore, he asked the following question.

QUESTION 1.1 [27, Question 1.3]. Let $(S_n)_{n\geq 0}$ be a linear recurrence of order $k \geq 2$ satisfying $S_n = a_1 S_{n-1} + a_2 S_{n-2} + \cdots + a_k S_{n-k}$ for all integers $n \geq k$, where $a_1, \ldots, a_k, S_0, \ldots, S_{k-1} \in \mathbb{Z}$, with $a_k \neq 0$. For which prime numbers p is the quotient set of $(S_n)_{n\geq 0}$ dense in $\mathbb{Q}_p$?

In [13], Garcia *et al.* studied the quotient sets of certain second-order recurrences: given two fixed integers r and s, let $(a_n)_{n\geq 0}$ be defined by $a_n = ra_{n-1} + sa_{n-2}$ for $n \geq 2$ with initial values $a_0 = 0$ and $a_1 = 1$, and let $(b_n)_{n\geq 0}$ be defined by $b_n = rb_{n-1} + sb_{n-2}$ for $n \geq 2$ with initial values $b_0 = 2$ and $b_1 = r$.

THEOREM 1.2 [13, Theorem 5.2]. *With the notation as above, let $A = \{a_n : n \geq 0\}$ and $B = \{b_n : n \geq 0\}$.*

(a) *If $p \mid s$ and $p \nmid r$, then $R(A)$ is not dense in $\mathbb{Q}_p$.*
(b) *If $p \nmid s$, then $R(A)$ is dense in $\mathbb{Q}_p$.*
(c) *For all odd primes p, $R(B)$ is dense in $\mathbb{Q}_p$ if and only if there exists a positive integer n such that $p \mid b_n$.*

We study ratio sets of some other linear recurrences over the set of integers. Our results give some answers to Question 1.1. Our first result gives a sufficient condition for the denseness of the ratio sets of certain kth-order recurrence sequences. Finding a general solution to Question 1.1 seems to be a difficult problem. Hence, in Theorem 1.3, we consider kth-order recurrence sequences for which $a_k = 1$ and with initial values $x_0 = \cdots = x_{k-2} = 0$, $x_{k-1} = 1$. Recall that a *Pisot number* is a positive algebraic integer greater than 1 all of whose conjugate elements have absolute value less than 1.

THEOREM 1.3. *Let $(x_n)_{n\geq 0}$ be a kth-order linear recurrence satisfying*

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \cdots + a_{k-1} x_{n-k+1} + x_{n-k}$$

for all integers $n \geq k$ with initial values $x_0 = x_1 = \cdots = x_{k-2} = 0, x_{k-1} = 1$ and $a_1, \ldots, a_{k-1} \in \mathbb{Z}$. Suppose that the characteristic polynomial of the recurrence sequence has a root $\pm\alpha$, where α is a Pisot number. If p is a prime such that the characteristic polynomial of the recurrence sequence is irreducible in $\mathbb{Q}_p$, then the quotient set of $(x_n)_{n\geq 0}$ is dense in $\mathbb{Q}_p$.

If we take $k = 3$ in Theorem 1.3, then we have the following corollary.

COROLLARY 1.4. *Let $(x_n)_{n\geq 0}$ be a third-order linear recurrence satisfying*

$$x_n = ax_{n-1} + bx_{n-2} + x_{n-3}$$

for all integers $n \geq 3$ with initial values $x_0 = x_1 = 0, x_2 = 1$, where the integers a and b are such that $(a + b)(b - a - 2) < 0$. If p is a prime such that the characteristic polynomial of the recurrence sequence is irreducible in $\mathbb{Q}_p$, then the quotient set of $(x_n)_{n\geq 0}$ is dense in $\mathbb{Q}_p$.

We discuss two examples as applications of Corollary 1.4.

