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THE EXPLICIT FOURIER DECOMPOSITION OF $L^2(SO(n)/SO(n-m))$

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1. Introduction. The decomposition of $L^2(SO(n)/SO(n-m))$ into a direct sum of irreducible representations of $SO(n)$ is given abstractly by the branching theorem and the Frobenius reciprocity theorem [**1**]. The goal of this paper is to obtain this decomposition explicitly, generalizing the theory of spherical harmonics ($m = 1$). The case $m = 2$ has been studied in Levine [**6**], and the case $2m \leqq n$ in Gelbart [**3**]. Our results shed more light on these cases as well as revealing new phenomena which only occur when $2m > n$.

Following Gelbart [**3**] we realize $SO(n)/SO(n-m)$ for $1 \leqq m < n$ as the Stiefel manifold $S_m{}^n = \{$real $n \times m$ matrices whose columns are orthonormal vectors in $\mathbf{R}^n\}$. The irreducible subspaces of $L^2(S_m{}^n)$ are realized as restrictions to $S_m{}^n$ of certain harmonic polynomials on real $n \times m$ matrix space. We now describe them.

Let $x_1, \ldots, x_m$ denote the columns of the $n \times m$ matrix x, so that each x is a vector in $\mathbf{R}^n$. Let $\mu = [n/2]$ and let $a_1, \ldots, a_\mu$ be vectors in $\mathbf{C}^n$ satisfying $a_j \cdot a_k = 0$ (bilinear dot product). If n is odd let $b \in \mathbf{C}^n$ satisfy $b \cdot a_j = 0$, $b \cdot b = 1$. One choice for the a_j's and b is

$$a_1 = \begin{bmatrix} 1 \\ i \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ i \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \text{ etc.,} \quad b = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}.$$

We shall refer to this as the canonical choice.

Let A denote any subset of $\{1, \ldots, m\}$, and let $|A|$ denote its cardinality. We define $M(A)$, a polynomial on matrix space, as follows ($M(A)$ depends on the choice of the a_j's and b):

(i) if $|A| \leqq \mu$, $M(A)$ is the determinant of the $|A| \times |A|$ matrix obtained from the $\mu \times m$ matrix $\{a_j \cdot x_k\}$ by selecting the first $|A|$ rows and those columns corresponding to $k \in A$.

(ii) if $|A| > \mu$, $M(A)$ is the determinant of the $|A| \times |A|$ matrix obtained be selecting the first μ rows and the last $|A| - \mu$ rows and those columns cor-

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responding to $k \in A$ from the $n \times m$ matrix

$$\begin{Bmatrix} a_j \cdot x_k \\ \bar{a}_j \cdot x_k \end{Bmatrix} \quad \text{if } n \text{ is even, or}$$

$$\begin{Bmatrix} a_j \cdot x_k \\ \bar{a}_j \cdot x_k \\ b \cdot x_k \end{Bmatrix} \quad \text{if } n \text{ is odd.}$$

Let $\mathscr{A}$ denote a finite sequence $A_1, A_2, \ldots, A_N$ of non-empty subsets of $\{1, \ldots, m\}$ of decreasing cardinality, $|A_j| \geqq |A_{j+1}|$, and satisfying $|A_1| + |A_2| \leqq n$. We write $f(\mathscr{A}) = \prod_{j=1}^N M(A_j)$, with the canonical choice for the a_j's and b. If n is even and $|A_1| = n/2$ we write $f^-(\mathscr{A})$ for the polynomial obtained from $f(\mathscr{A})$ by replacing a_μ by $\bar{a}_\mu$.

THEOREM 1. *$f(\mathscr{A})$ is a non-zero highest weight vector for an irreducible representation of $SO(n)$ of highest weight $\omega = (m_1, \ldots, m_\mu)$ given as follows:*

(a) if $|A_1| \leqq n/2$ then $m_j = |\{r: |A_r| \geqq j\}|$

(b) if $|A_1| > n/2$ then $m_j = 0$ if $j > n - |A_1|$ and $m_j = |\{r: |A_r| \geqq j\}|$ if $j \leqq n - |A_1|$. When defined, $f^-(\mathscr{A})$ is a non-zero highest weight vector with highest weight $(m_1, \ldots, m_{\mu-1}, -m_\mu)$.

The polynomial $f(\mathscr{A})$ is $SO(n)$-harmonic in the sense that it is annihilated by every $SO(n)$-invariant differential operator, namely $\nabla_{x_j} \cdot \nabla_{x_k} f(\mathscr{A}) = 0$ for all $j, k = 1, \ldots, m$. Its restriction to S_m^n is non-zero and so generates under the action of $SO(n)$ an irreducible subspace of $L^2(S_m^n)$ with highest weight ω. As $\mathscr{A}$ varies these subspaces span $L^2(S_m^n)$, but they are not linearly independent. There are two reasons why this is so. The first is that there may exist linear relations between the polynomials $f(\mathscr{A})$. An example of this is

$$M(\{1, 2\})M(\{3\}) + M(\{2, 3\})M(\{1\}) + M(\{3, 1\})M(\{2\}) = 0$$

for $m \geqq 3$ (this cannot happen when $m = 1$ or 2). Or it may even happen that while a set of $f(\mathscr{A})$'s is linearly independent, their restrictions to S_m^n are not. An example of this is

$$M(\{1, 2\})^2 + M(\{2, 3\})^2 + M(\{1, 3\})^2 = 0 \quad \text{on } S_3^4.$$

(This does not appear to happen unless $2m > n$.) In order to obtain a spanning linearly independent set of invariant subspace of $L^2(S_m^n)$ we restrict the set of sequences $\mathscr{A}$ as in the following definition:

A sequence $\mathscr{A} = A_1, \ldots, A_N$ is S_m^n *admissible* if

(1) $|A_j| \geqq |A_{j+1}|$;

(2) if $A_j = \{i_1, \ldots, i_p\}$ and $A_{j+1} = \{i_1', \ldots, i_q'\}$ with $i_1 < i_2 < \ldots$ and $i_1' < i_2' < \ldots$ then $i_k' \geqq i_k$ for $k \leqq q$ (note $q \leqq p$ by (1));

(3) for any $k \leqq m$ we have

$$|\{r: r \in A_1 \text{ and } r \leqq k\}| + |\{r: r \in A_2 \text{ and } r \leqq k\}| \leqq n + k - m$$

(if $\mathscr{A} = A_1$ drop the second summand). The empty sequence is also admissible.

THEOREM 2. *The irreducible subspaces generated by* $f(\mathscr{A})$ *restricted to* $L^2(S_m{}^n)$ *as* $\mathscr{A}$ *runs over all admissible sequences, and, when* n *is even, the restrictions of* $f^-(\mathscr{A})$ *when* $\mathscr{A}$ *is admissible and* $f^-(\mathscr{A})$ *defined, are linearly independent and span* $L^2(S_m{}^n)$.

These results are incomplete in some respects, for we do not obtain orthogonal subspaces. While in principle all that is required is to orthogonalize the $f(\mathscr{A})$ corresponding to a fixed highest weight ω, we have no explicit way of doing this (except when $m = 1$ or 2, or $n = 4$). It would also be of interest to verify the following conjectures:

(1) The irreducible spaces of polynomials in Theorem 1 span all $SO(n)$-harmonic polynomials.

(2) If $2m \leqq n$ then an $SO(n)$-harmonic polynomial is determined by its restriction to $S_m{}^n$.

These conjectures were proved for the case $m = 2$ by Levine [**6**]. They are related to more general results of Helgason [**4**] and Kostant [**5**].

We will prove Theorems 1 and 2 in the next two sections. We also indicate the modifications necessary to deal with the case $n = m$, where we obtain a simplification in that we may always have $|A_1| \leqq \mu$. Section 4 describes some special cases in more detail, and Section 5 deals with the symmetric space $SO(n)/SO(n - m) \times SO(m)$.

