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PAPER

Two-layered logics for probabilities and belief functions over Belnap–Dunn logic

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Abstract

This paper is an extended version of Bílková et al. ((2023b). *Logic, Language, Information, and Computation. WoLLIC 2023*, Lecture Notes in Computer Science, vol. 13923, Cham, Springer Nature Switzerland, 101–117.). We discuss two-layered logics formalising reasoning with probabilities and belief functions that combine the Łukasiewicz $[0,1]$-valued logic with Baaz $\triangle$ operator and the Belnap–Dunn logic. We consider two probabilistic logics – $\mathsf{Pr}^{\mathsf{Ł}^2}_{\triangle}$ (introduced by Bílková et al. 2023d. *Annals of Pure and Applied Logic*, 103338.) and $\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}_\triangle}$ (from Bílková et al. 2023b. *Logic, Language, Information, and Computation. WoLLIC 2023*, Lecture Notes in Computer Science, vol. 13923, Cham, Springer Nature Switzerland, 101–117.) – that present two perspectives on the probabilities in the Belnap–Dunn logic. In $\mathsf{Pr}^{\mathsf{Ł}^2}_{\triangle}$, every event ϕ has independent positive and negative measures that denote the likelihoods of ϕ and $\neg\phi$, respectively. In $\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}_\triangle}$, the measures of the events are treated as partitions of the sample into four exhaustive and mutually exclusive parts corresponding to pure belief, pure disbelief, conflict and uncertainty of an agent in ϕ. In addition to that, we discuss two logics for the paraconsistent reasoning with belief and plausibility functions from Bílková et al. ((2023d). *Annals of Pure and Applied Logic*, 103338.) – $\mathsf{Bel}^{\mathsf{Ł}^2}_{\triangle}$ and $\mathsf{Bel}^{\mathsf{NŁ}}$. Both these logics equip events with two measures (positive and negative) with their main difference being that in $\mathsf{Bel}^{\mathsf{Ł}^2}_{\triangle}$, the negative measure of ϕ is defined as the *belief in* $\neg\phi$ while in $\mathsf{Bel}^{\mathsf{NŁ}}$, it is treated independently as *the plausibility of* $\neg\phi$. We provide a sound and complete Hilbert-style axiomatisation of $\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}_\triangle}$ and establish faithful translations between it and $\mathsf{Pr}^{\mathsf{Ł}^2}_{\triangle}$. We also show that the validity problem in all the logics is coNP-complete.

Keywords: Two-layered logics; Łukasiewicz logic; non-standard probabilities; non-standard belief functions; paraconsistent logics

1. Introduction

Classical probability and Dempster–Shafer theories study probability measures and belief functions. These are maps from the set of events of a sample space W (i.e., from 2^W) to $[0,1]$ that are monotone w.r.t. $\subseteq$ with additional conditions. Probability measures satisfy the (finite or countable) additivity condition

$$\mu\left(\bigcup_{i\in I} E_i\right) = \sum_{i\in I}\mu(E_i) \qquad (I\subseteq 2^W, \forall i,j\in I: i\neq j \Rightarrow E_i\cap E_j=\varnothing)$$

and belief functions its weaker version (*total monotonicity* in the terminology of Zhou 2013)

$$\mathtt{bel}(W)=1 \qquad \mathtt{bel}\left(\bigcup_{1\le i\le k} E_i\right) \ge \sum_{\substack{J\subseteq\{1,\ldots,k\}\\ J\neq\varnothing}} (-1)^{|J|+1}\cdot\mathtt{bel}\left(\bigcap_{j\in J} E_j\right) \qquad \mathtt{bel}(\varnothing)=0$$

Above, the disjointness of E_i and E_j can be understood as their incompatibility. Most importantly, if a propositional formula ϕ is associated with an event $\|\phi\|$ (and interpreted as a statement about it), then

$$\mu(\|\phi\wedge\neg\phi\|)=\mathtt{bel}(\|\phi\wedge\neg\phi\|)=0$$

(since $\|\phi\|$ and $\|\neg\phi\|$ are incompatible) and $\|\phi\vee\neg\phi\|$ exhausts the entire sample space, whence

$$\mu(\|\phi\vee\neg\phi\|)=\mathtt{bel}(\|\phi\vee\neg\phi\|)=1$$

Paraconsistent uncertainty theory, on the other hand, assumes that the measure of an event represents not the likelihood of it happening but an agent's certainty therein which they infer from the information given by the sources. In this respect, it is close to the classical Dempster–Shafer theory that can also be considered as a generalisation of the subjective probability theory.

The main difference between the classical and paraconsistent approaches is the treatment of negation. Dempster–Shafer theory usually deals with contradictions *between different sources*. However, it is reasonable to assume that even a *single source* can give contradictory information (or give no information at all): for instance, a witness in court can provide a contradictory testimony; likewise, a newspaper article can fail to mention some event at all, without explicitly denying or confirming it. Thus, a 'contradictory' event $\|\phi\wedge\neg\phi\|$ can have a positive probability or belief assignment and $\|\phi\vee\neg\phi\|$ does not necessarily exhaust the sample space. Thus, a logic describing events should allow them to be both true and false (if the source gives contradictory information) or neither true nor false (when the source does not give information). Formally, this means that $\neg$ does not correspond to the complement in the sample space.

1.1 Belnap–Dunn logic

As one can see from the previous paragraph, we need a very special kind of negation: the one that allows for true contradictions (and thus, rejects the principle of explosion), and, additionally, invalidates the law of excluded middle. Thus, the following principles are no longer valid

$$\mathsf{EFQ}: p\wedge\neg p\models q \qquad\qquad \mathsf{LEM}: p\models q\vee\neg q$$

The logics that lack EFQ are called *paraconsistent*, those that do not have LEM are *paracomplete*, and those that fail both principles are *paradefinite* or *paranormal* (cf., e.g., Arieli and Avron 2017 for the terminology).

The simplest paradefinite logic to represent reasoning about information provided by sources is the Belnap–Dunn logic (BD) from Belnap (1977, 2019) and Belnap (1977, 2019). Originally, BD was presented as a four-valued propositional logic in the $\{\neg,\wedge,\vee\}$ language. The values (which we will henceforth call *Belnapian values*) represent different accounts a source can give regarding a statement ϕ:

- **T** stands for 'the source only says that ϕ is true'
- **F** stands for 'the source only says that ϕ is false'
- **B** stands for 'the source says both that ϕ is false and that ϕ is true'
- **N** stands for 'the source does not say that ϕ is false nor that it is true'.

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The interpretation of the truth values allows for reformulating BD semantics in terms of *two classical but independent valuations*. Namely,

	is true when	is false when
$\neg\phi$	ϕ is false	ϕ is true
$\phi_1 \wedge \phi_2$	ϕ_1 and ϕ_2 are true	ϕ_1 is false or ϕ_2 is false
$\phi_1 \vee \phi_2$	ϕ_1 is true or ϕ_2 is true	ϕ_1 and ϕ_2 are false

It is easy to see that there are no universally true nor universally false formulas in BD. Thus, BD satisfies the desiderata outlined above. Moreover, even if we define $\phi \supset \chi$ as $\neg\phi \vee \chi$, it is clear that neither Modus Ponens, nor the Deduction theorem will hold for $\supset$. That is, BD lacks the (definable) implication (cf. Omori and Wansing 2017, Section 5.1) for a detailed discussion of the implication in BD). This, however, is not problematic since we are going to use BD only to represent events and conditional statements which are usually formalised with an implication do not correspond to descriptions of events.

1.2 Probabilities and belief functions in BD

The original interpretation of the Belnapian truth values that we gave above is presented in terms of the information one has. In this approach, however, the information is assumed to be crisp. Theories of uncertainty over BD were introduced to formalise situations where one has access to graded information. For instance, the first source could tell that p is true with the likelihood 0.4 and the second that p is false with probability 0.7. If one follows BD and treats positive and negative evidence independently, one needs a non-classical notion of uncertainty measures to represent this information.

To the best of our knowledge, the earliest formalisation of probability theory in terms of BD was provided by Mares (1997). The formalisation is very similar to the one that we are going to use in this paper but bears one significant distinction. Namely, *normalised* measures (i.e., those where $\mu(W) = 1$ and $\mu(\varnothing) = 0$) are used by Mares (1997). This requirement, however, is superfluous since BD lacks universally (in)valid formulas.

Another formalisation is given by Dunn (2010). Dunn proposes to divide the sample space into four exhaustive and mutually exclusive parts depending on the Belnapian value of ϕ. An alternative approach was proposed by Klein et al. (2021). There, the authors propose two equivalent interpretations based on the two formulations of semantics. The first option (which we call here $\pm$*-probability*) is to give ϕ two *independent probability measures*: the one determining the likelihood of ϕ to be true and the other the likelihood of ϕ to be false. The second option (**4**-*probabilities*) follows Dunn and divides the sample space according to whether ϕ has value **T**, **B**, **N**, or **F** in a given state. Note that in both cases, the probabilities are interpreted *subjectively*.

The main difference between the approaches of Dunn (2010) and Klein et al. (2021) is that in the former, the probability of $\phi \wedge \phi'$ is entirely determined by those of ϕ and ϕ' which makes it compositional. On the other hand, the paraconsistent probabilities proposed by Klein et al. (2021) are not compositional w.r.t. conjunction. In this paper, we choose the latter approach since it can be argued (cf. Dubois 2008 for the details) that belief is not compositional when it comes to contradictory information.

A similar approach to paraconsistent probabilities was proposed by, for example, Bueno-Soler and Carnielli (2016) and Rodrigues et al. (2021). There, probabilities are defined over an extension of BD with classicality and non-classicality operators. It is worth mentioning that the proposed axioms of probability are very close to those from Klein et al. (2021): for example, both allow measures π s.t. $\pi(\phi) + \pi(\neg\phi) < 1$ (if the information regarding ϕ is incomplete) or $\pi(\phi) + \pi(\neg\phi) > 1$ (when the information is contradictory).

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Belief functions over BD were first defined by Zhou (2013). There, they were presented on the ordered sets of states. Each formula ϕ in this approach corresponds to *two* sets of states: $|\phi|^+$ (states where ϕ has value **T** or **B**) and $|\phi|^-$ (states where it is evaluated at **B** or **F**). Moreover, a logic formalising reasoning with belief functions was presented. A similar treatment of (non-normalised or *general* in the terminology of the paper) belief functions in BD was given by Bílková et al. (2023d). The main difference between the two treatments of belief functions was that Bílková et al. (2023d) considered two options for interpreting $\texttt{bel}(|\phi|^-)$: the first one was to treat it as the belief of $\neg\phi$, and the second one – as *the plausibility* of $\neg\phi$. Another distinction was in the formalisation: Zhou (2013) constructs a logic for reasoning about belief functions following Fagin et al. (1990): That is, incorporating the arithmetical operations and inequalities containing them into the language. Bílková et al. (2023d) utilised a different approach: instead of using arithmetic inequalities, the reasoning about belief functions is conducted in a paraconsistent expansion of the Łukasiewicz logic Ł capable of expressing arithmetic operations on $[0, 1]$.

1.3 Two-layered logics for uncertainty

Reasoning about uncertainty can be formalised via modal logics where the modality is understood as a measure of an event. The concrete semantics of the modality can be defined in two ways. First, using a modal language with Kripke semantics where the measure is defined on the set of states as done by, for example, Gärdenfors (1975),Delgrande and Renne (2015), Delgrande et al. (2019) for qualitative probabilities, by Dautović et al. (2021) for the quantitative ones, and by Rodriguez et al. (2022) for the possibility and necessity measures. Second, employing a two-layered formalism (cf. Baldi et al. 2020; Fagin et al. 1990; Fagin and Halpern 1991; Hájek et al. 1995, and Bílková et al. 2023a,b,d) for examples). There, the logic is split into two levels: the inner layer describes events, and the outer layer describes the reasoning with the measure defined on events. The measure is a *non-nesting* modality $\mathtt{M}$, and the outer-layer formulas are built from *'modal atoms'* – formulas of the form $\mathtt{M}\phi$ with ϕ being an inner-layer formula. The outer-layer formulas are then equipped with the semantics of a fuzzy logic that permits necessary operations (e.g., Łukasiewicz for the quantitative reasoning and Gödel for the qualitative).

An alternative to the two-layered logics is to use the language of linear inequalities to reason about measures of events. This is done by Fagin et al. (1990) for the classical reasoning about probabilities and by Zhou (2013) for the reasoning with the belief functions over BD. In both cases, it is established that the logics with inequalities and the two-layered logics are equivalent – cf. Baldi et al. (2020) for the case of classical probabilities and Bílková et al. (2023d) for the reasoning with belief functions and probabilities over BD.

In this paper, we study reasoning about uncertainty in a paraconsistent framework. For this, we choose two-layered logics. First, they are more modular than the usual Kripke semantics: as long as the logic of the event description is chosen, we can define different measures on top of it using different outer-layer logics. Second, two-layered logics provide a uniform way to prove completeness. This is done by translating the axioms of a given measure into formulas of the outer-layer logic (cf. Cintula and Noguera 2014 for more details). Third, the techniques that are used to establish the decidability of the outer-layer logic can be applied to the decidability proof of the two-layered logic. Finally, even though the traditional Kripke semantics is more expressive than two-layered logics, this expressivity is not necessary in many contexts. Indeed, people rarely say something like 'it is probable that it is probable that ϕ'. Moreover, it is considerably more difficult to motivate the assignment of truth values in the nesting case, in particular, when the same measure is applied both to a propositional and modalised formula as in, for example, $\mathtt{M}(p \wedge \mathtt{M}q)$.

We will also be formalising the *quantitative* reasoning. That is, we assume that the agents can assign numerical values to their certainty in a given proposition or say something like 'I think that rain is twice more likely than snow'. Thus, we need a logic that can express the paraconsistent counterparts of the additivity condition as well as basic arithmetic operations. We choose the

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Łukasiewicz logic (Ł) for the outer layer since it can define (truncated) addition and subtraction on $[0, 1]$.

We will focus on four logics. Two probabilistic ones: $\mathsf{Pr}^{\mathsf{Ł}^2}_{\triangle}$ (the logic of $\pm$-probabilities), $\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}_\triangle}$ (the logic of **4**-probabilities); and two logics for belief and plausibility functions over BD introduced by Bílková et al. (2023d) – $\mathsf{Bel}^{\mathsf{Ł}^2}_{\triangle}$ where belief and plausibility are defined via one another, and $\mathsf{Bel}^{\mathsf{NŁ}}$ where belief and plausibility are assumed to be independent.

1.4 Contributions and plan of the paper

This paper extends an earlier conference submission by Bílková et al. (2023b). We provide omitted details for the proofs of coNP-completeness of $\mathsf{Pr}^{\mathsf{Ł}^2}_{\triangle}$ and $\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}_\triangle}$ and of the finite strong completeness proof of $\mathscr{H}\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}_\triangle}$ – the Hilbert-style axiomatisation of $\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}_\triangle}$. The novel contribution of this manuscript is the proof of the coNP-completeness of $\mathsf{Bel}^{\mathsf{Ł}^2}_{\triangle}$ and $\mathsf{Bel}^{\mathsf{NŁ}}$. We obtain this result by establishing a correspondence between these logics and $\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}$ and $\mathsf{Pr}^{\mathsf{NŁ}}_{\mathbf{S5}}$ – two logics for reasoning about probabilities of *modal* BD formulas that we introduce in the present paper. To apply the technique from Hájek and Tulipani (2001), we establish a version of the small model property for the canonical models of $\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}$ and $\mathsf{Pr}^{\mathsf{NŁ}}_{\mathbf{S5}}$. Thus, we continue the study of uncertainty via paraconsistent logics proposed by Bílková et al. (2020) and carried on by Bílková et al. (2023a,c,d). The overarching goal of the project is to study logics that can express the following properties of beliefs.

1. Given two statements ϕ and χ, one can be more certain in ϕ than in χ but still, neither believe in ϕ *completely* nor consider χ completely impossible.
2. Given two trusted sources, one can still prefer one source to the other.
3. One can believe in a contradiction but still not believe in something else.
4. Given two statements, it is possible that one cannot always compare their degrees of certainty in them (if, e.g., these statements have no common content).

The remainder of the text is structured as follows. In Section 2, we recall two approaches to probabilities over BD from Klein et al. (2021) – $\pm$-probabilities and **4**-probabilities. In Section 3, we provide the semantics and axiomatisations of $\mathsf{Pr}^{\mathsf{Ł}^2}_{\triangle}$ and $\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}_\triangle}$ – two-layered logics for probabilities and establish their NP-completeness. In Section 4, we recall two treatments of belief functions over BD that were presented by Bílková et al. (2023d). In Section 5, we discuss the two-layered logics for belief functions ($\mathsf{Bel}^{\mathsf{Ł}^2}_{\triangle}$ and $\mathsf{Bel}^{\mathsf{NŁ}}$), establish their connections with modal probabilistic logics $\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}$ and $\mathsf{Pr}^{\mathsf{NŁ}}_{\mathbf{S5}}$, and provide a complexity evaluation of their validity problems. Finally, we summarise our results and set goals for future research in Section 6.

2. Probabilities over BD

In the previous section, we gave an informal presentation of BD as a four-valued logic. Since we are going to use it to describe events, we will formulate its semantics in terms of sets of states. The language of BD is given by the following grammar (with $\mathtt{Prop}$ being a countable set of propositional variables).

$$\mathscr{L}_{\mathsf{BD}} \ni \phi := p \in \mathtt{Prop} \mid \neg\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi)$$

Convention 1 (Notation). *In what follows,* $\mathtt{Prop}(\phi)$ *denotes the set of variables occurring in* ϕ *and* $\mathsf{Lit}(\phi)$ *stands for the set of literals (i.e., variables or their negations) occurring in* ϕ*. Moreover,* $\mathtt{Sf}(\phi)$ *is the set of all subformulas of* ϕ*. The length (i.e., the number of symbols) of* ϕ *is denoted with* $\ell(\phi)$*.*

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We are also going to use two kinds of formulas: the single- and the two-layered ones. To make the differentiation between them simpler, we use Greek letters from the end of the alphabet (ϕ, χ, ψ, etc.) to designate the first kind and the letters from the beginning of the alphabet (α, β, γ, ...) for the second kind. We use v (with indices) to stand for the valuations of single-layered formulas and e (with indices) for the two-layered formulas.

Finally, we will use angular brackets $\langle \ldots \rangle$ for tuples that designate models of logics and pairs of valuations and round brackets $(\ldots)$ for pairs of numbers that are values of formulas.

Definition 2 (Set semantics of BD). *Let $\phi, \phi' \in \mathscr{L}_{\mathsf{BD}}$, $W \neq \varnothing$, and $v^+, v^- : \texttt{Prop} \to 2^W$. For a model $\mathfrak{M} = \langle W, v^+, v^- \rangle$, we define notions of $w \vDash^+ \phi$ and $w \vDash^- \phi$ for $w \in W$ as follows.*

$$
\begin{array}{cc}
w \vDash^+ p \text{ iff } w \in v^+(p) & w \vDash^- p \text{ iff } w \in v^-(p) \\
w \vDash^+ \neg\phi \text{ iff } w \vDash^- \phi & w \vDash^- \neg\phi \text{ iff } w \vDash^+ \phi \\
w \vDash^+ \phi \wedge \phi' \text{ iff } w \vDash^+ \phi \text{ and } w \vDash^+ \phi' & w \vDash^- \phi \wedge \phi' \text{ iff } w \vDash^- \phi \text{ or } w \vDash^- \phi' \\
w \vDash^+ \phi \vee \phi' \text{ iff } w \vDash^+ \phi \text{ or } w \vDash^+ \phi' & w \vDash^- \phi \vee \phi' \text{ iff } w \vDash^- \phi \text{ and } w \vDash^- \phi'
\end{array}
$$

Given a model $\mathfrak{M}$, we denote the positive and negative extensions of a formula as follows:

$$
|\phi|^+_{\mathfrak{M}} := \{w \in W \mid w \vDash^+ \phi\} \qquad |\phi|^-_{\mathfrak{M}} := \{w \in W \mid w \vDash^- \phi\}.
$$

A sequent $\phi \vdash \chi$ is satisfied on $\mathfrak{M} = \langle W, v^+, v^- \rangle$ (denoted, $\mathfrak{M} \models [\phi \vdash \chi]$) iff $|\phi|^+ \subseteq |\chi|^+$. A sequent $\phi \vdash \chi$ is BD*-valid ($\phi \vDash_{\mathsf{BD}} \chi$) iff it is satisfied on every model. In this case, we will say that ϕ entails χ in* BD.