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EXAMPLE 1.5. For $a \in \mathbb{N}$, let ℓ be an odd positive integer less than $2a$. Let $(x_n)_{n\geq 0}$ be a linear recurrence satisfying $x_n = ax_{n-1} + (a-\ell)x_{n-2} + x_{n-3}$ for all integers $n \geq 3$ with initial values $x_0 = x_1 = 0, x_2 = 1$. Then a and $b := a - \ell$ satisfy $(a+b)(b-a-2) < 0$. The characteristic polynomial $p(x) = x^3 - ax^2 - (a-\ell)x - 1$ is irreducible in $\mathbb{Q}_2$ because $p(0) \neq 0$ and $p(1) = -2a + \ell \not\equiv 0 \pmod 2$. Therefore, by Theorem 1.3, $R((x_n)_{n\geq 0})$ is dense in $\mathbb{Q}_2$.

EXAMPLE 1.6. For $a \in \mathbb{N}$ such that $3 \nmid a$, let ℓ be an odd positive integer less than $2a$ and such that $3 \mid \ell$. Let $(x_n)_{n\geq 0}$ be a linear recurrence satisfying $x_n = ax_{n-1} + (a-\ell)x_{n-2} + x_{n-3}$ for all integers $n \geq 3$ with initial values $x_0 = x_1 = 0, x_2 = 1$. Then a and $b = a - \ell$ satisfy $(a+b)(b-a-2) < 0$. The characteristic polynomial $p(x) = x^3 - ax^2 - (a-\ell)x - 1$ is irreducible in $\mathbb{Q}_3$ because $p(0) \neq 0$, $p(1) = -2a + \ell \not\equiv 0 \pmod 3$ and $p(2) = -6a + 2\ell + 7 \not\equiv 0 \pmod 3$. Therefore, by Theorem 1.3, $R((x_n)_{n\geq 0})$ is dense in $\mathbb{Q}_3$.

Next, we consider recurrence sequences whose nth term depends on all the previous $n-1$ terms and obtain the following results.

THEOREM 1.7. *Let $(x_n)_{n\geq 0}$ be a linear recurrence satisfying*

$$x_n = x_{n-1} + 2x_{n-2} + \cdots + (n-1)x_1 + nx_0$$

for all integers $n \geq 1$ with initial value $x_0 = 1$. Then the quotient set of $(x_n)_{n\geq 0}$ is dense in $\mathbb{Q}_p$ for all primes p.

The recurrence relation given in Theorem 1.7 generates a subsequence of the Fibonacci sequence.

THEOREM 1.8. *Let $(x_n)_{n\geq 0}$ be a linear recurrence satisfying*

$$x_n = ax_{n-1} + arx_{n-2} + \cdots + ar^{n-1}x_0$$

for all integers $n \geq 1$, with $x_0, a, r \in \mathbb{Z}$. Then the quotient set of $(x_n)_{n\geq 0}$ is not dense in $\mathbb{Q}_p$ for all primes p.

In Theorem 1.2, Garcia *et al.* studied second-order recurrence relations with specific initial values. In the following result, we consider a particular second-order recurrence sequence with arbitrary initial values x_0 and x_1 in the set of integers.

THEOREM 1.9. *Let $(x_n)_{n\geq 0}$ be a second-order linear recurrence satisfying $x_n = 2ax_{n-1} - a^2x_{n-2}$ for all integers $n \geq 2$, where $a, x_0, x_1 \in \mathbb{Z}$. Then the quotient set of $(x_n)_{n\geq 0}$ is dense in $\mathbb{Q}_p$ for all primes p satisfying $p \nmid a(x_1 - ax_0)$.*

For a prime p, let v_p denote the p-adic valuation. The following theorem gives a set of linear recurrence sequences of order k whose ratio sets are not dense in $\mathbb{Q}_p$.