I am grateful to Professor Gelbert for interesting me in these problems, and to Robert Stanton for useful discussions concerning Section 5. Recently Tuong Ton-That [**9**] has announced a proof of conjecture (2) above.

2. Proof of Theorem 1.

LEMMA 1. $f(\mathscr{A})$ *and* $f^-(\mathscr{A})$ *are* $SO(n)$*-harmonic.*

Proof. If $|A_1| \leqq n/2$ the result is trivial, because then $f(\mathscr{A})$ is a sum of products of polynomials $a_j \cdot x_k$. Applying $\nabla_{x_r} \cdot \nabla_{x_s}$ produces factors $a_p \cdot a_q$, all of which vanish. Similarly for $f^-(\mathscr{A})$.

If $|A_1| > n/2$, then certain $\bar{a}_j \cdot x_k$ and $b \cdot x_k$ factors appear in $M(A_1)$. But the condition $|A_1| + |A_2| \leqq n$ implies that if $\bar{a}_j \cdot x_k$ occurs in $M(A_1)$ then a_j does not occur in $M(A_2), \ldots, M(A_N)$. Thus

$$\nabla_{x_r} \cdot \nabla_{x_s} f(\mathscr{A}) = 0 + [\nabla_{x_r} \cdot \nabla_{x_s} M(A_1)] \prod_{j=2}^{N} M(A_j).$$

Now assuming $r, s \in A_1$, $r \neq s$ (otherwise $\nabla_{x_r} \cdot \nabla_{x_s} M(A_1)$ is trivially zero) we expand the determinant $M(A_1)$ by cofactors along the columns corresponding to r and s. Notice that $a_j \cdot x_r \bar{a}_j \cdot x_s$ and $\bar{a}_j \cdot x_r a_j \cdot x_s$ occur with the same cofactor but with opposite sign, so that when $\nabla_{x_r} \cdot \nabla_{x_s}$ is applied these terms will cancel. All the other terms are trivially zero (note that b occurs at most once), since $a_j \cdot \bar{a}_k = 0$ if $j \neq k$.

Thus we observe that the invariant space of polynomials generated by $f(\mathscr{A})$ (or $f^-(\mathscr{A})$) consists of spherical harmonics of degree $\sum |A_j|$ in the $n \cdot m$ variables, because the ordinary Laplacian is $\sum_{j=1}^m \nabla_{x_j} \cdot \nabla_{x_j}$. Now the space of spherical harmonics of fixed degree has an especially simple positive definite inner product given by $(f, g) = f(D)\bar{g}$. Our proof of Theorem 1 consists in showing that $f(\mathscr{A})$ is orthogonal, with respect to this inner product, to any rotation of a polynomial with the same homogeneity that is a weight vector with a higher weight.

We write an arbitrary polynomial in terms of the basis

$$\prod (a_j \cdot x_k)^{r_{jk}} \prod (\bar{a}_j \cdot x_k)^{s_{jk}} \prod (b \cdot x_k)^{t_k}$$

with the canonical choice for the a_j's and b (if n is even the b terms do not occur). Each such term has homogeneity $\sum\sum r_{jk} + \sum\sum s_{jk} + \sum t_k$ and is a weight vector with weight $\omega = (m_1, \ldots, m_\mu)$, $m_j = \sum_k (r_{jk} - s_{jk})$. It is clear from this that $f(\mathscr{A})$ is a weight vector with weight given by Theorem 1. We must show that $f(\mathscr{A})$ is orthogonal to $g = \prod (a_j' \cdot x_k)^{r_{jk}} \prod (\bar{a}_j' \cdot x_k)^{s_{jk}} \prod (b' \cdot x_k)^{t_k}$ for any choice of the a_j''s and b' provided $\sum\sum r_{jk} + \sum\sum s_{jk} + \sum t_k = \sum |A_r|$ and $\omega' = (m_1', \ldots, m_\mu')$, $m_j' = \sum_k (r_{jk} - s_{jk})$ is a higher weight than ω.

We compute $g(D)\overline{f(\mathscr{A})}$ by applying Leibniz' formula. This produces a sum of terms, each of which we will show to be zero. The basic observation is that a derivative of a determinant is the sum of the determinants obtained by differentiating one column of the matrix. Thus the terms comprising $g(D)\overline{f(\mathscr{A})}$ are obtained by replacing x_k's in the determinants $M(A_r)$ by a_j', $\bar{a}_j'$ and b, exactly r_{jk}, s_{jk} and t_k times respectively. What we shall show is that this implies that one of the determinants must have two identical columns, hence be zero.

Consider first the case when $|A_1| \leqq n/2$. Because ω' is a higher weight than ω we have $m_1' \geqq m_1$ and hence $\sum_k r_{1k} \geqq m_1$. Now there are $\sum_k r_{1k}$ a_1's to be distributed over all the $\overline{M(A_r)}$'s, which number exactly m_1 by (a). Thus, one determinant must be hit twice unless $m_1' = m_1$ and $\sum_k s_{1k} = 0$, and the a_1's are distributed one to a determinant. Next we distribute the a_2''s. We have $m_2' \geqq m_2$ hence $\sum_k r_{2k} \geqq m_2$. But the number of $\overline{M(A_r)}$'s left is exactly m_2, because those with $|A_r| = 1$ were used up when the a_1''s were distributed. Thus we must have $m_2' = m_2$, $\sum_k s_{2k} = 0$ and the a_2''s must be distributed one to a determinant with $|A_r| \geqq 2$. Reasoning inductively we conclude $\omega' = \omega$ which contradicts the hypothesis that ω' is a higher weight.

Next consider the case $|A_1| > n/2$, and let $\lambda = n - |A_1|$. We may use the same reasoning as before to conclude that $m_j' = m_j$ for $j \leqq \lambda$, and that only $|A_1| - \lambda = n - 2\lambda$ columns of $\overline{M(A_1)}$ remain to be filled. In order to avoid repeating columns we must have $\sum_k r_{jk} \leqq 1$, $\sum_k s_{jk} \leqq 1$ and $\sum_k t_{jk} \leqq 1$ for all $j > \lambda$.

Suppose n is even. Then there are only $n/2 - \lambda$ values of $j > \lambda$, hence to fill $n - 2\lambda$ columns we must have $\sum_k r_{jk} = \sum_k s_{jk} = 1$ for $j > \lambda$ which implies $m_j' = 0$ for $j > \lambda$ hence $\omega' = \omega$.

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Suppose n is odd. Then there are only $(n - 1)/2 - \lambda$ values of $j > \lambda$, hence to fill $n - 2\lambda$ columns we must have $\sum_k r_{jk} = \sum_k s_{jk} = \sum t_k = 1$ for $j > \lambda$ which again implies $\omega' = \omega$.

Finally we consider $f^-(\mathscr{A})$ when n is even and $|A_1| = n/2$ so that $m_\mu > 0$. Under the larger group $O(n)$, $f(\mathscr{A})$ generates an invariant subspace with highest weight ω which contains $f^-(\mathscr{A})$. Now $f^-(\mathscr{A})$ is a weight vector with weight $\omega' = (m_1, \ldots, m_{\mu-1}, -m_\mu)$ and the weight vectors with this weight are one-dimensional. Thus upon splitting the representation of $O(n)$ into two irreducible representations of $SO(n)$ with highest weights ω and ω', we see that $f^-(\mathscr{A})$ must generate the space with highest weight ω'.

3. Proof of Theorem 2. We shall give an inductive proof, deriving the result for $S_m{}^n$ assuming it for $S_{m-1}{}^{n-1}$. For this purpose it is more convenient to have an inductive criterion for admissible sequences. We define the deletion $\delta(A)$ of $A \subseteq \{1, \ldots, m\}$ to be $A \cap \{1, \ldots, m - 1\}$.