Remark 3. *One can see that every $\phi \in \mathscr{L}_{\mathsf{BD}}$ can be turned into its negation normal form $\mathsf{NNF}(\phi)$ where $\neg$ is applied to variables only. This can be done via the following transformations:*

$$
\neg\neg p \rightsquigarrow p \qquad \neg(\chi \wedge \psi) \rightsquigarrow \neg\chi \vee \neg\psi \qquad \neg(\chi \vee \psi) \rightsquigarrow \neg\chi \wedge \neg\psi
$$

Moreover, it is clear that $|\phi|^+_{\mathfrak{M}} = |\mathsf{NNF}(\phi)|^+_{\mathfrak{M}}$ and $|\phi|^-_{\mathfrak{M}} = |\mathsf{NNF}(\phi)|^-_{\mathfrak{M}}$ in every model $\mathfrak{M}$ and that $\ell(\mathsf{NNF}(\phi)) = \mathcal{O}(\ell(\phi))$.

Note that the semantics in Definition 2 is a formalisation of the truth and falsity conditions of $\{\neg, \wedge, \vee\}$-formulas we saw in the table in Section 1.1. We can now use it to define probabilities on the models. We adapt the definitions from Klein et al. (2021).

Definition 4 (BD-models with $\pm$-probabilities). *A* BD*-model with a $\pm$-probability is a tuple $\mathfrak{M}_\pm = \langle \mathfrak{M}, \mu \rangle$ with $\mathfrak{M}$ being a* BD*-model and $\mu : 2^W \to [0, 1]$ satisfying:*

- **mon**: *if $X \subseteq Y$, then $\mu(X) \leq \mu(Y)$;*
- **neg**: $\mu(|\phi|^-_{\mathfrak{M}}) = \mu(|\neg\phi|^+_{\mathfrak{M}})$;
- **ex**: $\mu(|\phi \vee \chi|^+_{\mathfrak{M}}) = \mu(|\phi|^+_{\mathfrak{M}}) + \mu(|\chi|^+_{\mathfrak{M}}) - \mu(|\phi \wedge \chi|^+_{\mathfrak{M}})$.

To facilitate the presentation of the four-valued probabilities defined over BD-models, we introduce additional extensions of ϕ defined via $|\phi|^+$ and $|\phi|^-$.

Convention 5. *Let $\mathfrak{M} = \langle W, v^+, v^- \rangle$ be a* BD*-model, $\phi \in \mathscr{L}_{\mathsf{BD}}$. We set*

$$
|\phi|^{\mathsf{b}}_{\mathfrak{M}} = |\phi|^+_{\mathfrak{M}} \setminus |\phi|^-_{\mathfrak{M}} \quad |\phi|^{\mathsf{d}}_{\mathfrak{M}} = |\phi|^-_{\mathfrak{M}} \setminus |\phi|^+_{\mathfrak{M}} \quad |\phi|^{\mathsf{c}} = |\phi|^+ \cap |\phi|^-_{\mathfrak{M}} \quad |\phi|^{\mathsf{u}}_{\mathfrak{M}} = W \setminus (|\phi|^+_{\mathfrak{M}} \cup |\phi|^-_{\mathfrak{M}})
$$

We call these extensions, respectively, pure belief, pure disbelief, conflict, and uncertainty in ϕ, following Klein et al. (2021).

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One can observe that these extensions correspond to the Belnapian values of ϕ (recall Section 1.1):

- *pure belief extension* of ϕ is the set of states where ϕ has value **T** (i.e., *exactly true*, in other words, true and not false);
- *pure disbelief extension* of ϕ is the set of states where ϕ has value **F** (i.e., *exactly false*, in other words, false and not true);
- *conflict extension* of ϕ is the set of states where ϕ has value **B** (i.e., *both true and false*);
- *uncertainty extension* of ϕ is the set of states where ϕ has value **N** (i.e., *neither true nor false*).

Definition 6 (BD-models with **4**-probabilities). *A* BD*-model with a* **4***-probability is a tuple* $\mathfrak{M}_\mathbf{4} = \langle \mathfrak{M}, \mu_v \rangle$ *with* $\mathfrak{M}$ *being a* BD*-model and* $\mu_\mathbf{4} : 2^W \to [0, 1]$ *satisfying:*

part: $\mu_\mathbf{4}(|\phi|^{\mathsf{b}}_{\mathfrak{M}}) + \mu_\mathbf{4}(|\phi|^{\mathsf{d}}_{\mathfrak{M}}) + \mu_\mathbf{4}(|\phi|^{\mathsf{u}}_{\mathfrak{M}}) + \mu_\mathbf{4}(|\phi|^{\mathsf{c}}_{\mathfrak{M}}) = 1$;
neg: $\mu_\mathbf{4}(|\neg\phi|^{\mathsf{b}}_{\mathfrak{M}}) = \mu_\mathbf{4}(|\phi|^{\mathsf{d}}_{\mathfrak{M}})$, $\mu_\mathbf{4}(|\neg\phi|^{\mathsf{c}}_{\mathfrak{M}}) = \mu_\mathbf{4}(|\phi|^{\mathsf{c}}_{\mathfrak{M}})$;
contr: $\mu_\mathbf{4}(|\phi \wedge \neg\phi|^{\mathsf{b}}_{\mathfrak{M}}) = 0$, $\mu_\mathbf{4}(|\phi \wedge \neg\phi|^{\mathsf{c}}_{\mathfrak{M}}) = \mu_\mathbf{4}(|\phi|^{\mathsf{c}}_{\mathfrak{M}})$;
BCmon: *if* $\mathfrak{M} \models [\phi \vdash \chi]$, *then* $\mu_\mathbf{4}(|\phi|^{\mathsf{b}}_{\mathfrak{M}}) + \mu_\mathbf{4}(|\phi|^{\mathsf{c}}_{\mathfrak{M}}) \leq \mu_\mathbf{4}(|\chi|^{\mathsf{b}}_{\mathfrak{M}}) + \mu_\mathbf{4}(|\chi|^{\mathsf{c}}_{\mathfrak{M}})$;
BCex: $\mu_\mathbf{4}(|\phi|^{\mathsf{b}}) + \mu_\mathbf{4}(|\phi|^{\mathsf{c}}) + \mu_\mathbf{4}(|\psi|^{\mathsf{b}}) + \mu_\mathbf{4}(|\psi|^{\mathsf{c}}) = \mu_\mathbf{4}(|\phi \wedge \psi|^{\mathsf{b}}) + \mu_\mathbf{4}(|\phi \wedge \psi|^{\mathsf{c}}) + \mu_\mathbf{4}(|\phi \vee \psi|^{\mathsf{b}}) + \mu_\mathbf{4}(|\phi \vee \psi|^{\mathsf{c}})$.

Convention 7. *We will further omit lower indices in* $|\phi|^+_{\mathfrak{M}}$, $|\phi|^-_{\mathfrak{M}}$, $|\phi|^{\mathsf{b}}_{\mathfrak{M}}$, *etc. and write* $|\phi|^+$, $|\phi|^-$, $|\phi|^{\mathsf{b}}$, *etc. when the model is clear from the context.*

We will utilise the following naming convention:

- *we use the term '$\pm$-probability' to stand for μ from Definition 4;*
- *we call $\mu_\mathbf{4}$ from Definition 6 a '**4**-probability' or a 'four-valued probability'.*

Recall that $\pm$-probabilities are referred to as 'non-standard' by Klein et al. (2021) *and* Bílková et al. (2023). *As this term is too broad (four-valued probabilities are not 'standard' either), we use a different designation.*

Let us quickly discuss the measures defined above. First, observe that $\mu(|\phi|^+)$ and $\mu(|\phi|^-)$ are independent from one another. Thus, μ gives two measures to each ϕ, as desired. Second, recall (Klein et al. 2021, Theorems 2–3) that every **4**-probability on a BD-model induces a $\pm$-probability and vice versa. In the following sections, we will define two-layered logics for BD-models with $\pm$- and **4**-probabilities and show that they can be faithfully embedded into each other. It is instructive to note, moreover, that while $\pm$- and **4**-probabilities can be simulated by the classical ones, they do behave in a very different way.

Convention 8 (Notation in models). *Throughout the paper, we are going to give examples of various models. We use the following notation for values of variables in the states of a given model.*

notation	meaning
$w : p^+$	$w \models^+ p$ and $w \not\models^- p$
$w : p^-$	$w \models^- p$ and $w \not\models^+ p$
$w : p^\pm$	$w \models^+ p$ and $w \models^- p$
$w : \not{p}$	$w \not\models^+ p$ and $w \not\models^- p$

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$u_1 : \not{p}$ $u_2 : \not{p}$

Figure 1. A counterexample to (*IE*): $\mu(\{u_1\}) = \mu(\{u_2\}) = \frac{1}{3}$, $\mu(W) = 1$, and $\mu(\varnothing) = 0$. All variables have the same values exemplified by p.

$w_1 : p^+$ $w_2 : p^\pm$ $w'_1 : p^+$ $w'_2 : p^\pm$ $w'_3 : \not{p}$

Figure 2. The values of all variables coincide with the values of p state-wise. $\mu(X) = \frac{1}{2}$ for every $X \subseteq W$; $\pi(\varnothing) = \pi(\{w'_1\}) = 0$, $\pi(W') = 1$, $\pi(X') = \frac{1}{2}$ otherwise.

Example 9 (Paraconsistent vs. classical probabilities). *First of all, observe that a $\pm$-probability can be uniform. Indeed, for every $c \in [0, 1]$ and every* BD*-model $\langle W, v^+, v^- \rangle$, it is easy to check that the assignment $\forall X \subseteq W : \mu(X) = c$ is a $\pm$-probability.*

Second, the general import-export condition

$$\mu(X \cup Y) = \mu(X) + \mu(Y) - \mu(X \cap Y) \qquad \text{(IE)}$$

that is true for the classical probabilities does not hold if μ is a $\pm$-probability. For consider Fig. 1. One can see that μ is monotone and that $\mu(|\phi|^+) = \mu(|\phi|^-) = 0$ for every $\phi \in \mathscr{L}_{\mathsf{BD}}$ (whence, μ is a $\pm$-probability). On the other hand, it is clear that $\mu(W) \neq \mu(\{u_1\}) + \mu(\{u_2\}) - \mu(\varnothing)$.

This, however, is not a problem since for every BD*-model with a $\pm$-probability $\langle W, v^+, v^-, \mu \rangle$, there exists a* BD*-model $\langle W', v'^+, v'^-, \pi \rangle$ with a classical probability measure π s.t. $\pi(|\phi|^+) = \mu(|\phi|^+)$ (Klein et al. 2021, Theorem 4).*

First of all, for the case of the model shown in Fig. 1, one can either define $\pi(\{u_1\}) = \pi(\{u_2\}) = \frac{1}{2}$ or add a new state u_3 where all variables are neither true nor false. For a bit more refined example, consider Fig. 2 and set $W = \{w_1, w_2\}$ and $W' = \{w'_1, w'_2, w'_3\}$. It is clear that for every $\phi \in \mathscr{L}_{\mathsf{BD}}$, $\mu(|\phi|^+) = \pi(|\phi|^+) = \frac{1}{2}$.

Likewise, $\mu_{\mathbf{4}}$ is not necessarily monotone w.r.t. $\subseteq$ (which is required of the classical probabilities) since not every subset of a model is represented by an extension of an $\mathscr{L}_{\mathsf{BD}}$ formula. Again, it is not a problem since for every BD*-model with* **4***-probability $\langle W, v^+, v^-, \mu_{\mathbf{4}} \rangle$, there exist a* BD*-model $\langle W', v'^+, v'^-, \pi \rangle$ with a classical probability measure π s.t. $\pi(|\phi|^{\mathsf{x}}) = \mu_{\mathbf{4}}(|\phi|^{\mathsf{x}})$ for $\mathsf{x} \in \{\mathsf{b}, \mathsf{d}, \mathsf{c}, \mathsf{u}\}$ (Klein et al. 2021, Theorem 5).*

Thus, we will further assume w.l.o.g. that μ and $\mu_{\mathbf{4}}$ are classical probability measures on W.

3. Logics for paraconsistent probabilities

In the Introduction, we saw that there could be two views on formalising paraconsistent probabilities (or uncertainty measures in general). The first option is to define probabilities in a paraconsistent logic. This is done, for example, by Dunn (2010), Bueno-Soler and Carnielli (2016), and Klein et al. (2021). Another option is to employ a two-layered framework where the outer layer is not explosive as Flaminio et al. (2022) do. Our approach can be thought of as a combination of these two. Not only are our probabilities defined over a paraconsistent logic, but the logic with which we reason about them is also non-explosive.

In this section, we provide two-layered logics that are (weakly) complete w.r.t. BD-models with $\pm$- and **4**-probabilities. Since conditions on measures contain arithmetic operations on [0, 1], we choose an expansion of Łukasiewicz logic, namely, Łukasiewicz logic with $\triangle$ ($Ł_\triangle$), for the outer layer. Furthermore, $\pm$-probabilities work with both positive and negative extensions of formulas, whence it is reasonable to use $Ł^2_{(\triangle,\rightarrow)}$ – a paraconsistent expansion of Ł (cf. Bílková et al. 2020 and 2021 for details) with two valuations – v_1 (support of truth) and v_2 (support of falsity) – on [0, 1].

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3.1 Languages, semantics, and equivalence

Let us first recall the semantics of $Ł_\triangle$ and $Ł^2_{(\triangle,\rightarrow)}$. To facilitate the presentation, we begin with the definition of the standard $Ł_\triangle$ algebra on $[0, 1]$ that we will then use to define the semantics of both $Ł_\triangle$ and $Ł^2_{(\triangle,\rightarrow)}$.

Definition 10. *The standard $Ł_\triangle$ algebra is a tuple $\langle[0, 1], \sim_Ł, \triangle_Ł, \wedge_Ł, \vee_Ł, \rightarrow_Ł, \odot_Ł, \oplus_Ł, \ominus_Ł\rangle$ with the operations are defined as follows.*

$$\sim_Ł a := 1 - a \qquad \triangle_Ł a := \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{array}{lll} a \wedge_Ł b := \min(a, b) & a \vee_Ł b := \max(a, b) & a \rightarrow_Ł b := \min(1, 1 - a + b) \\ a \odot_Ł b := \max(0, a + b - 1) & a \oplus_Ł b := \min(1, a + b) & a \ominus_Ł b := \max(0, a - b) \end{array}$$

Definition 11 ($Ł_\triangle$) *The language of $Ł_\triangle$ is given via the following grammar*

$$\mathscr{L}_Ł \ni \phi := p \in \texttt{Prop} \mid \sim\phi \mid \triangle\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi) \mid (\phi \odot \phi) \mid (\phi \oplus \phi) \mid (\phi \ominus \phi)$$

We will also write $\phi \leftrightarrow \chi$ as a shorthand for $(\phi \rightarrow \chi) \odot (\chi \rightarrow \phi)$.

A valuation is a map $v : \texttt{Prop} \rightarrow [0, 1]$ that is extended to the complex formulas as expected: $v(\phi \circ \chi) = v(\phi) \circ_Ł v(\chi)$.

ϕ is $Ł_\triangle$-valid iff $v(\phi) = 1$ for every v. Γ entails χ (denoted $\Gamma \models_{Ł_\triangle} \chi$) iff for every v s.t. $v(\phi) = 1$ for all $\phi \in \Gamma$, it holds that $v(\chi) = 1$ as well.

Definition 12 ($Ł^2_{(\triangle,\rightarrow)}$). *The language is constructed using the following grammar.*

$$\mathscr{L}_{Ł^2_{(\triangle,\rightarrow)}} \ni \phi := p \in \texttt{Prop} \mid \neg\phi \mid \sim\phi \mid \triangle\phi \mid (\phi \rightarrow \phi)$$

The semantics is given by two valuations v_1 (support of truth) and v_2 (support of falsity) $v_1, v_2 : \texttt{Prop} \rightarrow [0, 1]$ that are extended as follows.

$$\begin{array}{rlrl} v_1(\neg\phi) &= v_2(\phi) & v_2(\neg\phi) &= v_1(\phi) \\ v_1(\sim\phi) &= \sim_Ł v_1(\phi) & v_2(\sim\phi) &= \sim_Ł v_2(\phi) \\ v_1(\triangle\phi) &= \triangle_Ł v_1(\phi) & v_2(\triangle\phi) &= \sim_Ł \triangle_Ł \sim_Ł v_2(\phi) \\ v_1(\phi \rightarrow \chi) &= v_1(\phi) \rightarrow_Ł v_1(\chi) & v_2(\phi \rightarrow \chi) &= v_2(\chi) \ominus_Ł v_2(\phi) \end{array}$$

We say that ϕ is $Ł^2_{(\triangle,\rightarrow)}$-valid iff for every v_1 and v_2, it holds that $v_1(\phi) = 1$ and $v_2(\phi) = 0$.

Convention 13. *When there is no risk of confusion, we will use $v(\phi) = (x, y)$ to stand for $v_1(\phi) = x$ and $v_2(\phi) = y$.*

Remark 14. *Note that $\sim$ and $\rightarrow$ can define all other binary connectives in $Ł_\triangle$ and $Ł^2_{(\triangle,\rightarrow)}$.*

$$\begin{array}{lll} \phi \vee \chi := (\phi \rightarrow \chi) \rightarrow \chi & \phi \wedge \chi := \sim(\sim\phi \vee \sim\chi) & \phi \oplus \chi := \sim\phi \rightarrow \chi \\ \phi \odot \chi := \sim(\phi \rightarrow \sim\chi) & \phi \ominus \chi := \phi \odot \sim\chi & \phi \leftrightarrow \chi := (\phi \rightarrow \chi) \odot (\chi \rightarrow \phi) \end{array}$$

We are now ready to present our two-layered logics for paraconsistent probabilities: $\mathsf{Pr}^{Ł^2}_{\triangle}$ – the logic of $\pm$-probabilities and $\mathbf{4}\mathsf{Pr}^{Ł_\triangle}$ – the logic of $\mathbf{4}$-probabilities.

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Definition 15 ($4\mathsf{Pr}^{Ł_\triangle}$: language and semantics). *The language of* $4\mathsf{Pr}^{Ł_\triangle}$ *is constructed via the following grammar:*

$$\mathscr{L}_{4\mathsf{Pr}^{Ł_\triangle}} \ni \alpha := \mathsf{Bl}\phi \mid \mathsf{Db}\phi \mid \mathsf{Cf}\phi \mid \mathsf{Uc}\phi \mid \sim\alpha \mid \triangle\alpha \mid (\alpha \to \alpha) \qquad (\phi \in \mathscr{L}_{\mathsf{BD}})$$

A $4\mathsf{Pr}^{Ł_\triangle}$*-model is a tuple* $\mathbb{M} = \langle \mathfrak{M}, \mu_4, e\rangle$ *s.t.*

- $\langle \mathfrak{M}, \mu_4\rangle$ *is a* BD*-model with* **4***-probability;*
- e *is a* $Ł_\triangle$ *valuation induced by* μ_4*, that is:*
  - $e(\mathsf{Bl}\phi) = \mu_4(|\phi|^{\mathsf{b}})$, $e(\mathsf{Db}\phi) = \mu_4(|\phi|^{\mathsf{d}})$, $e(\mathsf{Cf}\phi) = \mu_4(|\phi|^{\mathsf{c}})$, $e(\mathsf{Uc}\phi) = \mu_4(|\phi|^{\mathsf{u}})$;
  - *values of complex* $\mathscr{L}_{4\mathsf{Pr}^{Ł_\triangle}}$*-formulas are computed via Definition 11.*

We say that α *is* $4\mathsf{Pr}^{Ł_\triangle}$*-valid iff* $e(\alpha) = 1$ *in every* $4\mathsf{Pr}^{Ł_\triangle}$*-model. A set of formulas* Γ *entails* α $(\Gamma \models_{4\mathsf{Pr}^{Ł_\triangle}} \alpha)$ *iff there is no* $\mathbb{M}$ *s.t.* $e(\gamma) = 1$ *for every* $\gamma \in \Gamma$ *but* $e(\alpha) \neq 1$.