THEOREM 1.10. *Let $(x_n)_{n\geq 0}$ be a linear recurrence of order $k \geq 2$ satisfying*

$$x_n = a_1x_{n-1} + \cdots + a_kx_{n-k}$$

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for all integers $n \geq k$, where $x_0, \ldots, x_{k-1}, a_1, \ldots, a_k \in \mathbb{Z}$. If p is a prime such that $p \nmid a_k$ and $\min\{\nu_p(a_j) : 1 \leq j < k\} > \max\{\nu_p(x_m) - \nu_p(x_n) : 0 \leq m, n < k\}$, then the quotient set of $(x_n)_{n\geq 0}$ is not dense in $\mathbb{Q}_p$.

The next example is an application of Theorem 1.10. Given a prime p, this example gives infinitely many recurrence sequences of order $k \geq 2$ whose quotient sets are not dense in $\mathbb{Q}_p$.

EXAMPLE 1.11. Let $(x_n)_{n\geq 0}$ be a linear recurrence of order $k \geq 2$ satisfying

$$x_n = a_1 x_{n-1} + \cdots + a_k x_{n-k}$$

for all integers $n \geq k$, where $x_0 = x_1 = \cdots = x_{k-1} = 1$ and $a_1, \ldots, a_k \in \mathbb{Z}$. If p is a prime such that $p \mid a_j, 1 \leq j \leq k-1$, and $p \nmid a_k$, then by Theorem 1.10, the quotient set of $(x_n)_{n\geq 0}$ is not dense in $\mathbb{Q}_p$.

2. Preliminaries

Let p be a prime and r be a nonzero rational number. Then r has a unique representation of the form $r = \pm p^k a/b$, where $k \in \mathbb{Z}, a, b \in \mathbb{N}$ and $\gcd(a, p) = \gcd(p, b) = \gcd(a, b) = 1$. The p-adic valuation of r is $\nu_p(r) = k$ and its p-adic absolute value is $\|r\|_p = p^{-k}$. By convention, $\nu_p(0) = \infty$ and $\|0\|_p = 0$. The p-adic metric on $\mathbb{Q}$ is $d(x, y) = \|x - y\|_p$. The field $\mathbb{Q}_p$ of p-adic numbers is the completion of $\mathbb{Q}$ with respect to the p-adic metric. The p-adic absolute value can be extended to a finite normal extension field K over $\mathbb{Q}_p$ of degree n. For $\alpha \in K$, define $\|\alpha\|_p$ as the nth root of the determinant of the matrix of the linear transformation from the vector space K over $\mathbb{Q}_p$ to itself defined by $x \mapsto \alpha x$ for all $x \in K$. Also, define $\nu_p(\alpha)$ as the unique rational number satisfying $\|\alpha\|_p = p^{-\nu_p(\alpha)}$.

The following results will be used in the proofs of our theorems.

LEMMA 2.1 [13, Lemma 2.1]. *If S is dense in $\mathbb{Q}_p$, then for each finite value of the p-adic valuation, there is an element of S with that valuation.*

LEMMA 2.2 [13, Lemma 2.3]. *Let $A \subset \mathbb{N}$.*

(1) *If A is p-adically dense in $\mathbb{N}$, then $R(A)$ is dense in $\mathbb{Q}_p$.*
(2) *If $R(A)$ is p-adically dense in $\mathbb{N}$, then $R(A)$ is dense in $\mathbb{Q}_p$.*

THEOREM 2.3 [4, Theorem 1]. *Let $\alpha_1, \ldots, \alpha_n$ be units in Ω_p, the completion of the algebraic closure of $\mathbb{Q}_p$, which are algebraic over the rationals $\mathbb{Q}$ and whose p-adic logarithms are linearly independent over $\mathbb{Q}$. These logarithms are then linearly independent over the algebraic closure of $\mathbb{Q}$ in Ω_p.*

3. Proof of the theorems

PROOF OF THEOREM 1.3. Let $p(x) = x^k - a_1 x^{k-1} - a_2 x^{k-2} - \cdots - a_{k-1} x - 1$ be the characteristic polynomial of the recurrence. Let $\alpha_1, \ldots, \alpha_k$ be the k distinct roots of the