LEMMA 2. *A sequence* $\mathscr{A} = A_1, \ldots, A_N$ *is* $S_m{}^n$ *admissible if and only if*

(1′) $|A_j| \geqq |A_{j+1}|$,

(2′) $|A_1| + |A_2| \leqq n$,

(3′) $\delta(\mathscr{A}) = \delta(A_1), \ldots, \delta(A_M)$ *is* $S_{m-1}{}^{n-1}$ *admissible and* $\delta(A_j) = \emptyset$ *for* $j > M$. (If $\delta(A_1) = \emptyset$ then $\delta(\mathscr{A})$ is the empty sequence.)

Proof. Assume $\mathscr{A}$ is $S_m{}^n$ admissible. Then (1′) and (2′) above follow from (1) and (3) with $k = m$ of the definition of admissible. Now $\delta(A_j) = \emptyset$ if and only if $A_j = \{m\}$ and (2) implies such a set can only occur at the end of an admissible sequence. To complete the verification of (3′) above we write $A_j = \{i_1, \ldots, i_p\}$, $A_{j+1} = \{i_1', \ldots, i_q'\}$ in ascending order. By (2) we have $i_k' \geqq i_k$ for $k \leqq q$ and $q \leqq p$. Now $|\delta(A_j)| \geqq |\delta(A_{j+1})|$ unless $q = p$, $i_p = m$ and $i_p' \neq m$. But this contradicts $i_p' \geqq i_p$, proving (1) for $\delta(\mathscr{A})$. Similarly (2) and (3) hold for $\delta(\mathscr{A})$. Thus an $S_m{}^n$ admissible sequence satisfies the conditions of the lemma.

Conversely, assume the conditions of the lemma are satisfied. We must show that $\mathscr{A}$ is $S_m{}^n$ admissible. By (3′) we know that $\delta(\mathscr{A})$ is $S_{m-1}{}^{n-1}$ admissible. This, together with (1′) and (2′) easily yield conditions (1) and (3) of the definition of $S_m{}^n$ admissible for $\mathscr{A}$. To verify (2) write $A_j = \{i_1, \ldots, i_p\}$, $A_{j+1} = \{i_1', \ldots, i_q'\}$. Because $\delta(\mathscr{A})$ is $S_{m-1}{}^{n-1}$ admissible we have $i_k' \geqq i_k$ for $k \leqq q - 1$ and also for $k = q$ unless $i_q' = m$. But in that case $i_q' \geqq q$ trivially.

We now give the induction step in the proof of Theorem 2.

Let ω be a dominant weight for $SO(n)$ and ω' a dominant weight for $SO(n - 1)$. We say ω *intertwines* ω' if:

(a) $n = 2\mu$, $\omega = (m_1, \ldots, m_\mu)$, $\omega' = (m_1', \ldots, m_{\mu-1}')$, $m_1 \geqq m_1' \geqq m_2 \geqq m_2' \geqq \ldots \geqq m_{\mu-1}' \geqq |m_\mu|$;

(b) $n = 2\mu + 1$, $\omega = (m_1, \ldots, m_\mu)$, $\omega' = (m_1', \ldots, m_\mu')$, $m_1 \geqq m_1' \geqq m_2 \geqq m_2' \geqq \ldots \geqq m_\mu \geqq |m_\mu'|$.

Now the representation of $SO(n)$ on $L^2(SO(n)/SO(n - m))$ is the induced representation from the trivial representation of $SO(n-m)$. By the composition theorem for induced representations it may also be regarded as the induced representation from the representation of $SO(n-1)$ on $L^2(SO(n-1)/SO(n-m))$. In our induction argument we assume that the representation of $SO(n-1)$ on $L^2(SO(n-1)/(SO(n-m))$ is already decomposed into irreducibles by Theorem 2. By the Frobenius reciprocity theorem and the branching theorem we know that each irreducible subrepresentation of $SO(n-1)$ with highest weight ω' induces on $SO(n)$ a representation which decomposes into a direct sum of irreducible representations with highest weight ω, where each ω intertwines ω' and occurs with multiplicity one. This gives us, inductively, an exact formula for the multiplicity of any abstract representation in the decomposition of $L^2(SO(n)/SO(n-m))$. The next lemma will enable us to show that Theorem 2 gives the same multiplicity.

LEMMA 3. (a) *Let $\mathscr{A}$ be S_m^n admissible. If $f(\mathscr{A})$ has weight ω and $f(\delta(\mathscr{A}))$ has weight ω', then ω intertwines ω'.*

(b) *Let $\mathscr{A}'$ be S_{m-1}^{n-1} admissible, let $f(\mathscr{A}')$ have weight ω', and let ω intertwine ω', with $m_\mu \geqq 0$. Then there exists a unique S_m^n admissible sequence $\mathscr{A}$ satisfying $\delta(\mathscr{A}) = \mathscr{A}'$ and such that $f(\mathscr{A})$ has weight ω, unless n is odd and $m_\mu' > 0$. In that case there are exactly two such admissible sequences.*

Proof. (a) Let $\mathscr{A} = A_1, \ldots, A_N, \delta(\mathscr{A}) = \delta(A_1), \ldots, \delta(A_M)$ with $A_j = \{m\}$ for $j > M$. Because $|A| \geqq |\delta(A)|$ we have $m_j \geqq m_j'$. Because $|A| \leqq |\delta(A)| + 1$ we have $m_j' \geqq m_{j+1}$. Thus ω intertwines ω'.

(b) First assume $2m \leqq n$. Write $\mathscr{A}' = A_1', \ldots, A_M'$. Theorem 1 implies that $A_{m_{j+1}'+1}', \ldots, A_{m_j'}'$ are sets with cardinality j. Lemma 2 implies that for $\mathscr{A}$ to be S_m^n admissible and satisfy $\delta(\mathscr{A}) = \mathscr{A}'$ it must be of the form $\mathscr{A} = A_1, \ldots, A_N$ with $A_j = \{m\}$ for $j > M$ and $\delta(A_j) = A_j'$ for $j \leqq M$. In order to guarantee $|A_k| \geqq 1A_{k+1}|$ we must adjoin $\{m\}$ to some initial subsequence of $A_{m_{j+1}'+1}', \ldots, A_{m_j'}'$ for each j. That means we must choose integers M_{j+1} satisfying $m_{j+1}' \leqq M_{j+1} \leqq M_j'$ such that $A_k = A_k' \cup \{m\}$ for $m_{j+1}' + 1 \leqq k \leqq M_{j+1}$ and $A_k = A_k'$ for $M_{j+1} + 1 \leqq k \leqq m_j'$. This produces $f(\mathscr{A})$ with weight $(N, M_2, \ldots, M_\mu)$ with any $N \geqq M = m_1'$, giving exactly one admissible $\mathscr{A}$ for each weight that intertwines ω'.

The same reasoning applies if $2m > n$ but $|A_1'| < \mu$. Thus assume $|A_1'| \geqq \mu$ and set $\lambda = n - 1 - |A_1'|$. Then $A_2', \ldots, A_{m_\lambda'}'$ have cardinality λ and $m_j' = 0$ for $j > \lambda$. Assume $\lambda < n - 1 - |A_1'|$. In order to have $|A_1| + |A_2| \leqq n$ we may either adjoin $\{m\}$ to A_1' and leave $A_2', \ldots, A_{m_\lambda'}'$ alone, or we may leave A_1' alone and adjoin $\{m\}$ to an initial subsequence of $A_2', \ldots, A_{m_\lambda'}'$, but not both. The first option produces $f(\mathscr{A})$ with weight satisfying $m_\lambda = m_\lambda', m_{\lambda+1} = 0$. In the second case, if we adjoin $\{m\}$ to $A_2', \ldots, A_p'$ we obtain $m_\lambda = m_\lambda' - p - 1$, $m_{\lambda+1} = p + 1$. Thus once again we obtain one admissible sequence for every weight that intertwines ω'.