Definition 16 ($\mathsf{Pr}^{Ł^2}_\triangle$: language and semantics). *The language of* $\mathsf{Pr}^{Ł^2}_\triangle$ *is given by the following grammar*

$$\mathscr{L}_{\mathsf{Pr}^{Ł^2}_\triangle} \ni \alpha := \mathsf{Pr}\phi \mid \sim\alpha \mid \neg\alpha \mid \triangle\alpha \mid (\alpha \to \alpha) \qquad (\phi \in \mathscr{L}_{\mathsf{BD}})$$

A $\mathsf{Pr}^{Ł^2}_\triangle$*-model is a tuple* $\mathbb{M} = \langle \mathfrak{M}, \mu, e_1, e_2\rangle$ *s.t.*

- $\langle \mathfrak{M}, \mu\rangle$ *is a* BD*-model with* $\pm$*-probability;*
- e_1 *and* e_2 *are* $Ł^2_{(\triangle,\to)}$ *valuations induced by* μ*, that is:*
  - $e_1(\mathsf{Pr}\phi) = \mu(|\phi|^+)$, $e_2(\mathsf{Pr}\phi) = \mu(|\phi|^-)$;
  - *the values of complex* $\mathscr{L}_{\mathsf{Pr}^{Ł^2}_\triangle}$ *formulas are computed following Definition 12.*

We say that α *is* $4\mathsf{Pr}^{Ł_\triangle}$*-valid iff* $e(\alpha) = 1$ *in every* $4\mathsf{Pr}^{Ł_\triangle}$*-model. A set of formulas* Γ *entails* α $(\Gamma \models_{4\mathsf{Pr}^{Ł_\triangle}} \alpha)$ *iff there is no* $\mathbb{M}$ *s.t.* $e(\gamma) = 1$ *for every* $\gamma \in \Gamma$ *but* $e(\alpha) \neq 1$.

We note quickly that in Definitions 15 and 16, e (as well as e_1 and e_2) are valuations of *outer-layer* formulas. Thus, they are defined only on *modal atoms*, not propositional variables. Propositional variables, in turn, have their values assigned by v^+ and v^- valuations of the BD-model over which the $4\mathsf{Pr}^{Ł_\triangle}$- or $\mathsf{Pr}^{Ł^2}_\triangle$-model is built.

The following property of $\mathsf{Pr}^{Ł^2}_\triangle$ will be useful further in the section.

Lemma 17. *Let* $\alpha \in \mathscr{L}_{\mathsf{Pr}^{Ł^2}_\triangle}$*. Then,* α *is* $\mathsf{Pr}^{Ł^2}_\triangle$*-valid iff* $e_1(\alpha) = 1$ *in every* $\mathsf{Pr}^{Ł^2}_\triangle$*-model.*

Proof. Let $\mathbb{M} = \langle W, v^+, v^-, \mu, e_1, e_2\rangle$ be a $\mathsf{Pr}^{Ł^2}_\triangle$-model s.t. $e_2(\alpha) \neq 0$. We construct a model $\mathbb{M}^* = \langle W, (v^*)^+, (v^*)^-, \mu, e^*_1, e^*_2\rangle$ where $e^*_1(\alpha) \neq 1$. To do this, we define new BD valuations $(v^*)^+$ and $(v^*)^-$ on W as follows.

$$(v^*)^+ = W \setminus v^- \qquad\qquad (v^*)^- = W \setminus v^+$$

It can be easily checked by induction on $\phi \in \mathscr{L}_{\mathsf{BD}}$ that

$$|\phi|^+_{\mathbb{M}} = W \setminus |\phi|^-_{\mathbb{M}^*} \qquad\qquad |\phi|^-_{\mathbb{M}} = W \setminus |\phi|^+_{\mathbb{M}^*} \tag{1}$$

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$$w_1 : p^+, q^\pm \qquad w_2 : \not{p}, q^- \qquad w_3 : p^-, \not{q}$$

Figure 3. $\mu(\{w_1\}) = \frac{1}{3}, \mu(\{w_2\}) = \frac{1}{2}, \mu(\{w_3\}) = \frac{1}{6}$.

Now, since we can w.l.o.g. assume that μ is a (classical) probability measure on W (recall Example 9), we have that

$$e^*(\mathsf{Pr}\phi) = (1 - \mu(|\phi|^-), 1 - \mu(|\phi|^+)) = (1 - e_2(\mathsf{Pr}\phi), 1 - e_1(\mathsf{Pr}\phi)) \tag{2}$$

It remains to show that $e^*(\alpha) = (1 - e_2(\alpha), 1 - e_1(\alpha))$ for every $\alpha \in \mathscr{L}_{\mathsf{Pr}^{\text{Ł}^2}_{\triangle}}$. We proceed by induction on formulas. The basis case of $\alpha = \mathsf{Pr}\phi$ holds by (2).

Consider $\alpha = \beta \rightarrow \beta'$.

$$\begin{aligned} e^*_1(\beta \rightarrow \beta') &= \min(1, 1 - e^*_1(\beta) + e^*_1(\beta')) \\ &= \min(1, 1 - (1 - e_2(\beta)) + (1 - e_2(\beta'))) \quad (\text{ by } IH) \\ &= \min(1, 1 - e_2(\beta') + e_2(\beta)) \\ &= 1 - \max(0, e_2(\beta') - e_2(\beta)) \\ &= 1 - e_2(\beta \rightarrow \beta') \end{aligned}$$

$$\begin{aligned} e^*_2(\beta \rightarrow \beta') &= \max(0, e^*_2(\beta') - e^*_2(\beta)) \\ &= \max(0, 1 - e_1(\beta') - (1 - e_1(\beta))) \qquad (\text{ by } IH) \\ &= \max(0, e_1(\beta) - e_1(\beta')) \\ &= 1 - \min(1, 1 - e_1(\beta) + e_1(\beta')) \end{aligned}$$

The remaining cases of $\alpha = \neg\beta$, $\alpha = {\sim}\beta$, and $\alpha = \triangle\beta$ can be tackled similarly.

It is now clear that if $e(\alpha) = (1, y)$ for some $y > 0$, we have $e^*(\alpha) = (1 - y, 0)$, that is, $e^*_1(\alpha) < 1$, as required. The result follows. □

Example 18 (Values of formulas in models). *Consider Fig. 3 with a classical probability μ. Let us compute the values of the following (atomic) formulas:* $\mathsf{Pr}(p \wedge \neg q)$, $\mathsf{Db}(p \wedge \neg q)$, *and* $\mathsf{Bl}(q \vee \neg q)$.

We have $|p \wedge \neg q|^+ = \{w_1\}$ and $|p \wedge \neg p|^- = \{w_1, w_3\}$. Thus, $\mu(|p \wedge \neg q|^+) = \frac{1}{3}$ and $\mu(|p \wedge \neg q|^-) = \frac{1}{2}$ which gives us $e(\mathsf{Pr}(p \wedge \neg q)) = \left(\frac{1}{3}, \frac{1}{2}\right)$. Moreover, $|p \wedge \neg q|^{\mathsf{d}} = \{w_3\}$, whence $\mathsf{Db}(p \wedge \neg q) = \frac{1}{6}$. For $q \vee \neg q$, we have $|q \vee \neg q|^{\mathsf{b}} = \{w_2\}$, and thus, $e(\mathsf{Bl}(q \vee \neg q)) = \frac{1}{2}$.

Before establishing faithful translations between $\mathsf{Pr}^{\text{Ł}^2}_{\triangle}$ and $\mathbf{4}\mathsf{Pr}^{\text{Ł}_\triangle}$, let us recall that in the Introduction, we mentioned that quantitative statements about uncertainty expressed in the natural language may sound like 'I think that rain is twice more likely than snow'. We show how to formalise this statement.

Example 19 (Formalisation).

`twice`: *I think that rain is twice more likely than snow.*

We are going to translate this statement into $\mathscr{L}_{\mathbf{4}\mathsf{Pr}^{\text{Ł}_\triangle}}$ and treat 'I think that' as pure belief modality. We denote 'it is going to rain' with r and 'it is going to snow' with s. It remains to write down a formula $\phi_{\texttt{twice}}$ that has value 1 iff $e(\mathsf{Bl}r) = 2 \cdot e(\mathsf{Bl}s)$. Consider the following formula

$$\phi_{\texttt{twice}} = \triangle((\mathsf{Bl}r \ominus \mathsf{Bl}s) \leftrightarrow \mathsf{Bl}s)$$

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Note that a more intuitive formalisation – $\triangle(\mathsf{Bl}r \leftrightarrow (\mathsf{Bl}s \oplus \mathsf{Bl}s))$ *– would not work:* $\oplus$ *is a truncated sum, whence it, e.g., does not exclude the situation with both* $\mathsf{Bl}r$ *and* $\mathsf{Bl}s$ *having value* 1*. It can, however, be altered as follows:* $\triangle(\mathsf{Bl}r \leftrightarrow (\mathsf{Bl}s \oplus \mathsf{Bl}s)) \wedge {\sim}(\mathsf{Bl}s \odot \mathsf{Bl}s)$ *which will give the desired outcome.*

It is also clear that desiderata (1), (3), and (4) listed in Section 1.4 are satisfied by both $\mathbf{4}\mathsf{Pr}^{\L_\triangle}$ and $\mathsf{Pr}^{\L^2}_\triangle$. It is less straightforward, however, to see how we can formalise preferring one source to another (desideratum (2)). We explain this in the following remark.

Remark 20. *To represent different sources, one can consider* BD*-models with several measures, each representing a source, and, accordingly, expand the language with other modalities corresponding to these new measures. We can state, for example, that one source* (s_1) *considers* ϕ *to be more likely than the other source* (s_2) *does. In* $\mathbf{4}\mathsf{Pr}^{\L_\triangle}$ *we can interpret this as the value of* $\mathsf{Bl}_{s_1}\phi$ *being smaller than the value of* $\mathsf{Bl}_{s_2}\phi$*. This is formalised as follows:*

$${\sim}\triangle(\mathsf{Bl}_{s_2}\phi \to \mathsf{Bl}_{s_1}\phi)$$

Unfortunately, there seems to be no direct way of representing the degrees of trust the agent assigns to s_1 *and* s_2 *using only modalities treated as measures in the Łukasiewicz setting. A traditional way (cf., e.g., Shafer 1976, p. 252) of accounting for the degree of trust in a given source is to multiply the value a mass function gives to* $X \subseteq W$ *by some* $x \in [0, 1]$*. Thus, to model this approach, one would need a combination of Rational Pavelka and Product logics. Another option would be to redefine* $e(\mathsf{Bl}_s\phi) = (\mathsf{tr}_s \odot_{\L} \mu(|\phi|^{\mathsf{b}}))$ *and similarly for other modalities (with* $\mathsf{tr}_s \in [0, 1]$ *standing for the trust in source s). It is unclear, however, how to axiomatise this logic.*

It is possible, though, to make different modalities stand for different types of measures (e.g., Bl_{s_1} *can be generated by a* **4***-probability while* Bl_{s_2} *by a belief function). This represents the different ways of aggregating the data the agent can have.*

We finish the section by establishing faithful embeddings of $\mathbf{4}\mathsf{Pr}^{\L_\triangle}$ and $\mathsf{Pr}^{\L^2}_\triangle$ into one another. First, we introduce some technical notions that will simplify the proof.

Convention 21. *We say that* $\alpha \in \mathscr{L}_{\mathsf{Pr}^{\L^2}_\triangle}$ *is outer-*$\neg$*-free when* $\neg$*'s appear only inside modal atoms.*

Definition 22. *Let* $\phi \in \mathscr{L}_{\mathsf{BD}}$ *and* $\alpha \in \mathscr{L}_{\mathsf{Pr}^{\L^2}_\triangle}$*.* $\alpha^\neg$ *is produced from* α *by successively applying the following transformations.*

$$\neg\mathsf{Pr}\phi \rightsquigarrow \mathsf{Pr}\neg\phi \qquad \neg\neg\alpha \rightsquigarrow \alpha \qquad \neg{\sim}\alpha \rightsquigarrow {\sim}\neg\alpha$$
$$\neg(\alpha \to \alpha') \rightsquigarrow {\sim}(\neg\alpha' \to \neg\alpha) \qquad \neg\triangle\alpha \rightsquigarrow {\sim}\triangle{\sim}\neg\alpha$$

It is easy to see that $\alpha^\neg$ is outer-$\neg$-free. Using Definitions 12 and 16, one can also check that $e(\alpha) = e(\alpha^\neg)$ in every $\mathsf{Pr}^{\L^2}_\triangle$-model.

Definition 23. *Let* $\phi \in \mathscr{L}_{\mathsf{BD}}$ *and* $\alpha \in \mathscr{L}_{\mathsf{Pr}^{\L^2}_\triangle}$ *be outer-*$\neg$*-free. We define* $\alpha^{\mathbf{4}} \in \mathscr{L}_{\mathbf{4}\mathsf{Pr}^{\L_\triangle}}$ *as follows.*

$$(\mathsf{Pr}\phi)^{\mathbf{4}} = \mathsf{Bl}\phi \oplus \mathsf{Cf}\phi$$
$$(\heartsuit\alpha)^{\mathbf{4}} = \heartsuit\alpha^{\mathbf{4}} \qquad (\heartsuit \in \{\triangle, {\sim}\})$$
$$(\alpha \to \alpha')^{\mathbf{4}} = \alpha^{\mathbf{4}} \to \alpha'^{\mathbf{4}}$$

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Let $\beta \in \mathscr{L}_{4\mathsf{Pr}^{\text{Ł}_\triangle}}$. We define $\beta^{\pm}$ as follows.

$$
\begin{aligned}
(\mathsf{Bl}\phi)^{\pm} &= \mathsf{Pr}\phi \ominus \mathsf{Pr}(\phi \wedge \neg\phi) \\
(\mathsf{Cf}\phi)^{\pm} &= \mathsf{Pr}(\phi \wedge \neg\phi) \\
(\mathsf{Uc}\phi)^{\pm} &= {\sim}\mathsf{Pr}(\phi \vee \neg\phi) \\
(\mathsf{Db}\phi)^{\pm} &= \mathsf{Pr}\neg\phi \ominus \mathsf{Pr}(\phi \wedge \neg\phi) \\
(\heartsuit\beta)^{\pm} &= \heartsuit\beta^{\pm} \qquad\qquad (\heartsuit \in \{\triangle, \sim\}) \\
(\beta \rightarrow \beta')^{\pm} &= \beta^{\pm} \rightarrow \beta'^{\pm}
\end{aligned}
$$

Theorem 24. *$\alpha \in \mathscr{L}_{\mathsf{Pr}^{\text{Ł}^2}_\triangle}$ is $\mathsf{Pr}^{\text{Ł}^2}_\triangle$-valid iff $(\alpha^{\neg})^{\mathbf{4}}$ is $\mathbf{4}\mathsf{Pr}^{\text{Ł}_\triangle}$-valid.*

Proof. Let w.l.o.g. $\mathbb{M} = \langle W, v^+, v^-, \mu, e_1, e_2\rangle$ be a BD-model with $\pm$-probability where μ is a *classical* probability measure and let $e(\alpha) = (x, y)$. We show that in the BD-model $\mathbb{M}_{\mathbf{4}} = \langle W, v^+, v^-, \mu, e_1\rangle$ with *four-probability* μ and $e^{\mathbf{4}}$ induced by μ, $e^{\mathbf{4}}((\alpha^{\neg})^{\mathbf{4}}) = x$. This is sufficient to prove the result because if $\alpha^{\neg}$ is not $\mathsf{Pr}^{\text{Ł}^2}_\triangle$-valid, $\mathsf{Pr}^{\text{Ł}^2}_\triangle$-valid $\mathsf{Pr}^{\text{Ł}^2}_\triangle$-valid either.

We proceed by induction on $\alpha^{\neg}$. For the basis case, we have that

$$
\begin{aligned}
e_1(\mathsf{Pr}\phi) &= \mu(|\phi|^+) \\
&= \mu(|\phi|^{\mathsf{b}} \cup |\phi|^{\mathsf{c}}) \\
&= \mu(|\phi|^{\mathsf{b}}) \cup \mu(|\phi|^{\mathsf{c}}) && (|\phi|^{\mathsf{b}} \text{ and } |\phi|^{\mathsf{c}} \text{ are disjoint }) \\
&= e^{\mathbf{4}}(\mathsf{Bl}\phi) + e_{\mathbf{4}}(\mathsf{Cf}\phi) && (e^{\mathbf{4}} \text{ is induced by } \mu) \\
&= e^{\mathbf{4}}(\mathsf{Bl}\phi \oplus \mathsf{Cf}\phi) && (e^{\mathbf{4}}(\mathsf{Bl}\phi) + e_{\mathbf{4}}(\mathsf{Cf}\phi) \leq 1)
\end{aligned}
$$

The cases of connectives can be obtained by an application of the induction hypothesis.

Conversely, since the support of truth conditions in $\text{Ł}^2_{(\triangle,\rightarrow)}$ coincide with the semantics of $\text{Ł}_\triangle$ (cf. Definitions 11 and 12) and since $\alpha^{\neg}$ is outer-$\neg$-free, if $e(\alpha^{\neg}) < 1$ for some $\mathbf{4}\mathsf{Pr}^{\text{Ł}_\triangle}$-model $\mathbb{M}_{\mathbf{4}} = \langle W, v^+, v^-, \mu, e\rangle$, then $e_1(\alpha^{\neg}) < 1$ for $\mathbb{M} = \langle W, v^+, v^-, \mu, e_1, e_2\rangle$ with $e = e_1$.

We proceed by induction on $\alpha^{\neg}$ (recall that $e(\alpha) = e(\alpha^{\neg})$ in all $\mathsf{Pr}^{\text{Ł}^2}_\triangle$-models). If $\alpha = \mathsf{Pr}\phi$, then $e_1(\mathsf{Pr}\phi) = \mu(|\phi|^+) = \mu(|\phi|^{\mathsf{b}} \cup |\phi|^{\mathsf{c}})$. But $|\phi|^{\mathsf{b}}$ and $|\phi|^{\mathsf{c}}$ are disjoint, whence $\mu(|\phi|^{\mathsf{b}} \cup |\phi|^{\mathsf{c}}) = \mu(|\phi|^{\mathsf{b}}) + \mu(|\phi|^{\mathsf{c}})$, and since $\mu(|\phi|^{\mathsf{b}}) + \mu(|\phi|^{\mathsf{c}}) \leq 1$, we have that $e_1(\mathsf{Bl}\phi \oplus \mathsf{Cf}\phi) = \mu(|\phi|^{\mathsf{b}}) + \mu(|\phi|^{\mathsf{c}}) = e_1(\mathsf{Pr}\phi)$, as required.

The induction steps are straightforward since the semantic conditions of support of truth in $\text{Ł}^2_\triangle$ coincide with the semantics of $\text{Ł}_\triangle$ (cf. Definitions 12 and 11). □

Theorem 25. *$\beta \in \mathscr{L}_{4\mathsf{Pr}^{\text{Ł}_\triangle}}$ is $\mathbf{4}\mathsf{Pr}^{\text{Ł}_\triangle}$-valid iff $\beta^{\pm}$ is $\mathsf{Pr}^{\text{Ł}^2}_\triangle$-valid.*

Proof. Assume w.l.o.g. that $\mathbb{M} = \langle W, v^+, v^-, \mu_{\mathbf{4}}, e\rangle$ is a BD-model with a $\mathbf{4}$-probability where $\mu_{\mathbf{4}}$ is a classical probability measure and $e(\beta) = x$. It suffices to define a BD-model with $\pm$-probability $\mathbb{M}^{\pm} = \langle W, v^+, v^-, \mu_{\mathbf{4}}, e_1, e_2\rangle$ and show that $e_1(\beta^{\pm}) = x$. If $e(\beta) < 1$, then $e(\beta^{\pm}) < 1$ (and thus, is not valid). If $\beta^{\pm}$ is not valid, we have that $e_1(\beta^{\pm}) < 1$ by Lemma 17, whence $e(\beta) < 1$, as well.