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characteristic polynomial in its splitting field, say, K over $\mathbb{Q}_p$. The generating function of the sequence is

$$t(x) = \frac{x^{k-1}}{1 - a_1x - a_2x^2 - \cdots - x^k} = \sum_{i=1}^{k} \frac{1}{q(\alpha_i)} \sum_{n=0}^{\infty} \alpha_i^n x^n,$$

where $q(x) := p'(x)$, the derivative of the polynomial $p(x)$. Hence, the nth term of the sequence is given by

$$x_n = \sum_{i=1}^{k} \frac{1}{q(\alpha_i)} \alpha_i^n, \quad n \geq 0.$$

Since $p(0) = -1$, the roots of $p(x)$ are units in the ring formed by elements in K with p-adic absolute value less than or equal to 1. Following Sanna's proof of [27, Theorem 1.2], we can choose an even $t \in \mathbb{N}$ such that the function

$$G(z) := \sum_{i=1}^{k} \frac{1}{q(\alpha_i)} \exp_p(z \log_p(\alpha_i^t))$$

is analytic over $\mathbb{Z}_p$ and the Taylor series of $G(z)$ around 0 converges for all $z \in \mathbb{Z}_p$. Also, note that $x_{nt} = G(n)$ for $n \geq 0$.

We now use a variant of the following lemma which gives the multiplicative independence of any $k - 1$ roots among the k roots $\alpha_1, \ldots, \alpha_k$ of the characteristic polynomial $x^k - x^{k-1} - \cdots - x - 1$ of the k-generalised Fibonacci sequence in the field of complex numbers.

LEMMA 3.1 [11, Lemma 1]. *With the notation above, each set of $k - 1$ different roots $\alpha_1, \ldots, \alpha_{k-1}$ is multiplicatively independent, that is, $\alpha_1^{e_1} \cdots \alpha_{k-1}^{e_{k-1}} = 1$ for some integers $e_1, \ldots, e_{k-1}$ if and only if $e_1 = \cdots = e_{k-1} = 0$.*

Let $\sigma(\alpha_1) = \pm\alpha$, where α is a Pisot number with absolute value greater than 1, the other roots, $\sigma(\alpha_2), \ldots, \sigma(\alpha_k)$, having absolute values less than 1, where σ is an isomorphism from $\mathbb{Q}(\alpha_1, \ldots, \alpha_k)$ to the splitting field of $p(x)$ over $\mathbb{Q}$ in the field of complex numbers. Therefore, the proof of Lemma 3.1 holds true for the roots of $p(x)$, which are $\sigma(\alpha_1), \ldots, \sigma(\alpha_k)$, since $\log|\sigma(\alpha_1)|$ is positive and $\log|\sigma(\alpha_2)|, \ldots, \log|\sigma(\alpha_k)|$ are negative. Hence, $\sigma(\alpha_1), \ldots, \sigma(\alpha_{k-1})$ are multiplicatively independent, implying that $\alpha_1^t, \ldots, \alpha_{k-1}^t$ are multiplicatively independent. Thus, $\log_p(\alpha_1^t), \ldots, \log_p(\alpha_{k-1}^t)$ are linearly independent over $\mathbb{Z}$ and hence linearly independent over the algebraic numbers by Theorem 2.3.

Suppose $G'(0) = \sum_{i=0}^{k}(1/q(\alpha_i)) \log_p(\alpha_i^t) = 0$. Since $\log_p(\alpha_k^t) = -\log_p(\alpha_1^t) - \cdots - \log_p(\alpha_k^t)$ as the product of the roots is -1 and t is even, we obtain

$$\sum_{i=1}^{k-1} \left(\frac{1}{q(\alpha_i)} - \frac{1}{q(\alpha_k)} \right) \log_p(\alpha_i^t) = 0.$$