Finally, assume $\lambda = n - 1 - |A_1'|$. This occurs exactly in the exceptional case: n odd, $m_\mu' > 0$. Here $A_1', \ldots, A_{m_\mu'}'$ have cardinality $\mu = (n-1)/2$.

In order to have $|A_1| + |A_2| \leqq n$ we may adjoin $\{m\}$ only to A_1'. Whether or not we do so does not affect the weight of $f(\mathscr{A})$ which is already determined to be $(N, M_2, \ldots, M_\mu)$ by the previous choices. Thus we obtain two admissible sequences for each ω intertwining ω'.

As a consequence of the lemma it is sufficient to prove linear independence of the $f(\mathscr{A})$ in order to prove that they span. To see this we reason as follows:

Suppose n is even and ω satisfies $m_\mu \geqq 0$. Then $L^2(S_m{}^n)$ contains one irreducible subspace with highest weight ω for each irreducible subspace of $L^2(S_{m-1}{}^{n-1})$ with highest weight ω' such that ω intertwines ω'. By the induction hypotheses these subspaces of $L^2(S_{m-1}{}^{n-1})$ are in one-to-one correspondence with $S_{m-1}{}^{n-1}$ admissible sequences $\mathscr{A}'$ such that $f(\mathscr{A}')$ has highest weight ω'. By the lemma the mapping $\mathscr{A} \rightarrow \delta(\mathscr{A})$ puts into one-to-one correspondence the $S_m{}^n$ admissible sequences $\mathscr{A}$ such that $f(\mathscr{A})$ has highest weight ω with the $S_{m-1}{}^{n-1}$ admissible sequences $\mathscr{A}'$ such that $f(\mathscr{A}')$ has highest weight ω. This proves the contention in this case.

If n is even but $m_\mu < 0$ we reason as before replacing $f(\mathscr{A})$ with $f^-(\mathscr{A})$.

Suppose n is odd. Then $L^2(S_m{}^n)$ contains one irreducible subspace of L^2-$(S_{m-1}{}^{n-1})$ with highest weight ω' satisfying $m_\mu' = 0$, and two for each ω' satisfying $m_\mu' > 0$ (one for ω' and one for $(m_1', \ldots, m_{\mu-1}', -m_\mu'))$ such that ω intertwines ω'. We may thus reason as before.

We now give the proof of linear independence. Assume $\mathscr{A}^1, \mathscr{A}^2, \ldots$ are distinct $S_m{}^n$ admissible sequences, $f(\mathscr{A}^k)$ has weight ω, and $\sum \beta_k f(\mathscr{A}^k)$ vanishes on $S_m{}^n$. We must show that $\beta_1 = \beta_2 = \ldots = 0$. Note that since the highest weight space always has dimension one, this will prove that the subspaces generated by the $f(\mathscr{A}^k)$ are linearly independent.

Assume first that $2m \leqq n$. By Lemma 3, $\delta(\mathscr{A}^k)$ are distinct $S_{m-1}{}^{n-1}$ admissible sequences. Thus it suffices to show that $\sum \beta_k f(\delta(\mathscr{A}^k)) = 0$ on $S_{m-1}{}^{n-1}$ and apply the induction hypotheses.

To do this we let

$$x_m = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad x_j = \begin{bmatrix} x_j' \\ 0 \end{bmatrix} \quad \text{for} \quad j \leqq m-1,$$

where $(x_1', \ldots, x_{m-1}') \in S_{m-1}{}^{n-1}$. We do not substitute this directly in $\sum \beta_k f(\mathscr{A}^k)$ because it usually produces zero. Instead we first perform some rotation of variables to obtain non-zero terms. The principle is that if $F(x_1, \ldots, x_m)$ vanishes on $S_m{}^n$, then so does $F(Rx_1, \ldots, Rx_m)$ for any rotation R.

Consider the case n even. We set $R = R_{\theta_{\mu-1}}, R_{\theta_{\mu-2}} \ldots R_{\theta_1}$ where R_{θ_j} is a rotation through angle θ_j in the $a_j - a_\mu$ plane, sending $a_j \cdot x_k$ into $\cos \theta_j a_j \cdot x_k + \sin \theta_j a_\mu \cdot x_k$ and $a_\mu \cdot x_k$ into $- \sin \theta\, a_j \cdot x_k + \cos \theta_j a_\mu \cdot x_k$. We divide through by

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$\prod (\cos\theta_j)^{m_j}$ and obtain a polynomial in $\tan\theta_1, \ldots, \tan\theta_{\mu-1}$ because m_j is exactly the number of times that a_j occurs in $f(\mathscr{A}^k)$. We then make the substitution for $x_1, \ldots, x_m$ and equate to zero the coefficients of each monomial in $\tan\theta_1, \ldots, \tan\theta_{\mu-1}$. We order the monomials in lexicographic order and consider the lowest order terms.

Suppose $m \notin A$ (hence $|A| < \mu$). Then the contribution of $M(A)$ to the above is $M(\delta(A))$ + higher order terms (note $\delta(A) = A$), since $a_j \cdot x_k = a_j \cdot x_k'$ for $j < \mu$, $k < m$.

Suppose $m \in A$ and $|A| = \lambda$. Since $a_j \cdot x_m = 0$ for $j < \mu$ and $a_\mu \cdot x_m = i$, the lowest order term arising from $M(A)$ is obtained by selecting $a_j \cdot x_k$ over $\tan\theta_j a_\mu \cdot x_k$ in rows $1, \ldots, \lambda - 1$, and $\tan\theta_\lambda a_\mu \cdot x_k$ in row λ. The m-column becomes

$$\begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ i\tan\theta_\lambda \end{bmatrix}$$

and we obtain $M(\delta(A))i\tan\theta_\lambda$ + higher order terms.

Thus $f(\mathscr{A})$ is transformed into $f(\delta(\mathscr{A}))\prod(i\tan\theta_j)^{r_i}$ + higher order terms. The lowest order terms arising from $\sum\beta_k f(\mathscr{A}^k) = 0$ on $S_m^{\,n}$ give rise to $\sum'(i)^{\rho_k}\beta_k f(\delta(\mathscr{A}^k))$ on $S_{m-1}^{\,n-1}$ ($\sum'$ denoting a sum over a restricted set of k's). We thus have $\beta_k = 0$ for those values of k appearing in $\sum'$. These terms may be removed from the original sum and the argument repeated until all $\beta_k = 0$.

The case n odd but $m_\mu = 0$ may be handled by the same argument with just one modification: We must replace a_μ by

$$\tilde{a} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ i \end{bmatrix}$$

(note the condition $m_\mu = 0$ implies that a_μ does not occur in $f(\mathscr{A}^k)$). If we try this argument when $m_\mu > 0$ we encounter a new difficulty: If $m \notin A$ but $|A| = \mu$ then in place of $M(\delta(A))$ we have the same determinant with $\tilde{a} \cdot x_k$ in place of $a_\mu \cdot x_k$, and

$$\tilde{a} \cdot x_k = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \cdot x_k'.$$

We overcome this difficulty as follows:

We perform the rotation

$$T_\theta = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ \sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \end{bmatrix}$$

in the last three variables in $\mathbf{R}^n$ to replace a_μ with

$$\begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ \cos\theta \\ \sin\theta \\ i \end{bmatrix}.$$

In place of $M(\delta(A))$ we now have $\cos\theta\, M'(\delta(A)) + \sin\theta\, M''(\delta(A))$, where M' and M'' are obtained from M by replacing $a_\mu \in \mathbf{C}^{n-1}$ by

$$a' = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad a'' = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}.$$

Note that $M(\delta(A)) = M'(\delta(A)) + i\, M''(\delta(A))$.