We proceed by induction on β. If $\beta = \mathsf{Bl}\phi$, then $e(\mathsf{Bl}\phi) = \mu_{\mathbf{4}}(|\phi|^{\mathsf{b}})$. Now observe that $\mu_{\mathbf{4}}(|\phi|^+) = \mu(|\phi|^{\mathsf{b}} \cup |\phi|^{\mathsf{c}}) = \mu_{\mathbf{4}}(|\phi|^{\mathsf{b}}) + \mu_{\mathbf{4}}(|\phi|^{\mathsf{c}})$ since $|\phi|^{\mathsf{b}}$ and $|\phi|^{\mathsf{c}}$ are disjoint. But $\mu_{\mathbf{4}}(|\phi|^+) = e_1(\mathsf{Pr}\phi)$ and $\mu_{\mathbf{4}}(|\phi|^{\mathsf{c}}) = \mu_{\mathbf{4}}(|\phi \wedge \neg\phi|^+)$ since $|\phi \wedge \neg\phi|^+ = |\phi|^{\mathsf{c}}$. Thus, $\mu_{\mathbf{4}}(|\phi|^{\mathsf{b}}) = e_1(\mathsf{Pr}\phi \ominus \mathsf{Pr}(\phi \wedge \neg\phi))$ as required.

Other basis cases of $\mathsf{Cf}\phi$, $\mathsf{Uc}\phi$, and $\mathsf{Db}\phi$ can be tackled similarly. The induction steps are straightforward since the support of truth in $\text{Ł}^2_\triangle$ coincides with semantical conditions in $\text{Ł}_\triangle$. □

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We finish the section with a straightforward observation.

Lemma 26. *Let* $\alpha \in \mathscr{L}_{\mathsf{Pr}^{\text{Ł}^2}_{\triangle}}$ *and* $\beta \in \mathscr{L}_{\mathbf{4}\mathsf{Pr}^{\text{Ł}_{\triangle}}}$. *Then* $\ell((\alpha^{\neg})^{\mathbf{4}}) = \mathscr{O}(\ell(\alpha))$ *and* $\ell(\beta) = \mathscr{O}(\ell(\beta^{\pm}))$.

Proof. To simplify the presentation of the proof, we will further assume that $\ell(\mathsf{Y}\phi) = \ell(\phi) + 2$ for every $\mathsf{Y} \in \{\mathsf{Pr}, \mathsf{Bl}, \mathsf{Cf}, \mathsf{Uc}, \mathsf{Db}\}$. We begin with α. From Definition 22, it is clear that $\ell(\alpha^{\neg}) = \mathscr{O}(\ell(\alpha))$. It remains to show by induction on $\alpha^{\neg}$ that $\ell((\alpha^{\neg})^{\mathbf{4}}) = \mathscr{O}(\ell(\alpha^{\neg}))$.

The basis case of $\alpha^{\neg} = \mathsf{Pr}\phi$ is simple: $\ell(\mathsf{Pr}\phi) = \ell(\phi) + 2$, whence

$$\ell((\mathsf{Pr}\phi)^{\mathbf{4}}) = \ell(\mathsf{Bl}\phi \oplus \mathsf{Cf}\phi) = 2 \cdot \ell(\phi) + 7 = \mathscr{O}(\ell(\alpha^{\neg}))$$

Here, 7 is the sum of lengths of modalities, $\oplus$, and outer brackets in $(\mathsf{Pr}\phi)^{\mathbf{4}}$.

Now let $\alpha^{\neg} = (\alpha_1 \rightarrow \alpha_2)$. Then $\ell(\alpha^{\neg}) = \ell(\alpha_1) + \ell(\alpha_2) + 3$ (we count outer brackets here). Since $(\alpha_1 \rightarrow \alpha_2)^{\mathbf{4}} = \alpha_1^{\mathbf{4}} \rightarrow \alpha_2^{\mathbf{4}}$, we have $\ell(\alpha_1^{\mathbf{4}} \rightarrow \alpha_2^{\mathbf{4}}) = \ell(\alpha_1^{\mathbf{4}}) + \ell(\alpha_2^{\mathbf{4}}) + 3$. We apply the induction hypothesis and obtain that $\ell(\alpha_1^{\mathbf{4}} \rightarrow \alpha_2^{\mathbf{4}}) = \mathscr{O}(\ell(\alpha_1)) + \mathscr{O}(\ell(\alpha_2)) + 3 = \mathscr{O}(\ell(\alpha_1 \rightarrow \alpha_2))$.

The cases of $\alpha = \triangle\alpha'$ and $\alpha = {\sim}\alpha'$ can be dealt with in the same manner.

Consider now $\beta \in \mathscr{L}_{\mathbf{4}\mathsf{Pr}^{\text{Ł}_{\triangle}}}$. We show by induction on β that $\ell(\beta^{\pm}) = \mathscr{O}(\ell(\beta))$. Assume that $\beta = \mathsf{Bl}\phi$. Then $(\mathsf{Bl}\phi)^{\pm} = \mathsf{Pr}\phi \ominus \mathsf{Pr}(\phi \wedge \neg\phi)$. Observe that $\mathsf{Pr}\phi \ominus \mathsf{Pr}(\phi \wedge \neg\phi)$ is a shorthand for ${\sim}(\mathsf{Pr}\phi \rightarrow \mathsf{Pr}(\phi \wedge \neg\phi))$. Thus, we have

$$\ell({\sim}(\mathsf{Pr}\phi \rightarrow \mathsf{Pr}(\phi \wedge \neg\phi))) = 3 \cdot \ell(\phi) + 12 = \mathscr{O}(\ell(\mathsf{Bl}\phi))$$

Note again that 12 is the sum of the lengths of modalities, $\wedge$, $\neg$, $\rightarrow$, ${\sim}$, and brackets in ${\sim}(\mathsf{Pr}\phi \rightarrow \mathsf{Pr}(\phi \wedge \neg\phi))$. The cases of other modal atoms are similar. The cases of $\beta = \beta_1 \rightarrow \beta_2$, $\beta = \triangle\beta'$, and $\beta = {\sim}\beta'$ can be considered in the same way as we did $(\alpha^{\neg})^{\mathbf{4}}$. The result now follows since modalities do not nest. □

3.2 Axiomatisations

In this section, we present Hilbert-style calculi $\mathscr{H}\mathsf{Pr}^{\text{Ł}^2}_{\triangle}$ and $\mathscr{H}\mathbf{4}\mathsf{Pr}^{\text{Ł}_{\triangle}}$ that axiomatise $\mathsf{Pr}^{\text{Ł}^2}_{\triangle}$ and $\mathbf{4}\mathsf{Pr}^{\text{Ł}_{\triangle}}$. The completeness of $\mathscr{H}\mathsf{Pr}^{\text{Ł}^2}_{\triangle}$ w.r.t. finite theories (Ł is not compact, whence there are no finite calculi for expansions of Ł complete w.r.t. countable theories) was established by Bílková et al. (2023d). Here, we prove the completeness of $\mathscr{H}\mathbf{4}\mathsf{Pr}^{\text{Ł}_{\triangle}}$.

To construct the calculi, we translate the conditions on measures from Definitions 4 and 6 into $\mathscr{L}_{\mathsf{Pr}^{\text{Ł}^2}_{\triangle}}$ and $\mathscr{L}_{\mathbf{4}\mathsf{Pr}^{\text{Ł}_{\triangle}}}$ formulas. We then use these translated conditions as additional axioms that extend Hilbert-style calculi for $\text{Ł}_{\triangle}$ and $\text{Ł}_{\triangle}$ and $\text{Ł}^2_{(\triangle,\rightarrow)}$.

To facilitate the presentation, we recall these Hilbert calculi for $\text{Ł}_{\triangle}$. The axiomatisation of $\text{Ł}_{\triangle}$ can be obtained by adding $\triangle$-axioms and rules from Baaz (1996), Hájek (1998, Definition 2.4.5), or Běhounek et al. (2011, Chapter I, 2.2.1) to the Hilbert-style calculus for Ł from Metcalfe et al. (2008, Section 6.2).

Definition 27 ($\mathscr{H}\text{Ł}_{\triangle}$ – the Hilbert-style calculus for $\text{Ł}_{\triangle}$). *The calculus contains the following axioms and rules.*

w: $\phi \rightarrow (\chi \rightarrow \phi)$.
sf: $(\phi \rightarrow \chi) \rightarrow ((\chi \rightarrow \psi) \rightarrow (\phi \rightarrow \psi))$.
waj: $((\phi \rightarrow \chi) \rightarrow \chi) \rightarrow ((\chi \rightarrow \phi) \rightarrow \phi)$.
co: $({\sim}\chi \rightarrow {\sim}\phi) \rightarrow (\phi \rightarrow \chi)$.
MP: $\dfrac{\phi \quad \phi \rightarrow \chi}{\chi}$.
$\triangle 1$: $\triangle\phi \vee {\sim}\triangle\phi$.

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$\triangle$2: $\triangle\phi \to \phi$.
$\triangle$3: $\triangle\phi \to \triangle\triangle\phi$.
$\triangle$4: $\triangle(\phi \vee \chi) \to \triangle\phi \vee \triangle\chi$.
$\triangle$5: $\triangle(\phi \to \chi) \to \triangle\phi \to \triangle\chi$.
$\triangle$**nec**: $\dfrac{\phi}{\triangle\phi}$.

Proposition 28 (Finite strong completeness of $\mathscr{H}Ł_\triangle$). *Let Γ be finite. Then $\Gamma \models_{Ł_\triangle} \phi$ iff $\Gamma \vdash_{\mathscr{H}Ł_\triangle} \phi$.*

The calculus for $Ł^2_{(\triangle,\to)}$ is acquired by adding the axioms for $\neg$ (cf. Bílková et al. 2023 for details).

Definition 29 ($\mathscr{H}Ł^2_{(\triangle,\to)}$ – Hilbert-style calculus for $Ł^2_{(\triangle,\to)}$). *The calculus expands $\mathscr{H}Ł_\triangle$ with the following axioms and rules.*

$\neg\neg$: $\neg\neg\phi \leftrightarrow \phi$.
$\neg\sim$: $\neg{\sim}\phi \leftrightarrow {\sim}\neg\phi$.
${\sim}\neg\to$: $({\sim}\neg\phi \to {\sim}\neg\chi) \leftrightarrow {\sim}\neg(\phi \to \chi)$.
$\neg\triangle$: $\neg\triangle\phi \leftrightarrow {\sim}\triangle{\sim}\neg\phi$.
conf: $\dfrac{\phi}{{\sim}\neg\phi}$.

The completeness of $\mathscr{H}Ł^2_{(\triangle,\to)}$ was shown in Bílková et al. (2023, Lemma 4.16) (there, the logic is called $Ł^2$).

The calculi for $\mathsf{Pr}^{Ł^2}_\triangle$ and $\mathbf{4}\mathsf{Pr}^{Ł_\triangle}$ are as follows.

Definition 30 ($\mathscr{H}\mathsf{Pr}^{Ł^2}_\triangle$ – a Hilbert-style calculus for $\mathsf{Pr}^{Ł^2}_\triangle$). *The calculus has the following axioms and rules.*

$Ł^2_{(\triangle,\to)}$: $Ł^2_{(\triangle,\to)}$*-valid formulas and* $\mathscr{H}Ł^2_{(\triangle,\to)}$ *rules instantiated in* $\mathscr{L}_{\mathsf{Pr}^{Ł^2}_\triangle}$;
$\pm$mon: $\mathsf{Pr}\phi \to \mathsf{Pr}\chi$ *if* $\phi \models_{\mathsf{BD}} \chi$;
$\pm$neg: $\mathsf{Pr}\neg\phi \leftrightarrow \neg\mathsf{Pr}\phi$;
$\pm$ex: $\mathsf{Pr}(\phi \vee \chi) \leftrightarrow ((\mathsf{Pr}\phi \ominus \mathsf{Pr}(\phi \wedge \chi)) \oplus \mathsf{Pr}\chi)$.

It is important to note that $\pm$neg *is not* connected to the $\pm$-probability of the complement of an event. Indeed, it contains only BD negations and is, thus, a translation of the **neg** property from Definition 4.

Proposition 31 (Finite strong completeness of $\mathscr{H}\mathsf{Pr}^{Ł^2}_\triangle$ Bílková et al. 2023d, Theorem 4.24). *Let $\Xi \subseteq \mathscr{L}_{\mathsf{Pr}^{Ł^2}_\triangle}$ be finite. Then $\Xi \models_{\mathsf{Pr}^{Ł^2}_\triangle} \alpha$ iff $\Xi \vdash_{\mathscr{H}\mathsf{Pr}^{Ł^2}_\triangle} \alpha$.*

Definition 32 ($\mathscr{H}\mathbf{4}\mathsf{Pr}^{Ł_\triangle}$ – Hilbert-style calculus for $\mathbf{4}\mathsf{Pr}^{Ł_\triangle}$). *The calculus $\mathscr{H}\mathbf{4}\mathsf{Pr}^{Ł_\triangle}$ consists of the following axioms and rules.*

$Ł_\triangle$:	$Ł_\triangle$*-valid formulas and sound rules instantiated in* $\mathscr{L}_{\mathbf{4}\mathsf{Pr}^{Ł_\triangle}}$.
4equiv:	$\mathsf{X}\phi \leftrightarrow \mathsf{X}\chi$ *for every* $\phi, \chi \in \mathscr{L}_{\mathsf{BD}}$ *s.t.* $\phi \dashv\vdash \chi$ *is* BD*-valid and* $\mathsf{X} \in \{\mathsf{Bl}, \mathsf{Db}, \mathsf{Cf}, \mathsf{Uc}\}$.
4contr:	${\sim}\mathsf{Bl}(\phi \wedge \neg\phi)$; $\mathsf{Cf}\phi \leftrightarrow \mathsf{Cf}(\phi \wedge \neg\phi)$.
4neg:	$\mathsf{Bl}\neg\phi \leftrightarrow \mathsf{Db}\phi$; $\mathsf{Cf}\neg\phi \leftrightarrow \mathsf{Cf}\phi$.

4mon:	$(\mathsf{Bl}\phi \oplus \mathsf{Cf}\phi) \to (\mathsf{Bl}\chi \oplus \mathsf{Cf}\chi)$ *for every* $\phi, \chi \in \mathscr{L}_{\mathsf{BD}}$ *s.t.* $\phi \vdash \chi$ *is* BD-*valid.*
4part1:	$\mathsf{Bl}\phi \oplus \mathsf{Db}\phi \oplus \mathsf{Cf}\phi \oplus \mathsf{Uc}\phi$.
4part2:	$((\mathsf{X}_1\phi \oplus \mathsf{X}_2\phi \oplus \mathsf{X}_3\phi \oplus \mathsf{X}_4\phi) \ominus \mathsf{X}_4\phi) \leftrightarrow (\mathsf{X}_1\phi \oplus \mathsf{X}_2\phi \oplus \mathsf{X}_3\phi)$ *with* $\mathsf{X}_i \neq \mathsf{X}_j$, $\mathsf{X}_i \in \{\mathsf{Bl}, \mathsf{Db}, \mathsf{Cf}, \mathsf{Uc}\}$.
4ex:	$(\mathsf{Bl}(\phi \vee \chi) \oplus \mathsf{Cf}(\phi \vee \chi)) \leftrightarrow ((\mathsf{Bl}\phi \oplus \mathsf{Cf}\phi) \ominus (\mathsf{Bl}(\phi \wedge \chi) \oplus \mathsf{Cf}(\phi \wedge \chi)) \oplus (\mathsf{Bl}\chi \oplus \mathsf{Cf}\chi))$.

As one can see, the axioms in the calculi above are translations of properties from Definitions 4 and 6. In $\mathscr{H}\mathbf{4}\mathsf{Pr}^{Ł_\triangle}$, we split **4part** in two axioms to ensure that the values of $\mathsf{Bl}\phi$, $\mathsf{Db}\phi$, $\mathsf{Cf}\phi$, and $\mathsf{Uc}\phi$ sum up exactly to 1. Note, moreover, that 4mon and 4ex are translations of ±mon and ±ex (cf. Definition 23). However, since other $\mathscr{H}\mathbf{4}\mathsf{Pr}^{Ł_\triangle}$-axioms cannot be obtained as translations of $\mathscr{H}\mathsf{Pr}^{Ł^2}_\triangle$-axioms (nor, in fact, *any* $\mathsf{Pr}^{Ł^2}_\triangle$-valid formulas), soundness and completeness of $\mathscr{H}\mathbf{4}\mathsf{Pr}^{Ł_\triangle}$ cannot be established as a consequence of Theorem 24 and completeness of $\mathscr{H}\mathsf{Pr}^{Ł^2}_\triangle$ (Proposition 31).

In the remainder of this section, we prove the completeness of $\mathscr{H}\mathbf{4}\mathsf{Pr}^{Ł_\triangle}$ w.r.t. finite theories.

Theorem 33. *Let* $\Xi \subseteq \mathscr{L}_{\mathbf{4}\mathsf{Pr}^{Ł_\triangle}}$ *be finite. Then* $\Xi \models_{\mathbf{4}\mathsf{Pr}^{Ł_\triangle}} \alpha$ *iff* $\Xi \vdash_{\mathscr{H}\mathbf{4}\mathsf{Pr}^{Ł_\triangle}} \alpha$.

Proof. Soundness can be established by the routine check of the validity of the axioms. For example for 4equiv, observe that if ϕ and χ are equivalent in BD, then $|\phi|^+ = |\chi|^+$ and $|\phi|^- = |\chi|^-$, whence $|\phi|^{\mathsf{x}} = |\chi|^{\mathsf{x}}$ for every $\mathsf{x} \in \{\mathsf{b}, \mathsf{d}, \mathsf{c}, \mathsf{u}\}$. Thus, $e(\mathsf{X}\phi) = e(\mathsf{X}\chi)$ for each $\mathsf{X} \in \{\mathsf{Bl}, \mathsf{Db}, \mathsf{Cf}, \mathsf{Uc}\}$, from where $\mathsf{X}\phi \leftrightarrow \mathsf{X}\chi$ is valid. 4contr, 4neg, and 4mon are straightforward translations of **contr**, **neg**, and **BCmon**. 4ex is the translation of ±ex, whence are valid by Theorem 24.

Last, consider 4part1 and 4part2. We have $\sum\limits_{\mathsf{x}\in\{\mathsf{b},\mathsf{d},\mathsf{c},\mathsf{u}\}} |\phi|^{\mathsf{x}} = 1$. Hence, $\sum\limits_{\mathsf{X}\in\{\mathsf{Bl},\mathsf{Db},\mathsf{Cf},\mathsf{Uc}\}} e(\mathsf{X}\phi) = 1$. Thus, we have that 4part1 and 4part2 are valid.

Let us now prove the completeness. We reason by contraposition. Assume that $\Xi \nvdash_{\mathscr{H}\mathbf{4}\mathsf{Pr}^{Ł_\triangle}} \alpha$. Now, observe that $\mathscr{H}\mathbf{4}\mathsf{Pr}^{Ł_\triangle}$ proofs are, actually, $Ł_\triangle$ proofs with additional probabilistic axioms. Let Ξ^* stand for Ξ extended with probabilistic axioms built over all pairwise non-equivalent $\mathscr{L}_{\mathsf{BD}}$ formulas constructed from $\mathtt{Prop}[\Xi \cup \{\alpha\}]$. Clearly, $\Xi^* \nvdash_{\mathscr{H}\mathbf{4}\mathsf{Pr}^{Ł_\triangle}} \alpha$ either. Moreover, Ξ^* is finite as well since BD is tabular (and whence, there exist only finitely many pairwise non-equivalent $\mathscr{L}_{\mathsf{BD}}$ formulas over a finite set of variables). Now, by the weak completeness of $Ł_\triangle$ (Proposition 28), there exists an $Ł_\triangle$ valuation e s.t. $e[\Xi^*] = 1$ and $e(\alpha) \neq 1$.