By linear independence of $\log_p(\alpha_1^t), \ldots, \log_p(\alpha_{k-1}^t)$, we have $1/q(\alpha_1) = \cdots = 1/q(\alpha_k) = c$, for some p-adic number c. This gives k distinct roots $\alpha_1, \ldots, \alpha_k$ of the $(k-1)$-degree polynomial $q(x) - 1/c$, which is not possible.Therefore, $G'(0) \neq 0$. Since

$$G(z) = \sum_{j=0}^{\infty} \frac{G^{(j)}(0)}{j!} z^j$$

converges at $z = 1$, it follows that $\|G^{(j)}(0)/j!\|_p \to 0$. Hence, there exists an integer ℓ such that $\nu_p(G^{(j)}(0)/j!\,) \geq -\ell$ for all j. Thus, we obtain $G(mp^h) = G'(0)mp^h + d$ where $\nu_p(d) \geq 2h - \ell$ for all $m, h \geq 0$. Also, $G(0) = 0$ for $h > h_0 := \ell + \nu_p(G'(0))$ and hence

$$\nu_p\Big(\frac{G(mp^h)}{G(p^h)} - m\Big) \geq h - h_0.$$

This yields

$$\lim_{h\to\infty} \Big\|\frac{G(mp^h)}{G(p^h)} - m\Big\|_p = 0,$$

and hence $R(G(n)_{n\geq 0})$ is p-adically dense in $\mathbb{N}$. Since $x_{nt} = G(n), n \geq 0$, we find that $R((x_n)_{n\geq 0})$ is also p-adically dense in $\mathbb{N}$. Therefore, by Lemma 2.2, $R((x_n)_{n\geq 0})$ is dense in $\mathbb{Q}_p$. □

PROOF OF COROLLARY 1.4. Since $p(1)p(-1) = (-a-b)(b-a-2) > 0$ and $p(0) = -1$, by continuity of the polynomial function in $\mathbb{R}$, $p(x)$ has one real root with absolute value greater than 1 and two other roots with absolute values less than 1. Hence, the characteristic polynomial has a root $\pm\alpha$, where α is a Pisot number, and the corollary follows from Theorem 1.3. □

We need the following result to prove Theorem 1.7.

COROLLARY 3.2 [9, Corollary 2.2]. *The linear recurrence relation* $x_{n+1} = x_n + 2x_{n-1} + \cdots + nx_1 + (n+1)x_0, n \geq 0$, *with the initial data* $x_0 = 1$ *has the solution*

$$x_n = \frac{1}{\sqrt{5}}\Big(\Big(\frac{3+\sqrt{5}}{2}\Big)^n - \Big(\frac{3-\sqrt{5}}{2}\Big)^n\Big), \quad n \geq 1.$$

PROOF OF THEOREM 1.7. By Corollary 3.2, for $n \geq 1$,

$$x_n = \frac{1}{\sqrt{5}}\Big(\Big(\frac{3+\sqrt{5}}{2}\Big)^n - \Big(\frac{3-\sqrt{5}}{2}\Big)^n\Big) = \frac{\alpha^{2n} - \beta^{2n}}{\sqrt{5}} = F_{2n},$$

where $\alpha = (1+\sqrt{5})/2, \beta = (1-\sqrt{5})/2$ and F_n denotes the nth Fibonacci number which is obtained by the Binet formula. From [14], the ratio set of the Fibonacci numbers is dense in $\mathbb{Q}_p$ for all primes p. Therefore, by Lemma 2.1, $\nu_p(F_n)$ is

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not bounded. Hence, for any $j \in \mathbb{N}$, there exists F_m such that $\nu_p(F_m) \geq j$, that is, $(\alpha^m - \beta^m)/\sqrt{5} \equiv 0 \pmod{p^j}$ which gives $\alpha^m \equiv \beta^m \pmod{p^j}$. This yields

$$\alpha^{2mp^{j-1}(p-1)} = (\alpha^m\alpha^m)^{p^{j-1}(p-1)} \equiv (\alpha^m\beta^m)^{p^{j-1}(p-1)} \pmod{p^j}.$$