LEMMA 4. *Let P be a polynomial in two indeterminates of degree r, and let P_r denote the terms of homogeneity exactly r. If $P(\cos\theta, \sin\theta) = 0$ for all θ then $P_r(1, i) = 0$.*

Proof. Since $P(\cos(\theta + \pi), \sin(\theta + \pi)) = P(-\cos\theta, -\sin\theta)$ we have $P_r(\cos\theta, \sin\theta) = \sum_1^{r/2} P_{r-2j}(\cos\theta, \sin\theta) = 0$ by equating terms of equal parity. We multiply $P_{r-2j}(\cos\theta, \sin\theta)$ by $(\cos^2\theta + \sin^2\theta)^j$ and divide through by $(\cos\theta)^r$ to obtain

$$P_r(1, \tan\theta) + \sum_1^{r/2} (1 + \tan^2\theta)^j P_{r-2j}(1, \tan\theta) = 0.$$

We may now equate to zero the coefficients of $(\tan\theta)^k$ and sum them after multiplying by i^k. The result is to substitute i for $\tan\theta$ giving $P_r(1, i) = 0$ since $1 + i^2 = 0$.

We apply Lemma 4 to the equation

$$\sum{}' \beta_k \prod_{m \notin A_j^k,\ |A_j^k| = \mu} (\cos\theta M'(\delta(A_j^k)) + \sin\theta M''(\delta(A_j^k))) \times \prod_{\text{other } j\text{'s}} M(\delta(A_j^k)) = 0$$

on S_{m-1}^{n-1} to obtain $\sum''\beta_k f(\delta(\mathscr{A}^k)) = 0$ on S_{m-1}^{n-1}, the sum over those values of k for which $\{j: m \notin A_j^k$ and $|A_j^k| = \mu\}$ has maximal cardinality. We obtain $\beta_k = 0$ for these values of k and continue as before.

Next we consider the case n even and $2m > n$. If $m_\mu \neq 0$ then the argument given for $2m \leqq n$ may be applied without change (note that if $|A| = \mu$ then $M(A)$ is unchanged by the rotation R). If $m_\mu = 0$ let λ be the largest integer for which $m_\lambda > 0$. We choose $R = R_{\theta_\lambda} \ldots R_{\theta_1}$. If $|A| = n - \lambda$ the rotation R transforms $M(A)$ by changing the last row from $\bar{a}_\mu \cdot x_k$ to

$$-\sin\theta_1 \bar{a}_1 \cdot x_j - \cos\theta_1 \sin\theta_2 \bar{a}_2 \cdot x_j - \ldots - \cos\theta_1 \ldots \cos\theta_\lambda \bar{a}_\mu \cdot x_j.$$

Now if $m \notin A$ the $\bar{a}_\mu \cdot x_j$ term contributes zero because it duplicates the μth row since $\bar{a}_\mu \cdot x_j = a_\mu \cdot x_j = b \cdot x_j'$. The next-to-last term involving $\bar{a}_\lambda \cdot x_j$ produces $M(\delta(A))$ after interchanging the μth and last rows. Thus $M(A)$ becomes

$$\pm M(\delta(A)) \cos\theta_1 \ldots \cos\theta_{\lambda-1} \sin\theta_\lambda + \text{terms involving } \sin\theta_j \text{ for } j < \lambda.$$

If $m \in A$, on the other hand, then $a_\mu \cdot x_m = i$ and $\bar{a}_\mu \cdot x_m = -i$. Thus the $\cos\theta_1, \ldots, \cos\theta_\lambda$ term produces a determinant with the μth row $(b \cdot x_j', i)$ and the last row $(b \cdot x_j', -i)$. We add the last row to the μth row to reduce the last column to $\begin{bmatrix} 0 \\ 0 \\ -i \end{bmatrix}$, and so obtain $\pm 2i\ M(\delta(A))$. Thus $M(A)$ becomes $\pm 2i \cos\theta_1, \ldots, \cos\theta_\lambda M(\delta(A))$.

Now we divide by $(\cos\theta_1)^{m_1}$, select out the lowest power of $\tan\theta_1$, then divide by $(\cos\theta_1)^{m_2}$ and select out the lowest power of $\tan\theta_2$ and so forth. In this way the terms of lower homogeneity are discarded and we may repeat the previous argument.

Finally assume that n is odd and $2m > n$. Note that $f(\mathscr{A})$ will have different parity under the transformation $x_j \to -x_j$ (this preserves S_m^n even though it is an improper rotation) depending on whether or not $|A_1| > \mu$. Thus we may assume that in the sum $\sum \beta_k f(\mathscr{A}^k)$ either $|A_1^k| > \mu$ or not for all k. From the construction in Lemma 2 it follows that all $\delta(\mathscr{A}^k)$ are distinct. We may thus attempt an argument similar to those in the previous cases.

If $|A_1^k| \leqq \mu$ then we may repeat the argument given for the case n odd and $2m \leqq n$. Thus assume $|A_1^k| > \mu$. If $|A_1^k| = \mu + 1$ we choose $R = T_\theta R_{\theta_{\mu-1}}, \ldots, R_{\theta_1}$. Now the R_{θ_j} do not affect $M(A_1^k)$, but T_θ replaces a_μ with

$$\begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ \cos\theta \\ \sin\theta \\ i \end{bmatrix} \text{ and } b \text{ with } \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}.$$

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Thus if $m \notin A_1^k$ we obtain $\frac{1}{2}iM(\delta(A_1^k))$ while if $m \in A_1^k$ we obtain $\pm i(-\sin\theta\, M'(\delta(A_1^k)) + \cos\theta\, M''(\delta(A_1^k)))$. We may then repeat the argument given for the case $2m \leqq n$.

If $|A_1^k| > \mu + 1$, let $\lambda = n - |A_1^k|$ and choose $R = T_\theta R_{\theta_\lambda}, \ldots, R_{\theta_1}$. The $\bar{a}_\mu$ row of A_1^k is then transformed into $-\sin\theta_1\, \bar{a}_1 \cdot x_j - \ldots - \cos\theta_1, \ldots, \cos\theta_\lambda\, \bar{a}_\mu \cdot x_j$ as in the case n even. If $m \notin A_1^k$ then the $\bar{a}_\mu$ term contributes $-\frac{1}{2}\, i \cos\theta_1, \ldots, \cos\theta_\lambda\, M(\delta(A_1^k))$. If $m \in A_1^k$ then the $\bar{a}_\mu$ term contributes zero and the $\bar{a}_\lambda$ term contributes $-\frac{1}{2}i\cos\theta_1, \ldots, \cos\theta_{\lambda-1}\sin\theta_\lambda\, M(\delta(A_1^k))$. Thus we may repeat the argument given for the case n even and $2m > n$.

This completes the proof of the induction step. All that remains is to verify the theorem for the case $m = 1$, n arbitrary $n > 1$. But here we are dealing with the theory of spherical harmonics. The only choice for A_j is $\{1\}$, so $f(\mathscr{A})$ must be $(a_1 \cdot x_1)^k$ for some k. This generates the spherical harmonics of degree k for $n \geqq 3$. If $n = 2$ we must consider also $f^-(\mathscr{A}) = (\bar{a}_1 \cdot x_1)^k$, or, writing $x_1 = \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}$, we have $(a_1 \cdot x_1)^k = e^{ik\theta}$ and $(\bar{a}_1 \cdot x_1)^k = e^{-ik\theta}$. Thus the theorem for $m = 1$ is completely elementary.