It remains to construct a $\mathbf{4}\mathsf{Pr}^{Ł_\triangle}$-model $\Xi^* \models_{\mathbf{4}\mathsf{Pr}^{Ł_\triangle}} \alpha$ using e. We proceed as follows. First, we set $W = 2^{\mathsf{Lit}[\Xi^* \cup \{\alpha\}]}$, and for every $w \in W$ define $w \in v^+(p)$ iff $p \in w$ and $w \in v^-(p)$ iff $\neg p \in w$. We extend the valuations to $\phi \in \mathscr{L}_{\mathsf{BD}}$ in the usual manner. Then, for $\mathsf{X}\phi \in \mathtt{Sf}[\Xi^* \cup \{\alpha\}]$, we set $\mu_{\mathbf{4}}(|\phi|^{\mathsf{X}}) = e(\mathsf{X}\phi)$ according to modality X.

It remains to extend $\mu_{\mathbf{4}}$ to the whole 2^W. Observe, however, that any map from 2^W to $[0, 1]$ that extends $\mu_{\mathbf{4}}$ is, in fact, a **4**-probability. Indeed, all requirements from Definition 6 concern *only the extensions of formulas*. But the model is finite, BD is tabular, and Ξ^* contains all the necessary instances of probabilistic axioms and $e[\Xi^*] = 1$, whence all constraints on the formulas are satisfied. □

3.3 Complexity

Let us now establish the complexity of $\mathsf{Pr}^{Ł^2}_\triangle$ and $\mathbf{4}\mathsf{Pr}^{Ł_\triangle}$. Namely, we prove their NP-completeness. Since the mutual embeddings given in Definition 23 result in an at most linear increase in the size of the $\mathsf{Pr}^{Ł^2}_\triangle$ polynomial algorithm for $\mathsf{Pr}^{Ł^2}_\triangle$ since it contains only one modality which will simplify

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the presentation. Note, furthermore, that $Ł_\triangle$ (and thus, $Ł^2_{(\triangle,\to)}$) are NP-hard, thus, $\mathsf{Pr}^{Ł^2}_\triangle$ and $\mathbf{4}\mathsf{Pr}^{Ł_\triangle}$ are NP-hard as well. Thus, we only need to establish the upper bound.

For the proof, we adapt constraint tableaux for $Ł^2$ defined by Bílková et al. (2021) and expand them with FP(Ł) of Hájek and Tulipani (2001) to establish our result. of Hájek and Tulipani (2001) to establish our result.

Definition 34 (Constraint tableaux for $Ł^2_{(\triangle,\to)}$ – $\mathscr{T}(Ł^2_{(\triangle,\to)})$). *Branches contain labelled formulas of the form $\phi \leqslant_1 i$, $\phi \leqslant_2 i$, $\phi \geqslant_1 i$, or $\phi \geqslant_2 i$, and numerical constraints of the form $i \leq j$ with i and j (labels) being linear polynomials over* $[0, 1]$. *Each branch can be extended by an application of a rule below.*

$$\neg\leqslant_1 \frac{\neg\phi \leqslant_1 i}{\phi \leqslant_2 i} \qquad \neg\leqslant_2 \frac{\neg\phi \leqslant_2 i}{\phi \leqslant_1 i} \qquad \neg\geqslant_1 \frac{\neg\phi \geqslant_1 i}{\phi \geqslant_2 i} \qquad \neg\geqslant_2 \frac{\neg\phi \geqslant_2 i}{\phi \geqslant_1 i}$$

$$\sim\leqslant_1 \frac{\sim\phi \leqslant_1 i}{\phi \geqslant_1 1-i} \qquad \sim\leqslant_2 \frac{\sim\phi \leqslant_2 i}{\phi \geqslant_2 1-i} \qquad \sim\geqslant_1 \frac{\sim\phi \geqslant_1 i}{\phi \leqslant_1 1-i} \qquad \sim\geqslant_2 \frac{\sim\phi \geqslant_2 i}{\phi \leqslant_2 1-i}$$

$$\triangle\geqslant_1 \frac{\triangle\phi \geqslant_1 i}{i \leq 0 \left| \begin{array}{c} \phi \geqslant_1 j \\ j \geq 1 \end{array} \right.} \qquad \triangle\leqslant_1 \frac{\triangle\phi \leqslant_1 i}{i \geq 1 \left| \begin{array}{c} \phi \leqslant_1 j \\ j < 1 \end{array} \right.} \qquad \triangle\leqslant_2 \frac{\triangle\phi \leqslant_2 i}{i \geq 1 \left| \begin{array}{c} \phi \leqslant j \\ j \leq 0 \end{array} \right.} \qquad \triangle\geqslant_2 \frac{\triangle\phi \geqslant_2 i}{i \leq 0 \left| \begin{array}{c} \phi \geqslant j \\ j > 0 \end{array} \right.}$$

$$\to\leqslant_1 \frac{\phi_1 \to \phi_2 \leqslant_1 i}{i \geq 1 \left| \begin{array}{c} \phi_1 \geqslant_1 1-i+j \\ \phi_2 \leqslant_1 j \\ j \leq i \end{array} \right.} \qquad \to\leqslant_2 \frac{\phi_1 \to \phi_2 \leqslant_2 i}{\begin{array}{c} \phi_1 \geqslant_2 j \\ \phi_2 \leqslant_2 i+j \end{array}} \qquad \to\geqslant_1 \frac{\phi_1 \to \phi_2 \geqslant_1 i}{\begin{array}{c} \phi_1 \leqslant_1 1-i+j \\ \phi_2 \geqslant_1 j \end{array}} \qquad \to\geqslant_2 \frac{\phi_1 \to \phi_2 \geqslant_2 i}{i \leq 0 \left| \begin{array}{c} \phi_1 \leqslant_2 j \\ \phi_2 \geqslant_2 i+j \\ j \leq 1-i \end{array} \right.}$$

Let i's be linear polynomials that label the formulas in the branch and x's be variables ranging over $[0, 1]$. *We define the translation t from labelled formulas to linear inequalities as follows:*

$$t(\phi \leqslant_1 i) = x^L_\phi \leq i; \qquad t(\phi \geqslant_1 i) = x^L_\phi \geq i; \qquad t(\phi \leqslant_2 i) = x^R_\phi \leq i; \qquad t(\phi \geqslant_2 i) = x^R_\phi \geq i$$

Let $\triangledown \in \{\leqslant_1, \geqslant_1, \leqslant_2, \geqslant_2\}$. *A tableau branch*

$$\mathscr{B} = \{\phi_1 \triangledown i_1, \ldots, \phi_m \triangledown i_m, k_1 \leq l_1, \ldots, k_q \leq l_q\}$$

is closed if the system of inequalities

$$\mathscr{B}^t = \{t(\phi_1 \triangledown i_1), \ldots, t(\phi_m \triangledown i_m), k_1 \leq l_1, \ldots, k_q \leq l_q\}$$

does not have solutions. Otherwise, $\mathscr{B}$ is open. If no rule application adds new entries to $\mathscr{B}$, it is called complete. A tableau is closed if all its branches are closed. ϕ has a $\mathscr{T}\left(Ł^2_{(\triangle,\to)}\right)$ proof if the tableau beginning with $\{\phi \leqslant_1 c, c < 1\}$ is closed.

Remark 35. *Let us briefly remark on how to interpret rules and entries in the tableau. Consider, for instance, the rule $\to\leqslant_2$. It's meaning is: $v_2(\phi_1 \to \phi_2) \leq i$ iff there is $j \in [0, 1]$ s.t. $v_2(\phi_1) \geq j$ and $v_2(\phi_2) \leq i + j$. Thus, $\psi \leqslant_1 i$ means that $v_1(\phi) \leq i$, $\psi \geqslant_2 i$ that $v_2(\psi) \geq i$, etc.*

Furthermore, our tableaux rules for $\triangle$ are necessarily branching in contrast to the rules for standard Ł connectives proposed by Hähnle (1992, 1994). This is because all connectives Ł are continuous. On the other hand, $\triangle$ is not continuous, hence, there cannot be a non-branching rule for it.

Definition 36 (Satisfying valuation of a branch). *Let v_1 and v_2 be $Ł^2_{(\triangle,\to)}$ valuations. v_1 satisfies a labelled formula $\phi \leqslant_1 i$ (v_2 satisfies $\phi \geqslant_2 i$) iff $v_1(\phi) \leq i$ (resp., $v_2(\phi) \geq i$). v_1 satisfies a branch $\mathscr{B}$ iff v_1 satisfies every labelled formula in $\mathscr{B}$ (and similarly for v_2). A branch $\mathscr{B}$ is satisfiable iff there is a pair of valuations $\langle v_1, v_2\rangle$ that satisfies it.*

Theorem 37 (Completeness of tableaux).

1. *ϕ is $Ł_\triangle$-valid iff it has a $\mathscr{T}\left(Ł^2_{(\triangle,\to)}\right)$ proof.*
2. *ϕ is $Ł^2_{(\triangle,\to)}$-valid iff it has a $\mathscr{T}\left(Ł^2_{(\triangle,\to)}\right)$ proof.*

Proof. The proof follows Bílková et al. (2021, Theorem 1). For soundness, we show that if $\langle v_1, v_2\rangle$ satisfies the premise of a rule, then it also satisfies one of its conclusions. We consider the case of $\triangle\geqslant_1$ (the rules for other connectives are tackled by Bílková et al. (2021); the remaining rules for $\triangle$ can be dealt with analogously).

Let $\triangle\phi \geqslant_1 i$ be satisfied by $\langle v_1, v_2\rangle$. Then, $v_1(\triangle\phi) \geqslant i$. We have two cases: (i) $i = 0$ and (ii) $i > 0$. In the first case, the left conclusion is satisfied. In the second case, $v_1(\triangle\phi) = 1$. Hence, $v_1(\phi) = 1$, that is, the right conclusion is satisfied.

For completeness, note from Bílková et al. (2021, Proposition 1) that ϕ is $Ł^2_{(\triangle,\to)}$-valid iff $v_1(\phi) = 1$ for every v. Let us now show that every complete open branch can be satisfied. Assume that $\mathscr{B}$ is a complete open branch. We construct the satisfying valuation as follows. Let $\triangledown \in \{\leqslant_1, \geqslant_1, \leqslant_2, \geqslant_2\}$ and $p_1, \ldots, p_m$ be the propositional variables appearing in the atomic labelled formulas in $\mathscr{B}$.

Let $\{p_1 \triangledown j_1, \ldots, p_m \triangledown j_n\}$ and $\{k_1 \leq l_1, \ldots, k_q \leq l_q\}$ be the sets of all atomic labelled formulas and all numerical constraints in $\mathscr{B}$. Notice that one variable might appear in many atomic labelled formulas, hence we might have $m \neq n$. Since $\mathscr{B}$ is complete and open, the following system of linear inequalities over the set of variables $\{x^L_{p_1}, x^R_{p_1}, \ldots, x^L_{p_m}, x^R_{p_m}\}$ must have at least one solution under the constraints listed:

$$\not{t}(p_1 \triangledown i_1), \ldots, \not{t}(p_m \triangledown i_n), k_1 \leq l_1, \ldots, k_q \leq l_q$$

Let $c = (c^L_1, c^R_1, \ldots, c^L_m, c^R_m)$ be a solution to the above system of inequalities such that c^L_j (c^R_j) is the value of $x^L_{p_j}$ ($x^R_{p_j}$). Define valuations as follows: $v_1(p_j) = c^L_j$ and $v_2(p_j) = c^R_j$.

It remains to show by induction on ϕ that all formulas present at $\mathscr{B}$ are satisfied by v_1 and v_2. The basis case of variables holds by the construction of v_1 and v_2. We consider the case of $\triangle\phi \geqslant_1 i$ (the cases of other variables can be dealt with similarly).

Assume that $\triangle\phi \geqslant_1 i \in \mathscr{B}$. Since $\mathscr{B}$ is complete and open, we have that either (i) $i \leq 0 \in \mathscr{B}$ or (ii) $\{\phi \geqslant_1 j, j \geq 1\} \subseteq \mathscr{B}$. In the first case, since there is a solution for $\mathscr{B}^{\not{t}}$, we have that $i = 0$, and thus $\triangle\phi \geqslant_1 i$ is satisfied by any v_1. In the second case, we have that $\{\phi \geqslant_1 j, j \geq 1\}$ is satisfied by the induction hypothesis, whence $v_1(\phi) = 1$, and thus, $v_1(\triangle\phi) = 1$, as required. □

Definition 38 (Mixed-integer problem). *Let $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$ and $\mathbf{y} = (y_1, \ldots, y_m) \in \mathbb{Z}^m$ be variables, A and B be integer matrices, h an integer vector, and $f(\mathbf{x}, \mathbf{y})$ be a $k + m$-place linear function.*

1. *A general MIP is to find $\mathbf{x}$ and $\mathbf{y}$ s.t. $f(\mathbf{x}, \mathbf{y}) = \min\{f(\mathbf{x}, \mathbf{y}) : A\mathbf{x} + B\mathbf{y} \geq h\}$.*
2. *A bounded MIP (bMIP) is to find all solutions that belong to $[0, 1]$.*

Proposition 39. *Bounded MIP is* NP*-complete.*

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Remark 40. *The proof of Ł's* NP*-completeness by Hähnle (1992, 1994) uses the reduction of a Łukasiewicz formula ϕ to one instance of a bMIP of the size $\mathcal{O}(\ell(\phi))$. This is because the rules are linear. In our case (cf. Remark 35), we cannot avoid branching. This, however, does not affect the complexity of $Ł_\triangle$ and tableau and then solve the system of inequalities corresponding to it. Note that in our setting all variables can be evaluated over $\mathbb{R} \cap [0, 1]$; in addition, lengths of branches are polynomial in $\ell(\phi)$. Hence, we can solve the system of inequalities obtained from the branch in the time polynomial w.r.t. $\ell(\phi)$ (and if there is no solution, then the branch is closed and we need to guess again). Thus, $Ł_\triangle$ and $Ł_\triangle$ and $Ł^2_{(\triangle,\rightarrow)}$ are also* NP*-complete.*

Let us now proceed to the proof of NP-completeness of $\mathsf{Pr}^{Ł^2}_\triangle$. First, we show that we can *completely remove* ¬'s from $\mathscr{L}_{\mathsf{Pr}^{Ł^2}_\triangle}$ formulas while preserving their satisfiability.

Lemma 41. *For any outer-¬-free $\alpha \in \mathscr{L}_{\mathsf{Pr}^{Ł^2}_\triangle}$, $\mathsf{Pr}^{Ł^2}_\triangle$-valid $\mathsf{Pr}^{Ł^2}_\triangle$-valid iff α^* is $\mathsf{Pr}^{Ł^2}_\triangle$-valid.*

Proof. We construct α^* as follows. First, we take every modal atom $\mathsf{Pr}\phi$ (recall that $\phi \in \mathscr{L}_{\mathsf{BD}}$) and replace ϕ with its NNF. Second, we replace every literal $\neg p$ occurring in $\mathsf{Pr}(\mathsf{NNF}(\phi))$ with a corresponding fresh variable p'. That is, $\neg p$ is replaced with p', $\neg q$ with q', etc. Outer-layer connectives remain the same. Observe, that this increases the number of symbols in α *at most linearly*: indeed, the transformation of ϕ into $\mathsf{NNF}(\phi)$ is linear (recall Remark 3) and then we replace every occurrence of $\neg p$ with a fresh variable p^*. It remains to check that validity is preserved.

Let α be *not* $\mathsf{Pr}^{Ł^2}_\triangle$*-valid*, $\mathsf{Pr}^{Ł^2}_\triangle$-model. $\mathsf{Pr}^{Ł^2}_\triangle$-model. By Lemma 17, this is equivalent to $e_1(\alpha) \neq 1$ in some $\mathbb{M} = \langle W, v^+, v^-, \mu, e_1, e_2\rangle$. Now, let $\mathbb{M}^* = \langle W, v^{*+}, v^{*-}, \mu, e^*_1, e^*_2\rangle$ s.t. $v^{*+}(p) = v^+(p)$ and $v^{*+}(p') = v^-(p)$. It suffices to show that $e_1(\alpha) = e^*_1(\alpha^*)$.

We proceed by induction on α. Let $\alpha = \mathsf{Pr}\phi$ for some $\phi \in \mathscr{L}_{\mathsf{BD}}$. We have that $e_1(\alpha) = \mu(|\phi|^+) = \mu(|\mathsf{NNF}(\phi)|^+)$. We check that $|\mathsf{NNF}(\phi)|^+_{\mathbb{M}} = |\mathsf{NNF}(\phi)^*|^+_{\mathbb{M}^*}$ by induction on $\mathsf{NNF}(\phi)$.

If $\mathsf{NNF}(\phi) = p$, then $p^* = p$ and $v^+(p) = v^{*+}(p)$ by construction. If $\mathsf{NNF}(\phi) = \neg p$, then $(\neg p)^* = p'$ and $|\neg p|^+ = v^-(p) = v^{*+}(p')$ by construction. If $\mathsf{NNF}(\phi) = \chi \wedge \psi$, then $\mathsf{NNF}(\phi) = (\chi \wedge \psi)^* = \chi^* \wedge \psi^*$. By the induction hypothesis, $|\chi|^+ = |\chi^*|^+$ and $|\psi|^+ = |\psi^*|^+$. Hence, $|\chi \wedge \psi|^+_{\mathbb{M}} = |(\chi \wedge \psi)^*|^+_{\mathbb{M}^*}$. The case of $\mathsf{NNF}(\phi) = \chi \vee \psi$ can be considered in the same way.

It follows that $e_1(\mathsf{Pr}\phi) = e^*_1(\mathsf{Pr}(\mathsf{NNF}(\phi)^*))$. The cases of $Ł^2_{(\triangle,\rightarrow)}$ connectives can be proven by a straightforward application of the induction hypothesis.

For the converse, assume that α^* is not $\mathsf{Pr}^{Ł^2}_\triangle$-valid. Hence, $e_1(\alpha^*) \neq 1$ in some model $\mathbb{M} = \langle W, v^+, v^-, \mu, e_1, e_2\rangle$ by Lemma 17. We define $\mathbb{M}^\bullet = \langle W, v^{\bullet+}, v^{\bullet-}, \mu, e^\bullet_1, e^\bullet_2\rangle$ with $v^{\bullet-}(p) = v^+(p')$ and $v^{\bullet+}(p) = v^+(p)$. We can now show that $e_1(\alpha^*) = e^\bullet_1(\alpha)$ (i.e., $e^\bullet_1(\alpha) \neq 1$) as in the previous case. The result is as follows. □

We can now apply this lemma to adapt the proof of the NP-completeness of FP(Ł).

Theorem 42. *Validity of $\mathsf{Pr}^{Ł^2}_\triangle$ and $\mathbf{4}\mathsf{Pr}^{Ł_\triangle}$ is* coNP*-complete.*

Proof. Recall that $\mathsf{Pr}^{Ł^2}_\triangle$ and $\mathbf{4}\mathsf{Pr}^{Ł_\triangle}$ can be linearly embedded into one another (Theorems 24 and 25 and Lemma 26). Thus, it remains to provide a non-deterministic polynomial algorithm that decides whether a formula is falsifiable (i.e., *non-valid*) for one of these logics. We choose $\mathsf{Pr}^{Ł^2}_\triangle$ since it has only one modality.

Let $\alpha \in \mathscr{L}_{\mathsf{Pr}^{Ł^2}_\triangle}$ be over modal atoms $\mathsf{Pr}\phi_1, \ldots, \mathsf{Pr}\phi_n$. By Lemma 41, we can w.l.o.g. assume that α does $\mathsf{Pr}^{Ł^2}_\triangle$-valuation $\mathsf{Pr}^{Ł^2}_\triangle$-valuation for α.

First, we replace every modal atom $\mathsf{Pr}\phi_i$ with a fresh variable q_{ϕ_i}. Denote the new formula α^-. It is clear that $\ell(\alpha^-) = \mathcal{O}(\ell(\alpha))$. We construct a $\mathsf{Pr}^{Ł^2}_\triangle$ tableau beginning with $\{\alpha^- \leqslant_1 c, c < 1\}$.