Since $\alpha\beta = -1$, by using Euler's theorem, we find that

$$\alpha^{2mp^{j-1}(p-1)} \equiv (\alpha^m\beta^m)^{p^{j-1}(p-1)} \equiv 1 \pmod{p^j}.$$

This gives $\alpha^{2k} \equiv \beta^{2k} \equiv 1 \pmod{p^j}$, where $k = mp^{j-1}(p-1)$. Hence,

$$\frac{x_{kn}}{x_k} = \frac{F_{2kn}}{F_{2k}} = \frac{(\alpha^{2k})^n - (\beta^{2k})^n}{\alpha^{2k} - \beta^{2k}} = (\alpha^{2k})^{(n-1)} + (\alpha^{2k})^{n-2}\beta^{2k} + \cdots + (\beta^{2k})^{n-1},$$

which is congruent to n modulo p^j. Since, for $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $\|x_{kn}/x_k - n\|_p \leq p^{-j}$, $R((x_n)_{n\geq 0})$ is p-adically dense in $\mathbb{N}$. Therefore, by Lemma 2.2, $R((x_n)_{n\geq 0})$ is dense in $\mathbb{Q}_p$. □

We need the following results to prove Theorem 1.8.

THEOREM 3.3 [9, Theorem 3.1]. *The numbers x_n are solutions of the linear recurrence relation with constant coefficients in geometric progression $x_{n+1} = ax_n + aqx_{n-1} + \cdots + aq^{n-1}x_1 + aq^n x_0, n \geq 0$, with initial data x_0, if and only if they form the geometric progression given by the formula $x_n = ax_0(a+q)^{n-1}, n \geq 1$.*

LEMMA 3.4 [13, Lemma 2.2]. *If A is a geometric progression in $\mathbb{Z}$, then $R(A)$ is not dense in any $\mathbb{Q}_p$.*

PROOF OF THEOREM 1.8. By Theorem 3.3, $(x_n)_{n\geq 1}$ forms a geometric progression whose nth term is $ax_0(a+r)^{n-1}$ for $n \geq 1$. Hence, by Lemma 3.4, $R((x_n)_{n\geq 0})$ is not dense in $\mathbb{Q}_p$ for any prime p. □

To prove Theorem 1.9 we need some results on the uniform distribution of sequences of integers. Recall that a sequence $(x_n)_{n\geq 0}$ is said to be uniformly distributed modulo m if each residue occurs equally often, that is,

$$\lim_{N\to\infty} \frac{\#\{n \leq N \mid x_n \equiv t \pmod{m}\}}{N} = \frac{1}{m} \quad \text{for all } t \in \mathbb{Z}.$$

PROPOSITION 3.5 [8, Proposition 1]. *Suppose $(G_n)_{n\geq 0}$ is the sequence of integers determined by the recurrence relation $G_{n+1} = AG_n - BG_{n-1}$ with initial values G_0, G_1 where $A, B, G_0, G_1 \in \mathbb{Z}$. If $A = 2a, B = a^2$, then $(G_n)_{n\geq 0}$ is uniformly distributed modulo a prime p if and only if $p \nmid a(G_1 - aG_0)$.*

THEOREM 3.6 [8, Theorem]. *Suppose $(G_n)_{n\geq 0}$ is the sequence of integers determined by the recurrence relation $G_{n+1} = AG_n - BG_{n-1}$ with initial values G_0, G_1 where $A, B, G_0, G_1 \in \mathbb{Z}$. If $(G_n)_{n\geq 0}$ is uniformly distributed modulo p, then $(G_n)_{n\geq 0}$ is uniformly distributed modulo p^h with $h > 1$ if and only if*

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(1) $p > 3$; *or*
(2) $p = 3$ *and* $A^2 \not\equiv B \pmod 9$; *or*
(3) $p = 2, A \equiv 2 \pmod 4, B \equiv 1 \pmod 4$.