Remark. The case $m = n - 1$ gives the Peter-Weyl decomposition of $SO(n)$ because $SO(1)$ is the trivial group. Nevertheless it is interesting to obtain this decomposition in terms of functions on $SO(n)$ rather than on S_{n-1}^n. We can again represent $SO(n)$ as the component of the identity in $S_n^n = O(n)$. Theorem 1 remains true, since the condition $m < n$ was not used in the proof. Theorem 2 remains true if we add two additional conditions for $\mathscr{A}$ to be S_n^n admissible:

(4) $|A_1| \leqq \mu$,

(5) $1 \notin A_j$ for $j \geqq 2$ and if $1 \in A_1$ then $|A_1| \leqq n - 1 - \mu$.

To see why this is true we reason as follows: If $(x_1, \ldots, x_n) \in SO(n)$ then $(x_2, \ldots, x_n) \in S_{n-1}^n$ and x_1 is completely determined by $(x_2, \ldots, x_n)$. Thus the functions $f(\mathscr{A})$ and $f^-(\mathscr{A})$ on S_{n-1}^n may be regarded as functions on $SO(n)$ by replacing each $k \in A_j$ by $k + 1$. Theorem 2 for S_{n-1}^n translates into the same result for S_n^n with the additional condition:

(4′) $1 \notin A_j$ for any j.

That (4′) may be replaced by (4) and (5) follows from

LEMMA 5. *For $n = m$ we have $M(A) = cM(A^\sim)$ on $SO(n)$, where $A^\sim$ denotes the complement of $A \subseteq \{1, \ldots, n\}$, and c is a non-zero constant.*

Proof. Consider the complex $n \times n$ matrix z given as follows:

(1) If $n = 2\mu$ then

$$(-i)^{\mu/n} z = \begin{Bmatrix} \frac{1}{\sqrt{2}}\, a_j \cdot x_k \\ \frac{1}{\sqrt{2}}\, \bar{a}_j \cdot x_k \end{Bmatrix}.$$

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(2) If $n = 2\mu + 1$ then

$$(-i)^{\mu/n} z = \begin{Bmatrix} \frac{1}{\sqrt{2}} a_j \cdot x_k \\ \frac{1}{\sqrt{2}} \bar{a}_j \cdot x_k \\ b \cdot x_k \end{Bmatrix}.$$

It is easy to check that $x \in SO(n)$ implies $z \in SU(n)$. Now a generalization of Cramer's rule due to Jacobi [**2**, p. 58] states that if u is any complex $n \times n$ matrix with $\det u = 1$, if u_1 is any square submatrix and u_2 is the complementary square submatrix of ${}^t u^{-1}$ (obtained by selecting from ${}^t u^{-1}$ the rows and columns omitted from u in obtaining u_1) then $\det u_1 = \pm \det u_2$. We apply this to z, noting that $\det z = 1$ and ${}^t z^{-1} = \bar{z}$. If we choose the first $|A|$ rows of z (say $|A| \leqq \mu$) and the columns corresponding to $j \in A$ then $\det u_1 = cM(A)$. But then u_2 contains the last $n - |A|$ rows of $\bar{z}$ and the columns corresponding to $j \notin A$. After rearranging the rows we have the first μ rows and the last $n - |A| - \mu$ rows of z, hence $\det u_2 = c'M(A^\sim)$.

Thus if $\mathscr{A}$ is S_n^n admissible and $1 \in A_1$, then $f(\mathscr{A}) = cf(\mathscr{A}')$ on $SO(n)$ where $\mathscr{A}' = A_1^\sim, A_2, \ldots$. Thus it remains to show that $\mathscr{A}$ is S_n^n admissible if and only if $\mathscr{A}''$ is S_{n-1}^n admissible, where $\mathscr{A}''$ is obtained from $\mathscr{A}'$ by replacing each $k \in A_j$ by $k - 1$ (note $1 \notin A_j$).

Now let $p_k = |A_1 \cap \{1, \ldots, k\}|$ and $q_k = |A_2 \cap \{1, \ldots, k\}|$. Then (1) and (2) for $\mathscr{A}$ says exactly $q_k \leqq p_k$, while (3) says $p_k + q_k \leqq k$. On the other hand (1) and (2) for $\mathscr{A}''$ says

$$|A_1^\sim \cap \{2, \ldots, k\}| \geqq |A_2 \cap \{2, \ldots, k\}|,$$

in other words $k - p_k \geqq q_k$ or $p_k + q_k \leqq k$, while (3) says

$$|A_1^\sim \cap \{2, \ldots, k\} + |A_2 \cap \{2, \ldots, k\}| \leqq k,$$

in other words $k - p_k + q_k \leqq k$ or $q_k \leqq p_k$.

4. Some special cases. We wish to describe two cases in which we can obtain orthogonal irreducible subspaces by making use of an additional group action on S_m^n.

Case 1. $m = 2$. Let $z = x_1 + ix_2 \in \mathbf{C}^n$, $\bar{z} = x_1 - ix_2$. The condition $x \in S_2^n$ becomes $z \cdot z = 0$, $z \cdot \bar{z} = 2$ (bilinear inner product). The group $SO(2)$ acts on S_2^n by sending $x_1 \to \cos\theta x_1 + \sin\theta x_2$ and $x_2 \to -\sin\theta x_1 + \cos\theta x_2$. In terms of z, $\bar{z}$ coordinates it sends z_1 to $e^{i\theta}z$ and $\bar{z}$ to $e^{-i\theta}\bar{z}$. We will decompose $L^2(S_2^n)$ under the action of both $SO(n)$ and $SO(2)$.

Now each S_2^n admissible sequence $\mathscr{A}$ is specified by three non-negative integers r, s, t giving the number of occurrences of $\{1, 2\}$, $\{1\}$ and $\{2\}$, respectively. The only restriction on them is that $r = 0$ or 1 if $n = 3$. Now in place of $f(\mathscr{A})$ we consider $g(r, s, t) = (a_1 \cdot z\, a_2 \cdot \bar{z} - a_2 \cdot z a_1 \cdot \bar{z})^r (a_1 \cdot z)^s (a_1 \cdot \bar{z})^t$ (if $n = 3$ replace a_2 by b, and if $n = 4$ consider also $g^-(r, s, t)$ where a_2 is replaced by $\bar{a}_2$).

THEOREM 3. *For each choice of* r, s, t (with $r = 0$ or 1 if $n = 3$) *the function* $g(r, s, t)$ *is a non-zero highest weight vector of an irreducible representation of* $SO(n)$ *of highest weight* $\omega = (r + s + t, r)$ (if $n = 3$ the highest weight is $(r + s + t)$, and if $n = 4$ the highest weight for $g^-(r, s, t)$ is $(r + s + t, -r)$). *We also have* $g(e^{i\theta}z, e^{-i\theta}\bar{z}) = e^{ik\theta}g(z, \bar{z})$ *where* $k = s - t$. *No two distinct values of* (r, s, t) *give rise to the same values for* ω *and* k, *hence the spaces generated by the* $g(r, s, t)$ (and $g^-(r, s, t)$ when $n = 4$) *are orthogonal in any invariant inner product; furthermore their restrictions to* S_2^n *are non-zero and span* $L^2(S_2^n)$.

Proof. The proof of Theorem 1 can be repeated almost verbatim to show that $g(r, s, t)$ is a highest weight vector with the given weight. The fact that $g(e^{i\theta}z, e^{-i\theta}\bar{z}) = e^{ik\theta}g(z, \bar{z})$ is obvious from the definition of g. If (r, s, t) and (r', s', t') give rise to the same ω and k then we have $r = r'$, $r + s + t = r' + s' + t'$ and $s - t = s' - t'$ from which we conclude $(r, s, t) = (r', s', t')$. If $n = 3$ we must modify the argument somewhat since we are not given $r = r'$ but merely that r and r' are 0 or 1. But then if $r \neq r'$ they have different parity hence $s + t$ and $s' + t'$ have different parity which contradicts $s - t = s' - t'$. Thus $r = r'$ and we proceed as before.

