1 Introduction

This textbook presents an introduction to a set of mathematical tools that are extensively used in modern physics, and is mainly aimed at advanced undergraduate, graduate, and doctoral students in physics, engineering, and mathematics. The reader is ideally accompanied on a journey through a number of different, albeit related, topics.

Chapter 2 introduces group theory and related notions, including, in particular, group homomorphisms and isomorphisms, before discussing in detail the group of permutations and some other particularly interesting finite groups. Then, the formalism of Young diagrams is introduced, and an alternative definition of groups in terms of their presentation is given. The rest of the chapter is devoted to continuous groups and to groups acting on a set.

The next chapter, Chapter 3, discusses the different representations that groups can have: After a brief reminder of linear-algebra concepts, the definition of group representations is formulated. Then, the discussion focuses on the concept of reducibility of group representations, which leads to a classification of irreducible representations. Group characters are introduced and their use in the classification of inequivalent irreducible representations is explained. The chapter also discusses the properties of the regular representation, which is induced by the action of the group on itself through a translation. The final part of the chapter introduces dual vectors and tensors, and an example of application of these notions for a spin-chain system of relevance in quantum physics and condensed-matter theory.

In Chapter 4 we first introduce the concepts that allow one to endow a generic set with a topology, then we define manifolds and, finally, differential manifolds. The chapter discusses in detail calculus on manifolds, differential forms and their integration, and finally presents a formulation of classical mechanics in terms of differential forms.

The following chapter, Chapter 5, is devoted to Riemannian geometry: Topics such as metric tensors, the induced metric, affine connections, connection coefficients, and their transformation properties under coordinate changes are discussed in detail. The chapter presents a thorough exposition of the concepts relevant for the general relativity theory of gravitation and for gauge theories, including parallel transport and holonomy, covariant derivatives, geodesics, curvature, and torsion. The final sections of the chapter are devoted to the discussion of isometries and Killing vector fields.

Chapter 6 presents a discussion of semisimple Lie algebras (highlighting their relevance for different physical applications, from quantum mechanics to the theory of elementary particles in and beyond the Standard Model) and their unitary representations. After defining the Lie algebras of the generators of Lie groups,

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we introduce the concepts of roots, weights, and Cartan generators, and present the systematic classification of the algebras associated with the classical and exceptional simple Lie groups with the corresponding Dynkin diagrams. The chapter also discusses in detail the explicit construction of irreducible representations for Lie algebras of special unitary groups, using tensor methods and Young diagrams. The last part of the chapter describes the representations of products of unitary groups (such as those that describe the gauge interactions between fundamental particles), and the Lorentz and Lorentz–Poincaré groups relevant for the theory of special relativity and for quantum field theory.

Finally, Appendix A presents detailed solutions for a subset of the problems included at the end of each of the previous chapters. Other solutions are made available to the course instructors through the website of Cambridge University Press.

The book is ideally suited for a university course on mathematical methods for physics; the main emphasis is on geometrical and topological concepts, which are essential for the understanding of the symmetry principles and topological structures in modern physics. The book is largely self-contained, but some important mathematical prerequisites are assumed: In particular, it is assumed that the reader is already familiar with the basics of real and complex analysis and linear algebra.

In writing this book, we put a very strong emphasis on the *pedagogical aspects*. The book is primarily targeted at physics and engineering students and, following M. David Merrill's *application principle* in instructional design (which states that learning is promoted when the learner applies the new knowledge), its goal is to enable them not only to learn a collection of fundamental notions in different branches of mathematical physics, but also to directly *apply* these tools to concrete problems. To this purpose, the final section of each chapter includes a large collection of original problems and exercises. As in actual scientific research, some of these problems stimulate the readers to combine tools which are relevant for the different subjects presented in the various chapters, and to keep a broad perspective – rather than adopting a narrow, hyperspecialized approach.

There are numerous sources that we have used in the preparation of this textbook. Our main inspiration and influence comes from this short list of classic works that we highly recommend for further reading on the subjects covered herein. We list them here, mentioning the chapters for which they are most relevant and highly recommended for further reading on the subjects:

- Important references for Chapters 2 and 3 are the books by Hugh F. Jones [6], by Michael Tinkham [13], and by Morton Hamermesh [4].
- For further reading about the topics of Chapters 4 and 5, we recommend the books by Mikio Nakahara [10], by Charles Nash and Siddhartha Sen [11], by John M. Lee [8], and by Jeffrey M. Lee [7], as well as the books on the theory of general relativity by Charles W. Misner, Kip S. Thorne, and John A. Wheeler [9], by Robert M. Wald [14], and by Sean M. Carroll [1].
- Useful references for further reading on the topics discussed in Chapter 6 are the books by Howard Georgi [2] and by Francesco Iachello [5]. The applications for elementary particle theory can be found in many excellent textbooks; our discussion of the symmetry representations of the Standard Model is closest to that in the book by Mark Srednicki [12].

We conclude this Introduction by thanking all of the students who have taken the course, on which this textbook is based, for helping us to present the topics of this book as clearly as we could. We also thank our former teaching assistants and our colleagues Antti Kupiainen and Jouko Mickelsson, who also taught these subjects, for additional inspiration, as well as Klaus Larjo and Niko Jokela for assistance in converting our original handwritten notes into electronic format. We also thank Jarkko Järvelä and Saga Säppi, who generously shared their solutions to some of the problems. In addition, we are indebted to many colleagues with whom we have discussed about and from whom we have learned the topics presented herein. Finally, we acknowledge with gratitude the careful work and patience of the editorial staff at Cambridge University Press, especially Sarah Armstrong, Elliot Beck, Henry Cockburn, Sapphire Duveau, Kuttappan Suresh Kumar, Sarah Lambert, Beverley Lawrence, Róisín Munnelly, Aparna Nair, Mathew Rohit, and in particular Nicholas Gibbons, who followed this book project from its beginning to its completion.