As α^- does not contain $\neg$, every branch gives us an a system of linear inequalities that has a solution iff α^- is $Ł^2_{(\triangle,\rightarrow)}$-falsifiable: α^- is $Ł^2_{(\triangle,\rightarrow)}$-falsifiable iff at least one system of inequalities corresponding to a tableau branch has a solution. Clearly, if α^- is not $Ł^2_{(\triangle,\rightarrow)}$-falsifiable, it is not $\mathsf{Pr}^{Ł^2}_{\triangle}$-falsifiable either.

Now let $\mathfrak{Br} = \{\mathscr{B}_1, \ldots, \mathscr{B}_w\}$ be the set of all open branches in the $\mathscr{T}\left(Ł^2_{(\triangle,\rightarrow)}\right)$ tableau for α^-. We guess a $\mathscr{B} \in \mathfrak{Br}$ and consider the following system of (in)equalities:

$$(\mathrm{LI}(1)^{\mathscr{B}}) z_1 \triangledown t_1, \ldots, z_n \triangledown t_n, k_1 \leq k'_1, \ldots, k_r \leq k'_r, m_1 \geq 1, \ldots, m_s \geq 1, m'_1 \leq 0, \ldots, m'_t \leq 0$$

Here, z_i's correspond to the values of q_{ϕ_i}'s in α^- and t_i's are linear polynomials that label q_{ϕ_i}'s. Numerical constraints give us k's, k''s, m's, and m''s. Denote the number of inequalities and the number of variables in $(\mathrm{LI}(1)^{\mathscr{B}})$ with l_1 and l_2, respectively. It is clear that $l_1 = \mathscr{O}(\ell(\alpha^-))$ and $l_2 = \mathscr{O}(\ell(\alpha^-))$.

We need to check whether z_i's are coherent as probabilities of ϕ_i's. that is that there is a probability measure μ on $2^{\mathtt{Prop}(\alpha)}$ s.t. $\mu(|\phi_i|^+) = z_i$. Recall again that because of Klein et al. (2021, Theorem 4), we can assume that the probabilities are *classical*. Moreover, we can assume that there are n propositional variables in α (we can always add new superfluous variables or modal atoms to α to make their numbers equal).

Now, introduce 2^n variables u_v indexed by n-letter words over $\{0, 1\}$. These denote whether the variables of ϕ_i's are true under v^+ and thus correspond to subsets of $\mathtt{Prop}(\alpha)$. For example, if $\mathtt{Prop}(\alpha) = \{p_1, p_2, p_3, p_4\}$, then u_{1001} encodes $\{p_1, p_4\}$. Note that $\neg$ does not occur in α^- and thus we care only about e_1 and v^+. Furthermore, while n is the number of ϕ_i's, we can add extra modal atoms or variables to make it also the number of variables. We let $a_{i,v} = 1$ when ϕ_i is true under v^+ and $a_{i,v} = 0$ otherwise. Now add new equations to $\mathrm{LI}(1)^{\mathscr{B}}$, namely

$$\sum_v u_v = 1 \qquad\qquad \sum_v (a_{i,v} \cdot u_v) = z_i \qquad\qquad (\mathrm{LI}(2\exp)^{\mathscr{B}})$$

The new problem – $(\mathrm{LI}(1)^{\mathscr{B}}) \cup (\mathrm{LI}(2\exp)^{\mathscr{B}})$ – has a solution over $[0, 1]$ iff α is $\mathsf{Pr}^{Ł^2}_{\triangle}$-satisfiable since $(\mathrm{LI}(2\exp)^{\mathscr{B}})$ encodes a measure on $\mathtt{Prop}(\alpha)$ while the existence of a solution for $(\mathrm{LI}(1)^{\mathscr{B}})$ ensures that α is $Ł^2_{(\triangle,\rightarrow)}$-satisfiable. Furthermore, although there are $l_2 + 2^n + n$ variables in $(\mathrm{LI}(1)^{\mathscr{B}}) \cup (\mathrm{LI}(2\exp)^{\mathscr{B}})$, it has no more than $l_1 + n + 1$ (in)equalities. Thus by Fagin et al. (1990, Lemma 2.5), it has a solution with at most $l_1 + n + 1$ non-zero entries. We guess a list L of at most $l_1 + n + 1$ words v (its size is $n \cdot (l_1 + n + 1)$). We can now compute the values of $a_{i,v}$'s for $i \leq n$ and $v \in L$ and obtain a new system of inequalities:

$$\sum_{v \in L} u_v = 1 \qquad\qquad \sum_{v \in L} (a_{i,v} \cdot u_v) = z_i \qquad\qquad (\mathrm{LI}(2\mathrm{poly})^{\mathscr{B}})$$

It is clear that $(\mathrm{LI}(1)^{\mathscr{B}}) \cup (\mathrm{LI}(2\mathrm{poly})^{\mathscr{B}})$ is of polynomial size w.r.t. $\ell(\alpha^-)$ and thus can be solved in polynomial time. Moreover, $(\mathrm{LI}(1)^{\mathscr{B}}) \cup (\mathrm{LI}(2\mathrm{poly})^{\mathscr{B}})$ has a solution iff the values of $\mathsf{Pr}\phi_i$'s occurring on $\mathscr{B}$ are coherent as probabilities. If there is no solution for $(\mathrm{LI}(1)^{\mathscr{B}}) \cup (\mathrm{LI}(2\mathrm{poly})^{\mathscr{B}})$, we guess another open branch from the tableau and repeat the procedure. If there is no open branch $\mathscr{B}' \in \mathfrak{Br}$ s.t. its corresponding system of inequalities $\mathrm{LI}(1)^{\mathscr{B}'} \cup \mathrm{LI}(2\mathrm{poly})^{\mathscr{B}'}$ has a solution, then α is $\mathsf{Pr}^{Ł^2}_{\triangle}$-valid as well by Lemma 41. □

4. Belief and plausibility functions over *BD*

In this section, we introduce BD-models with belief and plausibility functions. We adapt the definitions from Zhou (2013) and Bílková et al. (2023d).

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Definition 43 (Belief function). *A belief function on* $W \neq \varnothing$ *is a map* $\texttt{bel} : 2^W \to [0, 1]$ *s.t.*

- $\texttt{bel}$ *is monotone w.r.t.* $\subseteq$*: if* $X \subseteq Y$*, then* $\texttt{bel}(X) \leq \texttt{bel}(Y)$;
- *for every* $X_1, \ldots, X_k \subseteq W$*, it holds that*

$$\texttt{bel}\left(\bigcup_{1 \leq i \leq k} X_i\right) \geq \sum_{\substack{J \subseteq \{1, \ldots, k\} \\ J \neq \varnothing}} (-1)^{|J|+1} \cdot \texttt{bel}\left(\bigcap_{j \in J} X_j\right)$$

- $\texttt{bel}(\varnothing) = 0$ *and* $\texttt{bel}(W) = 1$.

Definition 44 (Plausibility function). *A plausibility function on* $W \neq \varnothing$ *is a map* $\texttt{pl} : 2^W \to [0, 1]$ *s.t.*

- $\texttt{pl}$ *is monotone w.r.t.* $\subseteq$;
- *for every* $X_1, \ldots, X_k \subseteq W$*, it holds that*

$$\texttt{pl}\left(\bigcap_{1 \leq i \leq k} X_i\right) \leq \sum_{\substack{J \subseteq \{1, \ldots, k\} \\ J \neq \varnothing}} (-1)^{|J|+1} \cdot \texttt{pl}\left(\bigcup_{j \in J} X_j\right)$$

- $\texttt{pl}(\varnothing) = 0$ *and* $\texttt{pl}(W) = 1$.

Recall that every plausibility function $\texttt{pl}$ on W gives rise to a belief function $\texttt{bel}_{\texttt{pl}}$ and vice versa: given a belief function $\texttt{bel}$ on W, one can construct a plausibility function $\texttt{pl}_{\texttt{bel}}$.

$$\texttt{pl}_{\texttt{bel}}(X) = 1 - \texttt{bel}(W \setminus X) \qquad \texttt{bel}_{\texttt{pl}}(X) = 1 - \texttt{pl}(W \setminus X) \tag{3}$$

Moreover, it was shown in Bílková et al. (2023d, Lemmas 2.10 and 2.11) that a similar statement holds for BD-models. That is, belief and plausibility functions can be defined via one another even without set-theoretic complements.

$$\texttt{pl}_{\texttt{bel}}(|\phi|^+) = 1 - \texttt{bel}(|\phi|^-) \qquad \texttt{bel}_{\texttt{pl}}(|\phi|^+) = 1 - \texttt{pl}(|\phi|^-) \tag{4}$$

In what follows, we will be using two kinds of BD-models with belief functions introduced by Bílková et al. (2023): in the first one, belief and plausibility will be interdefinable via (4); in the second one, we will assume them to be independent.

Definition 45 (BD $\texttt{bel}$-models). *A* BD $\texttt{bel}$*-model is a tuple* $\mathfrak{M}_{\texttt{bel}} = \langle \mathfrak{M}, \texttt{bel} \rangle$ *with* $\mathfrak{M}$ *being a* BD*-model (recall Definition 2) and* $\texttt{bel}$ *a belief function on* W.

Definition 46 (BD $\texttt{bel}/\texttt{pl}$-models). *A* BD $\texttt{bel}/\texttt{pl}$*-model is a tuple* $\mathfrak{M}_{\texttt{bel}/\texttt{pl}} = \langle \mathfrak{M}, \texttt{bel}, \texttt{pl} \rangle$ *with* $\mathfrak{M}$ *being a* BD*-model (recall Definition 2),* $\texttt{bel}$ *a belief function on* W*, and* $\texttt{pl}$ *a plausibility function on* W.

Observe from Definitions 43 and 44 that the traditional axiomatisation of belief and plausibility functions is infinite. In the classical case, this can be circumvented following Godo et al. (2001, 2003) if we define belief in ϕ as *the probability of* $\Box\phi$ (with $\Box$ being the *non-nesting* **S5** modality). Then, we can use (3) to define the plausibility of ϕ as *the probability of* $\Diamond\phi$ since $\Diamond\phi \equiv {\sim}\Box{\sim}\phi$ (here, $\equiv$ is the classical equivalence and $\sim$ is the classical negation). In the remainder of the section, we show that the same can be done in BD.

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Let us first recall the classical result.

Definition 47. *A classical uncertainty model is a tuple* $\mathfrak{M} = \langle W, v, \mu \rangle$ *with* $W \neq \varnothing$, $\mu : 2^W \to [0, 1]$, *and* $v : \texttt{Prop} \to 2^W$ *extended to the satisfaction relation* $\mathfrak{M}, w \vDash \phi$ *as follows.*

$$\mathfrak{M}, w \vDash p \text{ iff } w \in v(p) \quad \mathfrak{M}, w \vDash {\sim}\phi \text{ iff } \mathfrak{M}, w \nvDash \phi \quad \mathfrak{M}, w \vDash \phi \wedge \chi \text{ iff } \mathfrak{M}, w \vDash \phi \text{ and } \mathfrak{M}, w \vDash \chi$$

If μ *is a belief function, plausibility, probability, etc., we call* $\mathfrak{M}$ *belief, plausibility, probabilistic, etc. model.*

The extension of a formula ϕ *is defined as* $\|\phi\| = \{w : \mathfrak{M}, w \vDash \phi\}$.

Definition 48 (Classical probabilistic Kripke model). *A classical probabilistic Kripke model is a tuple* $\mathfrak{M} = \langle W, R, v, \pi \rangle$ *with* $W \neq \varnothing$, *R being an equivalence relation on W,* π *being a classical probability measure on W and* $v : \texttt{Prop} \to 2^W$ *extended to the satisfaction relation* $\mathfrak{M}, w \vDash \phi$ *as follows.*

$$\mathfrak{M}, w \vDash p \text{ iff } w \in v(p)$$

$$\mathfrak{M}, w \vDash {\sim}\phi \text{ iff } \mathfrak{M}, w \nvDash \phi \qquad \mathfrak{M}, w \vDash \phi \wedge \chi \text{ iff } \mathfrak{M}, w \vDash \phi \text{ and } \mathfrak{M}, w \vDash \chi$$

$$\mathfrak{M}, w \vDash \Box\phi \text{ iff } \forall w' : wRw' \Rightarrow \mathfrak{M}, w' \vDash \phi \qquad \mathfrak{M}, w \vDash \Diamond\phi \text{ iff } \exists w' : wRw' \,\&\, \mathfrak{M}, w' \vDash \phi$$

Theorem 49 (Godo et al. 2001, Theorem 1). *Let* ϕ *be a propositional classical formula and let* $\mathfrak{M}_{\texttt{bel}} = \langle W, v, \texttt{bel} \rangle$ *and* $\mathfrak{M} = \langle W, v, \texttt{pl} \rangle$ *be, respectively, a belief and plausibility models, then:*

(i) there is a classical probabilistic model $\mathfrak{M}_\Box = \langle W_\Box, R_\Box, v_\Box, \pi_\Box \rangle$ *s.t.* $\pi_\Box(\|\Box\phi\|) = \texttt{bel}(\|\phi\|)$;

(ii) there is a classical probabilistic model $\mathfrak{M}_\Diamond = \langle W_\Diamond, R_\Diamond, v_\Diamond, \pi_\Diamond \rangle$ *s.t.* $\pi_\Diamond(\|\Diamond\phi\|) = \texttt{pl}(\|\phi\|)$.

We draw the attention of our readers to the fact that $\Box$ and $\Diamond$ do not have to be **S5**: it suffices that the underlying modal logic be normal. The original motivation for choosing **S5** was that it is locally finite. Note, however, that local finiteness can be achieved in other modal logics as well because $\Box$ and $\Diamond$ do not nest (cf. Flaminio et al. 2013, Section 6) for a detailed discussion).

Moreover, originally only (i) is proven by Godo et al. (2001, 2003). Observe, however, that (ii) can be obtained from (i) using (3) since

$$\begin{aligned}
\pi(\|\Diamond\phi\|) &= \pi(\|{\sim}\Box{\sim}\phi\|) \\
&= 1 - \pi(\|\Box{\sim}\phi\|) \\
&= 1 - \texttt{bel}(\|{\sim}\phi\|) \\
&= 1 - \texttt{bel}(W \setminus \|\phi\|) \\
&= \texttt{pl}_{\texttt{bel}}(\|\phi\|)
\end{aligned}$$

Let us now introduce probabilistic models for modal BD formulas. We borrow the definition of modalities in BD from Odintsov and Wansing (2017).

Definition 50 (BD probabilistic Kripke models). *A* BD *probabilistic Kripke model is a tuple* $\mathfrak{M} = \langle W, R, v^+, v^-, \pi \rangle$ *with* $W \neq \varnothing$, *R being an equivalence relation on W,* π *being a classical probability measure on W and* $v^+, v^- : \texttt{Prop} \to 2^W$ *extended to* $\mathfrak{M}, w \vDash^+ \phi$ *and* $\mathfrak{M}, w \vDash^- \phi$ *as in Definition 2 for propositional connectives and as below for modalities.*

$$\mathfrak{M}, w \vDash^+ \Box\phi \text{ iff } \forall w' : wRw' \Rightarrow \mathfrak{M}, w' \vDash^+ \phi \qquad \mathfrak{M}, w \vDash^+ \Diamond\phi \text{ iff } \exists w' : wRw' \,\&\, \mathfrak{M}, w' \vDash^+ \phi$$

$$\mathfrak{M}, w \vDash^- \Box\phi \text{ iff } \exists w' : wRw' \,\&\, \mathfrak{M}, w' \vDash^- \phi \qquad \mathfrak{M}, w \vDash^- \Diamond\phi \text{ iff } \forall w' : wRw' \Rightarrow \mathfrak{M}, w' \vDash^- \phi$$

Positive and negative extensions of ϕ *are as in Definition 2.*

We also call the $\langle W, R, v^+, v^- \rangle$ *reduct a* BD *Kripke model.*

Remark 51. *Since $\models^+$ conditions for $\Box$ and $\Diamond$ in the classical logic and in* BD *coincide, it is clear (cf.* Hájek 1996 *for a detailed discussion of the classical case) that given a probability π on a Kripke model $\mathfrak{M}$, there are a belief and plausibility functions* $\mathtt{bel}$ *and* $\mathtt{pl}$ *on $\mathfrak{M}$ s.t.* $\mathtt{bel}(|\phi|^+) = \pi(|\Box\phi|^+)$ *and* $\mathtt{pl}(|\phi|^+) = \pi(|\Diamond\phi|^+)$ *for every $\phi \in \mathscr{L}_{\mathsf{BD}}$.*

Theorem 52. *Let $\phi_1, \ldots, \phi_n \in \mathscr{L}_{\mathsf{BD}}$ and consider a* BD $\mathtt{bel}$*-model $\mathfrak{M}_{\mathtt{bel}} = \langle \mathfrak{M}, \mathtt{bel}\rangle$ and a* BD $\mathtt{bel}/\mathtt{pl}$*-model $\mathfrak{M}'_{\mathtt{bel}/\mathtt{pl}} = \langle \mathfrak{M}', \mathtt{bel}', \mathtt{pl}'\rangle$. Then, there exist*

1. *a* BD *probabilistic Kripke model $\mathfrak{M}_{\Box} = \langle W_{\Box}, R_{\Box}, v^+_{\Box}, v^-_{\Box}, \pi_{\Box}\rangle$ with $\mathtt{bel}(|\phi_i|^+) = \pi_{\Box}(|\Box\phi_i|^+)$ for each $i \in \{1, \ldots, n\}$;*
2. *a* BD *probabilistic Kripke model $\mathfrak{M}_{\Box,\Diamond} = \langle W_{\Box,\Diamond}, R_1, R_2, v^+_{\Box,\Diamond}, v^-_{\Box,\Diamond}, \pi_{\Box,\Diamond}\rangle$ with $\mathtt{bel}'(|\phi_i|^+) = \pi_{\Box,\Diamond}(|\Box_1\phi_i|^+)$ and $\mathtt{pl}'(|\phi_i|^+) = \pi_{\Box,\Diamond}(|\Diamond_2\phi_i|^+)$ for each $i \in \{1, \ldots, n\}$ (here, $\Box_1$ and $\Diamond_2$ are associated to R_1 and R_2, respectively).*

Proof. We prove (1) as (2) can be dealt with similarly. Let w.l.o.g. ϕ be in NNF and denote ϕ^* the result of replacing every negated variable $\neg p$ occurring in ϕ with a fresh variable p^*. Consider $\mathfrak{M}_{\mathtt{bel}} = \langle W, v^+, v^-, \mathtt{bel}\rangle$ and $\mathfrak{M}^*_{\mathtt{bel}}$ (cf. the proof of Lemma 41). It is easy to see that $\mathtt{bel}(|\phi_i|^+) = \mathtt{bel}(|\phi^*_i|^+)$ for every i.

Now define a *classical* belief model $\mathfrak{M}^{\mathsf{cl}}_{\mathtt{bel}} = \langle W, v^{\mathsf{cl}}, \mathtt{bel}\rangle$ with $v^{\mathsf{cl}}(q) = v^+(q)$ for every $q \in \mathtt{Prop}$. Clearly, $\mathtt{bel}(|\phi^*_i|^+) = \mathtt{bel}(\|\phi^*_i\|)$ for all ϕ^*_i's. Thus, by Theorem 49, we have that there is a classical Kripke probabilistic model $\mathfrak{M}_{\Box}=\langle W_{\Box},R_{\Box},v_{\Box},\pi_{\Box}\rangle$ s.t. $\pi_{\Box}(\|\Box\phi^*_i\|)=\mathtt{bel}(\|\phi^*_i\|)$ for every i. It remains to construct a suitable BD probabilistic Kripke model.

We define $\mathfrak{M}^{\mathsf{BD}}_{\Box} = \langle W_{\Box}, R_{\Box}, v^+_{\Box}, v^-_{\Box}, \pi_{\Box}\rangle$ with $v^+_{\Box}(p) = v_{\Box}(p)$ and $v^-_{\Box}(p) = v_{\Box}(p^*)$. One can show by induction on ϕ^*_i's that $\|\phi^*_i\| = |\phi_i|^+$, and thus $\|\Box\phi^*_i\| = |\Box\phi_i|^+$ for every i: indeed, cf. semantical conditions for $\Box$ in Definitions 48 and 50. Hence, $\pi_{\Box}(\|\Box\phi^*_i\|) = \pi_{\Box}(|\Box\phi_i|^+)$, as required. □

We finish the section with a brief observation.