We note in passing the permissible values of k given ω: if $n = 3$, $-m_1 \leqq k \leqq m_1$, and if $n \geqq 4$, $-m_1 + |m_2| \leqq k \leqq m_1 - |m_2|$ and k has the same parity as $m_1 - |m_2|$.

To show that $g(r, s, t)$ has non-zero restriction to S_2^n we evaluate it at

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \quad \text{and} \quad x_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix},$$

where $a_1 \cdot z = a_1 \cdot \bar{z} = 1$, $a_2 \cdot z = 1$ and $a_2 \cdot \bar{z} = -i$. Finally, the restrictions span $L^2(S_2^n)$ because the multiplicities agree with those in Theorem 2.

Remark. The case $n = 3$ can be further simplified using the Remark following Theorem 2. If $x = (x_{ij})$ denotes a 3×3 rotation matrix, then the functions $h(r, s, t)$

$$(x_{31} + ix_{32})^r (x_{11} + ix_{12} + ix_{21} - x_{22})^s (x_{11} + ix_{12} - ix_{21} + x_{22})^t$$

with $r = 0$ or 1, generate orthogonal irreducible subspaces of $L^2(SO(3))$ with highest weight $(s + t)$ which span $L^2(SO(3))$.

Case 2. $n = m = 4$. Here we set $z_1 = x_1 + ix_2$, $\bar{z}_1 = x_1 - ix_2$, $z_2 = x_3 + ix_4$ and $\bar{z}_2 = x_3 - ix_4$. We consider the maximal torus T^2 in $SO(4)$ acting by right multiplication. In z coordinates this action is given by $z_1 \to e^{i\theta_1}z_1$, $z_2 \to e^{i\theta_2}z_2$. We decompose $L^2(SO(4))$ with respect to the action of T^2 as well as $SO(4)$ and obtain multiplicity one, hence orthogonality.

Let $r_1, r_2, r_3, s_1, s_2, s_3, s_4$ be non-negative integers satisfying $r_1r_2 = s_2s_1 = 0$. We define

$$g(r, s) = (a_1 \cdot z_1a_2 \cdot z_2 - a_2 \cdot z_1a_2 \cdot z_2)^{r_1}(a_1 \cdot \bar{z}_1a_2 \cdot \bar{z}_2 - a_2 \cdot \bar{z}_1a_1 \cdot \bar{z}_2)^{r_2} \cdot (a_1 \cdot z_1a_2 \cdot \bar{z}_1 - a_2 \cdot z_1a_1 \cdot \bar{z}_2)^{r_3}(a_1 \cdot z_1)^{s_1}(a_1 \cdot \bar{z}_1)^{s_2}(a_1 \cdot z_2)^{s_3}(a_1 \cdot \bar{z}_2)^{s_4}$$

and, if $r_1 + r_2 + r_3 \neq 0$ we define $g^-(r, s)$ by replacing a_2 with $\bar{a}_2$ and z_2 with $\bar{z}_2$.

THEOREM 4. *The function $g(r, s)$ is a non-zero highest weight vector for an irreducible representation of $SO(4)$ of highest weight $\omega = (m_1, m_2)$ where $m_1 = \sum r_j + \sum s_j$ and $m_2 = \sum r_j$, and furthermore*

$$g(e^{i\theta_1}z_1, e^{-i\theta_1}\bar{z}_1, e^{i\theta_2}z_2, e^{-i\theta_2}\bar{z}_2) = e^{ik_1\theta_1}e^{ik_2\theta_2}g(z_1, \bar{z}_1, z_2, \bar{z}_2)$$

where $k_1 = r_1 - r_2 + s_1 - s_2$ and $k_2 = r_1 - r_2 + s_3 - s_4$. Similarly for $g^-(r, s)$, where $m_1 = \sum r_j + \sum s_j$, $m_2 = -\sum r_j$, $k_1 = r_1 - r_2 + s_1 - s_2$ and $k_2 = -r_1 + r_2 + s_3 - s_4$. Given any dominant weight $\omega = (m_1, m_2)$, the values of k_1, k_2 that arise are exactly those satisfying $|k_1 + k_2| \leqq m_1 + m_2$, $|k_1 - k_2| \leqq m_1 - m_2$ and $k_1 + k_2$ has the same parity as $m_1 + m_2$. The restrictions of $g(r, s)$ and $g^-(r, s)$ to $SO(4)$ are non-zero, the spaces they generate under the (left) action of $SO(4)$ are orthogonal (or coincident) and span $L^2(SO(4))$.

Proof. The fact that $g(r, s)$ is a highest weight vector with weight ω is proved as before. The transformation under the action of T^2 is obvious. If $m_2 \geqq 0$, the relation $|k_1 + k_2| \leqq m_1 + m_2$ follows from $k_1 + k_2 = 2r_1 - 2r_2 + s_1 - s_2 + s_3 - s_4$ and $m_1 + m_2 = 2\sum r_j + \sum s_j$. Since $m_1 + m_2 - (k_1 + k_2) = 4r_2 + 2r_3 + 2s_2 + 2s_4$ it follows that $m_1 + m_2$ and $k_1 + k_2$ have the same parity. Similarly we prove $|k_1 - k_2| \leqq m_1 - m_2$, and handle the case $m_2 < 0$.

To construct $g(r, s)$ or $g^-(r, s)$ given ω and k_1, k_2 we proceed by induction. First we consider the case $m_2 = 0$. Here we must have $r_1 = r_2 = r_3 = 0$, $\sum s_j = m_1$, $s_1 - s_2 = k_1$, $s_3 - s_4 = k_2$. If $k_1 \geqq 0$ we set $s_1 = k_1$, $s_2 = 0$. If $k_1 < 0$ we set $s_1 = 0$, $s_2 = -k_1$. In either case we solve the remaining equations, obtaining $s_3 = \frac{1}{2}(m_1 + k_2 - |k_1|)$, $s_4 = \frac{1}{2}(m_1 - k_2 - |k_1|)$. Next we assume the result true for $\omega' = (m_1 - 1, m_2 - 1)$ with $m_2 - 1 \geqq 0$ and prove it for $\omega = (m_1, m_2)$. Let k_1, k_2 be given. Suppose first $|k_1 + k_2| \leqq m_1 + m_2 - 2$. Then ω' and k_1, k_2 satisfy the hypotheses of the theorem. Thus by the induction hypothesis there exists $g(r', s')$ with weights ω' and k_1, k_2. But then $g(r, s)$ has weights ω and k_1, k_2 if we set $r_1 = r_1'$, $r_2 = r_2'$, $r_3 = r_3' + 1$, $s_j = s_j'$.

In the remaining cases $k_1 + k_2 = \pm(m_1 + m_2)$. Assume $k_1 + k_2 = m_1 + m_2$, the other case being treated similarly. Then ω' and $k_1 - 1$, $k_2 - 1$ satisfy the hypotheses of the theorem, hence by the induction hypothesis there exists

$g(r', s')$ with weights ω' and $k_1 - 1$, $k_2 - 1$. By setting $r_1 = r_1' + 1$, $r_2 = r_2'$, $r_3 = r_3'$, $s_j = s_j'$ we obtain $g(r, s)$ with weights ω and k_1, k_2.

Finally we assume the result for $\omega' = (m_1 - 1, m_2 + 1)$ with $m_2 + 1 \leqq 0$ and prove it for $\omega = (m_1, m_2)$. The argument is similar to the above. If $|k_1 - k_2| \leqq m_1 - m_2 - 2$ we apply the induction hypothesis to ω' and k_1, k_2 and then increase r_3' by one. If not, say $k_1 - k_2 = m_1 - m_2$, then we apply the induction hypothesis to ω' and $k_1 - 1$, $k_2 + 1$ and then increase r_1' by one.