Proof of Theorem 1.9. Let p be a prime. The given recurrence sequence $(x_n)_{n\geq 0}$ satisfies the hypotheses of Proposition 3.5, and hence $(x_n)_{n\geq 0}$ is uniformly distributed modulo p. If $p > 3$, then by Theorem 3.6(1), $(x_n)_{n\geq 0}$ is uniformly distributed modulo p^k with $k > 1$, that is,

$$\lim_{N\to\infty} \frac{\#\{n \leq N \mid x_n \equiv t \pmod{p^k}\}}{N} = \frac{1}{p^k} > 0.$$

Therefore, for all $t \in \mathbb{N}$ and for all $k > 1$, there exists x_n such that $\|x_n - t\|_p \leq p^{-k}$. Hence, $R((x_n)_{n\geq 0})$ is p-adically dense in $\mathbb{N}$. Therefore, by Lemma 2.2, $R((x_n)_{n\geq 0})$ is dense in $\mathbb{Q}_p$.

We next consider the remaining primes $p = 2, 3$. Since $p \nmid a(x_1 - ax_0)$, we have $p \nmid a$. It is easy to check that $p = 3$ satisfies the condition given in Theorem 3.6(2) and $p = 2$ satisfies the condition given in Theorem 3.6(3). The rest of the proof follows similarly as shown in the case $p > 3$. This completes the proof of the theorem. □

We need the following lemma to prove Theorem 1.10.

Lemma 3.7 [26, Lemma 3.3]. *Let $(r_n)_{n\geq 0}$ be a linearly recurring sequence of order $k \geq 2$ given by $r_n = a_1 r_{n-1} + \cdots + a_k r_{n-k}$ for each integer $n \geq k$, where $r_0, \ldots, r_{k-1}$ and $a_1, \ldots, a_k$ are all integers. Suppose that there exists a prime number p such that $p \nmid a_k$ and $\min\{\nu_p(a_j) : 1 \leq j < k\} > \max\{\nu_p(r_m) - \nu_p(r_n) : 0 \leq m, n < k\}$. Then $\nu_p(r_n) = \nu_p(r_{n \pmod k})$ for each nonnegative integer n.*

Proof of Theorem 1.10. By Lemma 3.7,

$$\nu_p(x_n/x_m) = \nu_p(x_{n \pmod k}) - \nu_p(x_{m \pmod k}) \leq M$$

for all $n, m \in \mathbb{N} \cup \{0\}$, where $M = \max\{\nu_p(x_i) : i = 0, 1, \ldots, k-1\}$. Therefore, by Lemma 2.1, $R((x_n)_{n\geq 0})$ is not dense in $\mathbb{Q}_p$. □

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References

[1] D. Antony and R. Barman, 'p-adic quotient sets: cubic forms', *Arch. Math.* **118**(2) (2022), 143–149.
[2] D. Antony, R. Barman and P. Miska, 'p-adic quotient sets: diagonal forms', *Arch. Math.* **119**(5) (2022), 461–470.

[3] B. Brown, M. Dairyko, S. R. Garcia, B. Lutz and M. Someck, 'Four quotient set gems', *Amer. Math. Monthly* **121**(7) (2014), 590–599.

[4] A. Brumer, 'On the units of algebraic number fields', *Mathematika* **14** (1967), 121–124.

[5] J. Bukor and P. Csiba, 'On estimations of dispersion of ratio block sequences', *Math. Slovaca* **59**(3) (2009), 283–290.

[6] J. Bukor, P. Erdős, T. Šalát and J. T. Tóth, 'Remarks on the (R)-density of sets of numbers. II', *Math. Slovaca* **4**(5) (1997), 517–526.

[7] J. Bukor and J. T. Tóth, 'On accumulation points of ratio sets of positive integers', *Amer. Math. Monthly* **103**(6) (1996), 502–504.

[8] R. T. Bumby, 'A distribution property for linear recurrence of the second order', *Proc. Amer. Math. Soc.* **50** (1975), 101–106.