Remark 53. *Just as in the classical case, $\Box$ and $\Diamond$ do not need to be* **S5***. We choose* BD *version of* **S5** *because of the following property: if $\mathfrak{M} = \langle W, R, v^+, v^-, \pi\rangle$ is a* BD *probabilistic Kripke model and ψ is a modal formula where all propositional variables are in the scope of a modality, and wRw', then $\mathfrak{M}, w \models^+ \psi$ iff $\mathfrak{M}, w' \models^+ \psi$ and $\mathfrak{M}, w \models^- \psi$ iff $\mathfrak{M}, w' \models^- \psi$.*

5. Logics for belief and plausibility functions over *BD*

In this section, we recall two-layered logics $\mathsf{Bel}^{\text{Ł}^2}_{\triangle}$ and $\mathsf{Bel}^{\mathsf{NŁ}}$ for reasoning with belief and plausibility functions over BD that were presented by Bílková et al., (2023). We then combine the technique of Hájek and Tulipani (2001) with the results of Godo et al. (2001, 2003) and Theorem 52 to obtain the coNP-completeness of $\mathsf{Bel}^{\text{Ł}^2}_{\triangle}$ and $\mathsf{Bel}^{\mathsf{NŁ}}$.

5.1 Languages and semantics

Definition 54 ($\mathsf{Bel}^{\text{Ł}^2}_{\triangle}$: language and semantics). *The language of* $\mathsf{Bel}^{\text{Ł}^2}_{\triangle}$ *is constructed using the grammar below.*

$$\mathscr{L}_{\mathsf{Bel}^{\text{Ł}^2}_{\triangle}} \ni \alpha := \mathsf{B}\phi \mid \neg\alpha \mid \alpha \to \alpha \mid {\sim}\alpha \mid \triangle\alpha \qquad (\phi \in \mathscr{L}_{\mathsf{BD}})$$

A $\mathsf{Bel}^{\text{Ł}^2}_{\triangle}$*-model is a tuple $\mathbb{M}_{\mathtt{bel}} = \langle \mathfrak{M}, \mathtt{bel}, e_1, e_2\rangle$ with*

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- $\langle\mathfrak{M},\texttt{bel}\rangle$ *being a* BD $\texttt{bel}$*-model (cf. Definition 45);*
- e_1 *and* e_2 *being* $Ł^2_{(\triangle,\to)}$ *valuations induced by* $\texttt{bel}$:
  - $e_1(\mathsf{B}\phi)=\texttt{bel}(|\phi|^+)$, $e_2(\mathsf{B}\phi)=\texttt{bel}(|\phi|^-)$;
  - *values of complex* $\mathscr{L}_{\mathsf{Bel}^{Ł^2}_{\triangle}}$*-formulas are computed via Definition 12.*

We say that α *is* $\mathsf{Bel}^{Ł^2}_{\triangle}$*-valid iff* $e_1(\alpha)=1$ *and* $e_2(\alpha)=0$ *in all models. A set of formulas* Γ *entails* α ($\Gamma\models_{\mathsf{Bel}^{Ł^2}_{\triangle}}\alpha$) *iff there is no* $\mathsf{Bel}^{Ł^2}_{\triangle}$*-model s.t.* $e(\gamma)=(1,0)$ *for all* $\gamma\in\Gamma$ *but* $e(\alpha)\neq(1,0)$.

Remark 55. *Note that we cannot utilise the technique from Lemma 17 to reduce* $\mathsf{Bel}^{Ł^2}_{\triangle}$*-validity to checking whether* $e_1(\alpha)=1$ *in every model. This is because belief functions are not additive. Thus, even though we can construct* $(v^*)^+$ *and* $(v^*)^-$ *s.t. (1) holds, we cannot use it to infer the following counterpart of (2):*

$$e^*(\mathsf{B}\phi)=(1-\texttt{bel}(|\phi|^-),1-\texttt{bel}(|\phi|^+))=(1-e_2(\mathsf{B}\phi),1-e_1(\mathsf{Pr}\phi)) \tag{5}$$

Indeed, (5) does not hold in general since it is not necessarily the case that $\texttt{bel}(W\setminus X)=1-\texttt{bel}(X)$ *for* $X\subseteq W$.

$\mathsf{Bel}^{\mathsf{N}Ł}$ logic (NŁ) that is (NŁ) that is inspired by Nelson's paraconsistent logic from Nelson (1949). The logic was introduced by Bílková et al. (2021) and further investigated by Bílková et al. (2023d). We recall its language and semantics below.

Definition 56 (NŁ: language and semantics). *The language is constructed via the following grammar.*

$$\mathscr{L}_{\mathsf{N}Ł}\ni\phi:=p\mid{\sim_{\mathsf{N}}}\phi\mid\neg\phi\mid(\phi\wedge\phi)\mid(\phi\rightarrowtriangle\phi)$$

The support of truth and support of falsity conditions are given by the following extensions of $v_1,v_2:\texttt{Prop}\to[0,1]$ *(NŁ valuations) to the complex formulas.*

$$\begin{aligned} v_1(\neg\phi)&=v_2(\phi) & v_2(\neg\phi)&=v_1(\phi)\\ v_1({\sim_{\mathsf{N}}}\phi)&={\sim_{Ł}}v_1(\phi) & v_2({\sim_{\mathsf{N}}}\phi)&=v_1(\phi)\\ v_1(\phi\wedge\chi)&=v_1(\phi)\wedge_{Ł}v_1(\chi) & v_2(\phi\wedge\chi)&=v_2(\phi)\vee_{Ł}v_2(\chi)\\ v_1(\phi\rightarrowtriangle\chi)&=v_1(\phi)\to_{Ł}v_1(\chi) & v_2(\phi\rightarrowtriangle\chi)&=v_1(\phi)\odot_{Ł}v_2(\chi)\end{aligned}$$

We say that α *is* $\mathsf{Bel}^{\mathsf{N}Ł}$*-valid iff* $e_1(\alpha)=1$ *for every* $\mathsf{Bel}^{\mathsf{N}Ł}$*-model.* Γ *entails* α ($\Gamma\models_{\mathsf{Bel}^{\mathsf{N}Ł}}\alpha$) *iff there is no* $\mathsf{Bel}^{\mathsf{N}Ł}$-model *s.t.* $e_1(\gamma)=1$ *for every* $\gamma\in\Gamma$ *and* $e_1(\alpha)\neq1$.

Definition 57 ($\mathsf{Bel}^{\mathsf{N}Ł}$: language and semantics). *The language is given by the grammar below.*

$$\mathscr{L}_{\mathsf{Bel}^{\mathsf{N}Ł}}\quad\alpha:=\mathsf{B}\phi\mid\mathsf{Pl}\phi\mid\neg\alpha\mid\alpha\wedge\alpha\mid\alpha\rightarrowtriangle\alpha\mid{\sim_{\mathsf{N}}}\alpha\qquad(\phi\in\mathscr{L}_{\mathsf{BD}})$$

A $\mathsf{Bel}^{\mathsf{N}Ł}$*-model is a tuple* $\mathbb{M}_{\texttt{bel/pl}}=\langle\mathfrak{M},\texttt{bel},\texttt{pl},e_1,e_2\rangle$ *with*

- $\langle\mathfrak{M},\texttt{bel},\texttt{pl}\rangle$ *being a* BD $\texttt{bel/pl}$*-model (cf. Definition 46);*
- e_1 *and* e_2 *being* NŁ *valuations induced by* $\texttt{bel}$ *and* $\texttt{pl}$:
  - $e_1(\mathsf{B}\phi)=\texttt{bel}(|\phi|^+)$, $e_2(\mathsf{B}\phi)=\texttt{pl}(|\phi|^-)$, $e_1(\mathsf{Pl}\phi)=\texttt{pl}(|\phi|^+)$, $e_2(\mathsf{Pl}\phi)=\texttt{bel}(|\phi|^-)$;
  - *values of complex* $\mathscr{L}_{\mathsf{Bel}^{\mathsf{N}Ł}}$*-formulas are computed via Definition 56.*

We say that α is $\mathsf{Bel}^{\mathsf{N}Ł}$-valid iff $e_1(\alpha)=1$ for every $\mathsf{Bel}^{\mathsf{N}Ł}$-model. Γ entails α ($\Gamma\models_{\mathsf{Bel}^{\mathsf{N}Ł}}\alpha$) iff there is no $\mathsf{Bel}^{\mathsf{N}Ł}$-model s.t. $e_1(\gamma)=1$ for every $\gamma\in\Gamma$ and $e_1(\alpha)\neq 1$.

Note that using (4), we can define a plausibility operator Pl_{B} in $\mathsf{Bel}^{Ł^2}_{\triangle}$ as $\mathsf{Pl}_{\mathsf{B}}\phi :=\sim\mathsf{B}\neg\phi$. The main difference between it and Pl in $\mathsf{Bel}^{\mathsf{N}Ł}$ is that the latter is *independent of belief*. We refer readers to Bílková et al. (2023d) for a more detailed discussion of differences between $\mathsf{Bel}^{Ł^2}_{\triangle}$ and $\mathsf{Bel}^{\mathsf{N}Ł}$.

Moreover, $\mathsf{Pr}^{Ł^2}_{\triangle}$ is complete w.r.t. models with *classical* probability measures (recall Klein et al. 2021, Theorems 3–4 and Bílková et al. 2023, Theorem 4.24). Thus, as belief and plausibility functions are generalisations of probabilities, it means that if a statement about belief functions is valid, then it is valid about probabilities as well. Formally, let $\alpha\in\mathscr{L}_{\mathsf{Bel}^{Ł^2}_{\triangle}}$ and let further α^{Pr} be the result of the replacement of $\mathsf{B}\phi$'s with $\mathsf{Pr}\phi$'s. Then if $\mathsf{Bel}^{Ł^2}_{\triangle}\models\alpha$, it follows that $\mathsf{Pr}^{Ł^2}_{\triangle}\models\alpha^{\mathsf{Pr}}$. This means that $\mathsf{Pr}^{Ł^2}_{\triangle}$ can be thought of as an extension or a semantical restriction of $\mathsf{Bel}^{Ł^2}_{\triangle}$.

In the previous section, we showed (Theorem 52) that belief and plausibility in BD can be represented as probabilities of modal formulas. We are going to introduce two logics – $\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}$ and $\mathsf{Pr}^{\mathsf{N}Ł}_{\mathbf{S5}}$ – $\mathsf{Bel}^{\mathsf{N}Ł}$ – that deal with those and show that $\mathsf{Bel}^{Ł^2}_{\triangle}$ and $\mathsf{Bel}^{\mathsf{N}Ł}$

Definition 58 ($\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}$: language and semantics). *The language is constructed as follows.*

$$\mathscr{L}_{\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}}\ni\alpha:=\mathsf{Pr}\heartsuit\phi\mid\sim\alpha\mid\neg\alpha\mid\triangle\alpha\mid(\alpha\to\alpha)\qquad(\phi\in\mathscr{L}_{\mathsf{BD}},\heartsuit\in\{\Box,\Diamond\})$$

A $\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}$-model is a tuple $\mathbb{M}_\pi=\langle\mathfrak{M},\pi,e_1,e_2\rangle$ s.t.

- *$\langle\mathfrak{M},\pi\rangle$ is a BD probabilistic Kripke model;*
- *e_1 and e_2 are $Ł^2_{(\triangle,\to)}$ valuations induced by π:*
  - *$e_1(\mathsf{Pr}\heartsuit\phi)=\pi(|\heartsuit\phi|^+)$, $e_2(\mathsf{Pr}\heartsuit\phi)=\pi(|\heartsuit\phi|^-)$ with $\heartsuit\in\{\Box,\Diamond\}$;*
  - *values of complex $\mathscr{L}_{\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}}$-formulas are computed via Definition 12.*

We say that α is $\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}$-valid iff $e_1(\alpha)=1$ and $e_2(\alpha)=0$ in all $\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}$-models. A set of formulas Γ entails α ($\Gamma\models_{\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}}\alpha$) iff there is no $\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}$-model s.t. $e(\gamma)=(1,0)$ for all $\gamma\in\Gamma$ but $e(\alpha)\neq(1,0)$.

The next statements can be proven in the same manner as Lemmas 17 and 41, respectively.

Lemma 59. *Let $\alpha\in\mathscr{L}_{\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}}$. Then α is $\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}$-valid iff $e_1(\alpha)=1$ in all $\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}$-models.*

Lemma 60. *Let $\alpha\in\mathscr{L}_{\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}}$. Then there is α^* where $\neg$ does not occur at all s.t. α^* is $\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}$-valid iff α is $\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}$-valid.*

Definition 61 ($\mathsf{Pr}^{\mathsf{N}Ł}_{\mathbf{S5}}$: language and semantics). *The language is generated by the following grammar.*

$$\mathscr{L}_{\mathsf{Pr}^{\mathsf{N}Ł}_{\mathbf{S5}}}\ni\alpha:=\mathsf{Pr}_i\heartsuit_i\phi\mid\neg\alpha\mid\alpha\wedge\alpha\mid\alpha\rightarrowtriangle\alpha\mid\sim_{\mathsf{N}}\alpha\qquad\phi\in\mathscr{L}_{\mathsf{BD}},\heartsuit\in\{\Box,\Diamond\},i\in\{1,2\})$$

A $\mathsf{Pr}^{\mathsf{N}Ł}_{\mathbf{S5}}$-model is a tuple $\mathbb{M}^{\mathsf{N}Ł}_{\pi_1,\pi_2}=\langle W,R_1,R_2,v^+,v^-,\pi_1,\pi_2,e_1,e_2\rangle$ s.t.

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- $\langle W, R_1, R_2, v^+, v^-, \pi_1, \pi_2\rangle$ *is a* BD *probabilistic Kripke model with two relations and two measures;*
- e_1 *and* e_2 *are* NŁ *valuations induced by* π_1 *and* π_2:
  - $e_1(\mathsf{Pr}_1\heartsuit_1\phi) = \pi_1(|\heartsuit_1\phi|^+)$, $e_2(\mathsf{Pr}_1\heartsuit_1\phi) = \pi_1(|\heartsuit_1\phi|^-)$ *with* $\heartsuit \in \{\Box, \Diamond\}$;
  - $e_1(\mathsf{Pr}_2\heartsuit_2\phi) = \pi_2(|\heartsuit_2\phi|^+)$, $e_2(\mathsf{Pr}_2\heartsuit_2\phi) = \pi_2(|\heartsuit_2\phi|^-)$ *with* $\heartsuit \in \{\Diamond, \Box\}$;
  - *values of complex* $\mathscr{L}_{\mathsf{Pr}^{\mathsf{NŁ}}_{\mathbf{S5}}}$*-formulas are computed via Definition 56.*

We say that α *is* $\mathsf{Pr}^{\mathsf{NŁ}}_{\mathbf{S5}}$*-valid iff* $e_1(\alpha) = 1$ *in all* $\mathsf{Pr}^{\mathsf{NŁ}}_{\mathbf{S5}}$*-models. A set of formulas* Γ *entails* α ($\Gamma \models_{\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}} \alpha$) *iff there is no* $\mathsf{Pr}^{\mathsf{NŁ}}_{\mathbf{S5}}$*-model s.t.* $e(\gamma) = 1$ *for all* $\gamma \in \Gamma$ *but* $e(\alpha) \neq 1$.

5.2 Embedding of belief logics into probabilistic logics

Theorem 52 shows that beliefs can be represented as probabilities of modal formulas. In this section, we use this result to prove that we can faithfully embed $\mathsf{Bel}^{Ł^2}_{\triangle}$ and $\mathsf{Bel}^{\mathsf{NŁ}}$ into $\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}$ and $\mathsf{Pr}^{\mathsf{NŁ}}_{\mathbf{S5}}$. To simplify the presentation, observe that all $\mathscr{L}_{\mathsf{Bel}^{Ł^2}_{\triangle}}$- and $\mathscr{L}_{\mathsf{Bel}^{\mathsf{NŁ}}}$-formulas can be transformed into an outer-$\neg$-free form since $Ł^2_{(\triangle,\to)}$ and NŁ permit NNF's and since the following holds.

$$e(\neg\mathsf{B}\phi) = e(\mathsf{B}\neg\phi) \qquad\qquad (\text{ in } \mathsf{Bel}^{Ł^2}_{\triangle})$$
$$e(\neg\mathsf{B}\phi) = e(\mathsf{Pl}\neg\phi) \qquad e(\neg\mathsf{Pl}\phi) = e(\mathsf{B}\neg\phi) \qquad (\text{ in } \mathsf{Bel}^{\mathsf{NŁ}})$$

Definition 62 (Embedding of $\mathsf{Bel}^{Ł^2}_{\triangle}$ into $\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}$). *Let* $\alpha \in \mathscr{L}_{\mathsf{Bel}^{Ł^2}_{\triangle}}$ *be outer-*$\neg$*-free, we define* $\alpha^{\boxplus}$ *and* $\alpha^{\boxminus}$ *as follows.*

$$(\mathsf{B}\phi)^{\boxplus} = \mathsf{Pr}(\Box\phi) \qquad (\mathsf{B}\phi)^{\boxminus} = \mathsf{Pr}(\Box\neg\phi)$$
$$(\triangle\beta)^{\boxplus} = \triangle(\beta^{\boxplus}) \qquad (\triangle\beta)^{\boxminus} = {\sim}\triangle{\sim}(\beta^{\boxminus})$$
$$({\sim}\beta)^{\boxplus} = {\sim}(\beta^{\boxplus}) \qquad ({\sim}\beta)^{\boxminus} = {\sim}(\beta^{\boxminus})$$
$$(\beta\to\gamma)^{\boxplus} = \beta^{\boxplus}\to\gamma^{\boxplus} \qquad (\beta\to\gamma)^{\boxminus} = \gamma^{\boxminus}\ominus\beta^{\boxminus}$$

Definition 63 (Embedding of $\mathsf{Bel}^{\mathsf{NŁ}}$ into $\mathsf{Pr}^{\mathsf{NŁ}}_{\mathbf{S5}}$). *Let* $\alpha \in \mathscr{L}_{\mathsf{Bel}^{\mathsf{NŁ}}}$ *be outer-*$\neg$*-free, we define* $\alpha^{\Box,\Diamond}$ *as follows.*

$$(\mathsf{B}\phi)^{\Box,\Diamond} = \mathsf{Pr}_1(\Box_1\phi) \qquad (\mathsf{Pl}\phi)^{\Box,\Diamond} = \mathsf{Pr}_2(\Diamond_2\phi)$$
$$({\sim}_{\mathsf{N}}\beta)^{\Box,\Diamond} = {\sim}_{\mathsf{N}}(\beta^{\Box,\Diamond}) \qquad (\beta\circ\gamma)^{\Box,\Diamond} = \beta^{\Box,\Diamond}\circ\gamma^{\Box,\Diamond} \qquad (\circ\in\{\wedge,\rightarrowtriangle\})$$

We can now prove the following statement.

Theorem 64.

1. $\alpha \in \mathscr{L}_{\mathsf{Bel}^{Ł^2}_{\triangle}}$ *is* $\mathsf{Bel}^{Ł}_{\triangle}$*-valid iff (a)* $e_1(\alpha^{\boxplus}) = 1$ *in every* $\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}$*-model and (b)* $e_1(\alpha^{\boxminus}) = 0$ *in every* $\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}$*-model.*
2. $\alpha \in \mathscr{L}_{\mathsf{Bel}^{\mathsf{NŁ}}}$ *is* $\mathsf{Bel}^{\mathsf{NŁ}}$*-valid iff* $\alpha^{\Box,\Diamond}$ *is* $\mathsf{Pr}^{\mathsf{NŁ}}_{\mathbf{S5}}$*-valid.*

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Proof. We begin with (1). Consider the 'only if' direction: we assume that either (i) $e_1(\alpha^{\boxplus}) = x < 1$ in some $\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}$-model or (ii) $e_1(\alpha^{\boxminus}) = y > 0$ in some $\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}$-model. We prove by induction that in this case, there are some $\mathsf{Bel}^{Ł^2}_{\triangle}$-models where $e'_1(\alpha) = x$ or $e'_2(\alpha) = y$, respectively.