To show the restriction of $g(r, s)$ and $g^-(r, s)$ to $SO(4)$ is non-zero we evaluate at

$$x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Here

$$z_1 = \begin{bmatrix} 1 \\ 0 \\ i \\ 0 \end{bmatrix} \quad \text{and} \quad z_2 = \begin{bmatrix} 0 \\ i \\ 0 \\ 1 \end{bmatrix},$$

$a_1 \cdot z_1 = a_1 \cdot \bar{z}_1 = -a_1 \cdot z_2 = a_1 \cdot \bar{z}_2 = 1, a_2 \cdot z_1 = -a_2 \cdot \bar{z}_1 = a_2 \cdot z_2 = a_2 \cdot \bar{z}_2 = i$, hence $(a_1 \cdot z_1 a_2 \cdot \bar{z}_1 - a_2 \cdot z_1 a_1 \cdot \bar{z}_1) = -2i$, $(a_1 \cdot z_1 a_2 \cdot z_2 - a_2 \cdot z_1 a_1 \cdot z_2) = 2i$ and $(a_1 \cdot \bar{z}_1 a_2 \cdot \bar{z}_2 - a_2 \cdot \bar{z}_1 a_1 \cdot \bar{z}_2) = 2i$, hence $g(r, s) \neq 0$. Also $\bar{a}_2 \cdot z_1 = -\bar{a}_2 \cdot \bar{z}_1 = -\bar{a}_2 \cdot z_2 = -\bar{a}_2 \cdot \bar{z}_2 = i$ hence $(a_1 \cdot z_1 \bar{a}_2 \cdot \bar{z}_1 - \bar{a}_2 \cdot z_1 a_1 \cdot \bar{z}_1) = 2i$, $(a_1 \cdot z_1 \bar{a}_2 \cdot \bar{z}_2 - \bar{a}_2 \cdot z_1 a_1 \cdot \bar{z}_2) = -2i$ and $(a_1 \cdot \bar{z}_1 \bar{a}_2 \cdot z_2 - \bar{a}_2 \cdot \bar{z}_1 a_1 \cdot z_2) = -2i$, hence $g^-(r, s) \neq 0$.

Now the multiplicity of the irreducible subspaces of highest weight $\omega = (m_1, m_2)$ in $L^2(SO(4))$ is equal to the dimension of the representation, which is known to be $(m_1 + m_2 + 1)(m_1 - m_2 + 1)$ (see [**1**]). But the number of pairs k_1, k_2 for ω is exactly $(m_1 + m_2 + 1)(m_1 - m_2 + 1)$. Thus the spaces generated by $g(r, s)$ and $g^-(r, s)$ span $L^2(SO(4))$ and are orthogonal or coincident according as the associated weights ω and k_1, k_2 are distinct or not.

Remark. We could use similar ideas to decompose $L^2(S_m{}^n)$ with respect to the right action of the maximal torus in $SO(m)$. However the known multiplicity formulas (see [**3**]) indicate that we do not obtain multiplicity one except in the cases considered above.

5. The symmetric space $SO(n)/SO(n - m) \times SO(m)$. The space $SO(n)/SO(n - m) \times SO(m)$ is a compact symmetric space, and, as is well-known, the irreducible representations of $SO(n)$ that appear in the Fourier decomposition of $L^2(SO(n)/SO(n - m) \times SO(m))$ occur with multiplicity one. These have been identified (see Sugiura [**7**] and Takeuchi [**8**]) as follows:

Without loss of generality we may assume $2 \leqq m \leqq \mu$. Then a representation with highest weight $\omega = (m_1, \ldots, m_\mu)$ occurs if and only if

(a) $m_l = 0$ for all $k > m$; and

(b) the integers $m_1, \ldots, m_\mu$ all have the same parity (hence they must all be even unless $m = \mu$).

Now $L^2(SO(n)/SO(n-m) \times SO(m))$ may be realized as the subspace of $L^2(S_m^n)$ consisting of functions invariant under the action of right multiplication by matrices in $SO(m)$. In this realization we will construct explicitly the highest weight vector of every irreducible representation that occurs.

For each positive integer k satisfying $k \leqq m$ and $k < \mu$ we define

$$F_k = \sum_{|A|=k} M(A)^2 = \frac{1}{m!}\sum_{i_1=1}^{m} \ldots \sum_{i_k=1}^{m} M(\{i_1, \ldots, i_k\})^2.$$

If $m = \mu$ we also define

$$G_+ = M(\{1, \ldots, \mu\})$$

and similarly G_- by replacing a_μ by $\bar{a}_\mu$.

LEMMA 6. *Let $R \in SO(m)$. Then $F_k(xR) = F_k(x)$ and $G_\pm(xR) = G_\pm(x)$ when $m = \mu$.*

Proof. Recall that $M(\{1, \ldots, \mu\}) = \det(\{a_j \cdot x_k\})$. Thus $G_+(xR) = \det(\{a_j \cdot x_k\}R)$ so $G_+(xR) = G_+(x)$ since $\det R = 1$. Similarly $G_-(xR) = G_-(x)$.

Next we observe that

$$M(\{i_1, \ldots, i_k\})(xR) = \sum_{j_1=1}^{m} \ldots \sum_{j_k=1}^{m} R_{j_1 i_1} \ldots R_{j_k i_k} M(\{j_1, \ldots, j_k\})(x)$$

so that

$$F_k(xR) = \frac{1}{m!}\sum_{i_1=1}^{m} \ldots \sum_{i_k=1}^{m}\sum_{j_1=1}^{m} \ldots \sum_{j_k=1}^{m}\sum_{r_1=1}^{m} \ldots \sum_{r_k=1}^{m} R_{j_1 i_1} \ldots R_{j_k i_k} R_{r_1 i_1} \ldots \times R_{r_k i_k} \cdot M(\{j_1, \ldots, j_k\}) M(\{r_1, \ldots, r_k\}).$$

Summing first over the i's and using the orthonormality of the rows of R we see that all terms cancel unless $j_1 = r_1, \ldots, j_k = r_k$, and for these terms the coefficients sum to one. Thus $F_k(xR) = R_k(x)$.

THEOREM 5. *The subspaces generated under the action of $SO(n)$ by the restriction to S_m^n of the following functions give the orthogonal decomposition of L^2-$(SO(n)/SO(n-m) \times SO(m))$ into irreducible subspaces:*

(i) *if $m < \mu$, $\prod_{k=1}^{m} F_k^{r_k}$ for non-negative integers $r_1, \ldots, r_m$;*

(ii) *if $m = \mu$, $\prod_{k=1}^{m-1} F_k^{r_k} G_\pm^{s}$ for non-negative integers $r_1, \ldots, r_{m-1}, s$.*

The highest weights of these representations are given by

(i) $m_j = \sum_{k=j}^{m} 2r_k$ *if* $j \leqq m$, $m_j = 0$ *if* $j > m$.

(ii) $m_j = s + \sum_{k=j}^{m-1} 2r_k$ *if* $j \leqq m - 1$, $m_m = \pm s$.

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Proof. By taking

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \text{ etc.,}$$

it is easy to see that these functions have non-trivial restrictions to S_m^n. By Lemma 6 they belong to $L^2(SO(n)/SO(n-m) \times SO(m))$ and by Theorem 1 they are highest weight vectors for irreducible representations of $SO(n)$ with the given highest weight. But it is clear that the weights given by (i) and (ii) above coincide with the weights satisfying (a) and (b) above, so we have the complete decomposition of $L^2(SO(n)/SO(n-m) \times SO(m))$.

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