[9] M. I. Cîrnu, 'Linear recurrence relations with the coefficients in progression', *Ann. Math. Inform.* **42** (2013), 119–127.

[10] C. Donnay, S. R. Garcia and J. Rouse, '*p*-adic quotient sets II: quadratic forms', *J. Number Theory* **201** (2019), 23–39.

[11] C. Fuchs, C. Hutle, F. Luca and L. Szalay, 'Diophantine triples and *k*-generalized Fibonacci sequences', *Bull. Malays. Math. Sci. Soc.* **41** (2018), 1449–1465.

[12] S. R. Garcia, 'Quotients of Gaussian primes', *Amer. Math. Monthly* **120**(9) (2013), 851–853.

[13] S. R. Garcia, Y. X. Hong, F. Luca, E. Pinsker, C. Sanna, E. Schechter and A. Starr, '*p*-adic quotient sets', *Acta Arith.* **179**(2) (2017), 163–184.

[14] S. R. Garcia and F. Luca, 'Quotients of Fibonacci numbers', *Amer. Math. Monthly* **123** (2016), 1039–1044.

[15] S. R. Garcia, D. E. Poore, V. Selhorst-Jones and N. Simon, 'Quotient sets and Diophantine equations', *Amer. Math. Monthly* **118**(8) (2011), 704–711.

[16] S. Hedman and D. Rose, 'Light subsets of $\mathbb{N}$ with dense quotient sets', *Amer. Math. Monthly* **116**(7) (2009), 635–641.

[17] D. Hobby and D. M. Silberger, 'Quotients of primes', *Amer. Math. Monthly* **100**(1) (1993), 50–52.

[18] F. Luca, C. Pomerance and Š. Porubský, 'Sets with prescribed arithmetic densities', *Unif. Distrib. Theory* **3**(2) (2008), 67–80.

[19] A. Micholson, 'Quotients of primes in arithmetic progressions', *Notes Number Theory Discrete Math.* **18**(2) (2012), 56–57.

[20] L. Mišík, 'Sets of positive integers with prescribed values of densities', *Math. Slovaca* **52**(3) (2002), 289–296.

[21] P. Miska, 'A note on *p*-adic denseness of quotients of values of quadratic forms', *Indag. Math. (N.S.)* **32** (2021), 639–645.

[22] P. Miska, N. Murru and C. Sanna, 'On the *p*-adic denseness of the quotient set of a polynomial image', *J. Number Theory* **197** (2019), 218–227.

[23] P. Miska and C. Sanna, '*p*-adic denseness of members of partitions of $\mathbb{N}$ and their ratio sets', *Bull. Malays. Math. Sci. Soc.* **43**(2) (2020), 1127–1133.

[24] T. Šalát, 'On ratio sets of natural numbers', *Acta Arith.* **15** (1968/1969), 273–278.

[25] T. Šalát, 'Corrigendum to the paper "On ratio sets of natural numbers"', *Acta Arith.* **16** (1969/1970), 103.

[26] C. Sanna, 'The *p*-adic valuation of Lucas sequences', *Fibonacci Quart.* **54**(2) (2016), 118–124.

[27] C. Sanna, 'The quotient set of *k*-generalised Fibonacci numbers is dense in $\mathbb{Q}_p$', *Bull. Aust. Math. Soc.* **96**(1) (2017), 24–29.

[28] B. D. Sittinger, 'Quotients of primes in an algebraic number ring', *Notes Number Theory Discrete Math.* **24**(2) (2018), 55–62.

[29] P. Starni, 'Answers to two questions concerning quotients of primes', *Amer. Math. Monthly* **102**(4) (1995), 347–349.

[30] O. Strauch and J. T. Tóth, 'Asymptotic density of $A \subset \mathbb{N}$ and density of the ratio set $R(A)$', *Acta Arith.* **87**(1) (1998), 67–78.

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