Let $\alpha^{\boxplus} = \mathsf{Pr}(\Box\phi)$ and $e_1(\mathsf{Pr}(\Box\phi)) = x < 1$. This means that $\pi(|\mathsf{Pr}(\Box\phi)|^+) = x$. Using Remark 51, we obtain that there is a belief function $\mathtt{bel}$ s.t. $\mathtt{bel}(|\phi|^+) = x < 1$. Thus, $e^{\mathtt{bel}}_1(\mathsf{B}\phi) = x < 1$ for the evaluation $e^{\mathtt{bel}}_1$ induced by $\mathtt{bel}$, as required. The induction steps can be obtained by a simple application of the induction hypothesis since $\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}$ and $\mathsf{Bel}^{Ł^2}_{\triangle}$ use $Ł^2_{(\triangle,\to)}$ as their outer-layer logic. Thus, (i) is tackled.

For (ii), we proceed similarly. Let $\alpha^{\boxminus} = \mathsf{Pr}(\Box\neg\phi) = y > 0$. Again, we obtain that there is a belief function $\mathtt{bel}$ s.t. $\mathtt{bel}(|\neg\phi|^+) = x > 0$ (i.e., $\mathtt{bel}(|\phi|^-) = y > 0$). Hence, $e^{\mathtt{bel}}_2(\mathsf{B}\phi) = x > 0$ for the evaluation induced by $\mathtt{bel}$. The cases of propositional connectives can be tackled similarly, so, we only consider $\alpha^{\boxminus} = \gamma^{\boxminus} \ominus \beta^{\boxminus}$. Let $e_1(\gamma^{\boxminus} \ominus \beta^{\boxminus}) = y > 0$. Thus, $z{=}e_1(\gamma^{\boxminus}){>}e_1(\beta^{\boxminus}){=}z'$ with $z{-}z'{=}y$. Applying the induction hypothesis, we have that $e'_2(\gamma){=}z$, $e'_2(\beta){=}z'$, and $z{-}z'{=}y$. Hence, $e'_2(\beta \to \gamma) = y > 0$.

Let us now deal with the 'if' direction. Assume that α is not $\mathsf{Bel}^{Ł^2}_{\triangle}$-valid, $\mathsf{Bel}^{Ł^2}_{\triangle}$-model where $e_1(\alpha) = x < 1$ or (ii) there is a model where $e_2(\alpha) = y > 0$. We prove by induction that there is where $e_1(\alpha) = x < 1$ or (ii) there is a model where $e_2(\alpha) = y > 0$. We prove by induction that there is a $\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}$-model s.t. $e'_1(\alpha^{\boxplus}) = x < 1$ or $e'_1(\alpha^{\boxminus}) = y > 0$.

First, if $\alpha = \mathsf{B}\phi$ and $e_1(\mathsf{B}\phi) = x < 1$ in some $\mathsf{Bel}^{Ł^2}_{\triangle}$-model, then we have that there is a BD $\mathtt{bel}$-model s.t. $\mathtt{bel}(|\phi|^+) = x$. Hence, by Theorem 52, there is a BD probabilistic Kripke model s.t. $\pi(|\Box\phi|^+) = x$, and thus, $e^{\pi}_1(\mathsf{Pr}(\Box\phi)) = x < 1$ for the induced valuation e^{π}_1. In the second case, we have $\mathsf{Bel}^{Ł^2}_{\triangle}$-model. Thus, $\mathtt{bel}(|\phi|^-) = y$ which is equivalent to $\mathtt{bel}(|\neg\phi|^+) = x$. Again, applying Theorem 52, we obtain that there is a probabilistic Kripke model with $\pi(|\Box\neg\phi|^+) = y > 0$. Hence, $e^{\pi}_1(\mathsf{Pr}(\Box\neg\phi)) = y > 0$, as required. The cases of propositional connectives can be obtained by simple applications of the induction hypothesis as in the 'only if' direction. hypothesis as in the 'only if' direction.

(2) can be shown similarly. Assume that $e_1(\alpha^{\Box,\Diamond}) = x < 1$ in some $\mathsf{Pr}^{\mathsf{N}Ł}_{\mathbf{S5}}$-model. Applying Remark 51, one can show by induction that there is a $\mathsf{Bel}^{\mathsf{N}Ł}$-model with $e'_1(\alpha) = x$. Conversely, using Theorem 52 one obtains that if α *is not* $\mathsf{Bel}^{\mathsf{N}Ł}$-valid, then α is not $\mathsf{Pr}^{\mathsf{N}Ł}_{\mathbf{S5}}$-valid either. □

One can show that both translations are linear.

Lemma 65.

1. *Let* $\alpha \in \mathscr{L}_{\mathsf{Bel}^{Ł^2}_{\triangle}}$. *Then* $\ell(\alpha^{\boxplus}) = \mathcal{O}(\ell(\alpha))$ *and* $\ell(\alpha^{\boxminus}) = \mathcal{O}(\ell(\alpha))$.
2. *Let* $\beta \in \mathscr{L}_{\mathsf{Bel}^{\mathsf{N}Ł}}$. *Then* $\ell(\beta^{\Box,\Diamond}) = \mathcal{O}(\ell(\alpha))$.

Proof. Analogously to Lemma 26 □

5.3 Complexity

Theorem 64 and Lemma 65 give us a polynomial reduction from validity in $\mathsf{Bel}^{Ł^2}_{\triangle}$ and $\mathsf{Bel}^{\mathsf{N}Ł}$ to $\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}$ and $\mathsf{Pr}^{\mathsf{N}Ł}_{\mathbf{S5}}$. It is, thus, sufficient to prove that $\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}$-validity and $\mathsf{Pr}^{\mathsf{N}Ł}_{\mathbf{S5}}$-validity are coNP-complete. Recall that in the proof of Theorem 42, we used canonical models of BD built over

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the powerset of all literals occurring in a formula which we encoded via u_v variables. We define canonical models for BD with **S5** modalities.

Definition 66 (Clusters). *Let $\mathfrak{M} = \langle W, R, v^+, v^- \rangle$ be a* BD *Kripke model. A cluster is any subset $X \subseteq W$ closed under R. That is, if $w \in X$ and wRw', then $w' \in X$.*

Definition 67 (Bisimilarity). *Two* BD *Kripke models $\mathfrak{M} = \langle W, R, v^+, v^- \rangle$ and $\mathfrak{M}' = \langle W', R', v'^+, v'^- \rangle$ are called bisimilar (denoted $\mathfrak{M} \simeq \mathfrak{M}'$) if there is a relation $\mathscr{Z} \subseteq W \times W'$ s.t. for every $p \in \mathtt{Prop}$, $x \in W$, and $x' \in W$, the following conditions hold:*

1. *$\mathfrak{M}, x \vDash^+ p$ iff $\mathfrak{M}', x' \vDash^+ p$ and $\mathfrak{M}, x \vDash^- p$ iff $\mathfrak{M}', x' \vDash^- p$ for every $x \in W$ and $x' \in W'$ s.t. $x\mathscr{Z}x'$;*
2. *if wRx and $w\mathscr{Z}w'$, then there is $x' \in W'$ s.t. $w'R'x'$ and $x\mathscr{Z}x'$;*
3. *if $w'R'x'$ and $w\mathscr{Z}w'$, then there is $x \in X$ s.t. wRx and $x\mathscr{Z}x'$.*

Bisimilarity of Kripke models with two relations are defined similarly but conditions (2) and (3) have to hold for both relations.

The following statement is straightforward to establish.

Proposition 68. *Let $\mathfrak{M} \simeq \mathfrak{M}'$, $\mathscr{Z}$ be the bisimilarity relation, and $w\mathscr{Z}w'$. Then for every σ over $\{\neg, \wedge, \vee, \Box, \Diamond\}$, it follows that*

$$\mathfrak{M}, w \vDash^+ \sigma \text{ iff } \mathfrak{M}', w' \vDash^+ \sigma \qquad\qquad \mathfrak{M}, w \vDash^- \sigma \text{ iff } \mathfrak{M}', w' \vDash^- \sigma$$

Definition 69 (Canonical model for BD with **S5** $\Box$). *Let $\Gamma = \{\Box\phi_1, \ldots \Box\phi_m\}$ and $\mathsf{Lit}[\Gamma] = \{l_1, \ldots, l_n\}$. The canonical model of Γ is the disjoint union of all pairwise non-bisimilar clusters on $2^{\mathsf{Lit}[\Gamma]}$, that is the following structure:*

$$\mathfrak{M}^{\mathsf{BDS5}}_{\Gamma} = \biguplus_{\mathsf{S} \subseteq 2^{\mathsf{Lit}[\Gamma]}} \langle \mathsf{S}, \mathsf{S} \times \mathsf{S}, v^+, v^- \rangle \qquad\qquad (\mathsf{S} \neq \varnothing)$$

with valuations v^+ and v^- defined as below:

$$w \in v^+(p) \text{ iff } p \in w \qquad\qquad w \in v^-(p) \text{ iff } \neg p \in w$$

Importantly, as the next statements show, we can prove a version of the small model property for probabilities of $\Box\phi$ and $\Diamond\phi$ formulas where $\phi \in \mathscr{L}_{\mathsf{BD}}$. Namely, we show that in arbitrary models, we only need clusters whose size is linearly bounded by the number of formulas and that we can assume the measure to be positive only on clusters with linearly bounded size in $\mathfrak{M}^{\mathsf{BDS5}}$.

Lemma 70 (Small model property). *Let $\mathfrak{M} = \langle W, R, v^+, v^-, \pi \rangle$ be a* BD *probabilistic Kripke model, $\phi_1, \ldots, \phi_m \subseteq \mathscr{L}_{\mathsf{BD}}$, $1 \leq k \leq m$ and $\Gamma = \{\Box\phi_i \mid i \leq k\} \cup \{\Diamond\phi_i \mid k < i \leq m\}$. Then there is a model $\mathfrak{M}_{\mathsf{S}} = \langle W_{\mathsf{S}}, R_{\mathsf{S}}, v^+_{\mathsf{S}}, v^-_{\mathsf{S}}, \pi_{\mathsf{S}} \rangle$ s.t. $W_{\mathsf{S}} \subseteq W$, $R_{\mathsf{S}} = R_{|W_{\mathsf{S}}}$, v^+_{S} and v^-_{S} are restrictions of v^+ and v^- on W_{S}, every cluster of $\mathfrak{M}_{\mathsf{S}}$ contains at most $2m + 1$ states, and for every $\sigma \in \Gamma$, it holds that*

$$\pi(|\sigma|^+) = \pi_{\mathsf{S}}(|\sigma|^+) \qquad\qquad \pi(|\sigma|^-) = \pi_{\mathsf{S}}(|\sigma|^-)$$

Proof. Observe that $\mathfrak{M}$ is a disjoint union of all its clusters and that for every $\sigma \in \Gamma$ and every cluster $X \subseteq W$ the following two statements hold:

$$\text{either } \forall x \in X : \mathfrak{M}, x, \vDash^+ \sigma \text{ or } \forall x \in X : \mathfrak{M}, x, \nvDash^+ \sigma;$$

$$\text{either } \forall x \in X : \mathfrak{M}, x, \vDash^- \sigma \text{ or } \forall x \in X : \mathfrak{M}, x, \nvDash^- \sigma.$$

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We show how to remove 'redundant' *states* from clusters containing more than $2m+1$ states and redefine the measure accordingly.

Now let X be a cluster in W with more than $2m+1$ states. Let further, $\Phi^{\forall}_X, \Phi^{\exists}_X \subseteq \Gamma$ be such that

$$\Phi^{\forall}_X = \{\Box\phi \mid \forall x \in X : \mathfrak{M}, x \vDash^{+} \Box\phi\} \cup \{\Diamond\phi \mid \forall x \in X : \mathfrak{M}, x \vDash^{-} \Diamond\phi\}$$
$$\Phi^{\exists}_X = \{\Box\phi \mid \forall x \in X : \mathfrak{M}, x \vDash^{-} \Box\phi\} \cup \{\Diamond\phi \mid \forall x \in X : \mathfrak{M}, x \vDash^{+} \Diamond\phi\}$$

It is clear from the semantics of BD with **S5** modalities (recall Definition 50) that no matter which and how many states we remove from X

1. all $\Box$-formulas from $\Phi^{\forall}_X$ will remain true in every state non-removed of X while all $\Box$-formulas from $\Gamma \setminus \Phi^{\exists}_X$ will remain non-false in every non-removed state of X;
2. dually, all $\Diamond$-formulas from $\Phi^{\forall}_X$ will remain false in every state non-removed of X while all $\Diamond$-formulas from $\Gamma \setminus \Phi^{\exists}_X$ will remain non-true in every non-removed state of X.

Let us now show how to construct a cluster X_s containing at most $2m+1$ states s.t. $\Phi^{\forall}_X = \Phi^{\forall}_{X_s}$ and $\Phi^{\exists}_X = \Phi^{\exists}_{X_s}$.

First, we take any state $x \in X$. Clearly, all $\Box$- and $\Diamond$-formulas from $\Phi^{\forall}_X$ will remain true (or, respectively, false) in x. Likewise, $\Box$-formulas from $\Gamma \setminus \Phi^{\exists}_X$ will remain non-false in x and $\Diamond$-formulas from $\Gamma \setminus \Phi^{\exists}_X$ will remain non-true.

Now let $\Box\phi \in \Phi^{\exists}_X$ be a formula s.t. it was *false* in X but became non-false in x. This means that there was some $x_{\Box\phi} \in X$ s.t. $\mathfrak{M}, x_{\Box\phi} \vDash^{-} \phi$ but $x_{\Box\phi} \neq x$. We set $X_{\Box\phi} = \{x, x_{\Box\phi}\}$. It is clear that $\Box\phi$ is false in $X_{\Box\phi}$ but all other formulas remained true or non-false because they were true or non-false in X and $X_{\Box\phi} \subseteq X$. Now we take the next formula, say, $\Box\phi' \in \Phi^{\exists}_X$ that was false in X but became non-false in x. If it is still non-false in $X_{\Box\phi}$, it means that there is some $x_{\Box\phi'} \in X \setminus X_{\Box\phi}$ s.t. $\mathfrak{M}, x_{\Box\phi'} \vDash^{-} \phi'$. Thus, we set $X_{\Box\phi'} = X_{\Box\phi} \cup \{x_{\Box\phi'}\}$. We repeat this procedure until we tackle all $\Box$-formulas from $\Phi^{\exists}_X$ that became non-false in x. Then we do the same with $\Diamond$-formulas from $\Phi^{\exists}_X$ that became *non-true* in x. Since there are at most m formulas in $\Phi^{\exists}_X$, we will have added at most m states to $\{x\}$. Denote the resulting cluster X_{-}.

After that, we do the same with $\Box$-formulas from $\Gamma \setminus \Phi^{\forall}_X$ that became true in x and $\Diamond$-formulas from $\Gamma \setminus \Phi^{\forall}_X$ that became false in x. This will also add at most m states to X_{-}. The resulting cluster which we call X_s will, thus, contain at most $2m+1$ states. Moreover, we have that $\Phi^{\forall}_X = \Phi^{\forall}_{X_s}$ and $\Phi^{\exists}_X = \Phi^{\exists}_{X_s}$, as required.

It remains to define π_s on X_s. As $X_s \subseteq X$, let $X_s = X' \cup \{x''\}$ and define

$$\pi_s(\{x'\}) = \begin{cases} \pi(\{x'\}) & \text{if } x' \in X' \\ \sum\limits_{y \in X \setminus X_s} \pi(\{y\}) & \text{if } x' = x'' \end{cases}$$

We repeat this procedure with every cluster. It is clear that the obtained model will consist of clusters containing at most $2m+1$ states and that $\pi(|\sigma|^{+}) = \pi_s(|\sigma|^{+})$ and $\pi(|\sigma|^{-}) = \pi_s(|\sigma|^{-})$ for every $\sigma \in \Gamma$, it holds that

$$\pi(|\sigma|^{+}) = \pi_s(|\sigma|^{+}) \qquad \pi(|\sigma|^{-}) = \pi_s(|\sigma|^{-})$$ □

The next statement shows that any measure assignment to formulas of the form $\Box\phi$ and $\Diamond\phi$ on a given BD probabilistic Kripke model can be transferred to a measure assignment on $\mathfrak{M}^{\mathsf{BDS5}}$.

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Theorem 71.

1. *Let* $\phi_1, \ldots, \phi_m \in \mathscr{L}_{\mathsf{BD}}$, $\Gamma = \{\Box\phi_1, \ldots \Box\phi_l, \Diamond\phi_{l+1}, \ldots, \Diamond\phi_m\}$, *and* $\mathfrak{M} = \langle W, R, v^+, v^-, \pi\rangle$ *be a probabilistic Kripke model. Then there is a probability measure* π^{C} *on* $\mathfrak{M}^{\mathsf{BDS5}}_\Gamma$ *s.t.*
   (a) $\pi^{\mathsf{C}}(|\sigma|^+) = \pi(|\sigma|^+)$ *and* $\pi^{\mathsf{C}}(|\sigma|^-) = \pi(|\sigma|^-)$ *for every* $\sigma \in \Gamma$;
   (b) *if* X *is a cluster in* $\mathfrak{M}^{\mathsf{BDS5}}_\Gamma$ *s.t.* $|X| > 2m + 1$, *then* $\pi^{\mathsf{C}}(X) = 0$.
2. *Let* $\{\phi_i \mid i \leq k\} \cup \{\chi_j \mid j \leq l\} \subseteq \mathscr{L}_{\mathsf{BD}}$, $k + l = m$, $k' \leq k$, $l' \leq l$, $\Gamma_1 = \{\Box_1\phi_i \mid i \leq k'\} \cup \{\Diamond_1\phi_i \mid k' < i \leq k\}$, $\Gamma_2 = \{\Box_2\chi_j \mid j \leq l'\} \cup \{\Diamond_2\chi_j \mid l' < j \leq l\}$, $\Gamma = \Gamma_1 \cup \Gamma_2$, *and* $\mathfrak{M} = \langle W, R_1, R_2, v^+, v^-, \pi_1, \pi_2\rangle$ *be a probabilistic Kripke model. Then there are probability measures* π^{C}_1 *and* π^{C}_2 *on* $\mathfrak{M}^{\mathsf{BDS5}}_\Gamma$ *s.t.*
   (a) $\pi^{\mathsf{C}}_1(|\sigma|^+) = \pi_1(|\sigma|^+)$, $\pi^{\mathsf{C}}_1(|\sigma|^-) = \pi_1(|\sigma|^-)$, $\pi^{\mathsf{C}}_2(|\tau|^+) = \pi_2(|\tau|^+)$, *and* $\pi^{\mathsf{C}}_2(|\tau|^-) = \pi_2(|\tau|^-)$ *for every* $\sigma \in \Gamma_1$ *and* $\tau \in \Gamma_2$;
   (b) *if* X *is a cluster in* $\mathfrak{M}^{\mathsf{BDS5}}_\Gamma$ *s.t.* $|X| > 2m + 1$, *then* $\pi^{\mathsf{C}}_1(X) = \pi^{\mathsf{C}}_2(X) = 0$.

Proof. We begin with (1). As there are only m formulas, it is clear that $\mathsf{Lit}[\Gamma]$ is finite. We let $|\mathsf{Lit}[\Gamma]| = k$. Thus, $\mathfrak{M}$ contains at most $\sum\limits_{j=1}^{2^k}\binom{2^k}{j} = 2^{2^k} - 1$ *pairwise non-bisimilar* clusters (recall Definition 66 for the notion of cluster). In addition to that, using Lemma 70, we can w.l.o.g. assume that these clusters contain at most $2m + 1$ states. Moreover, W is the disjoint union of *all* clusters, whence, $\sum\limits_{X \text{ is a cluster}} \pi(X) = 1$, and furthermore, for every cluster $X \subseteq \mathfrak{M}$, there is a cluster $X^{\mathsf{C}} \subseteq \mathfrak{M}^{\mathsf{BDS5}}_\Gamma$ s.t. $X \simeq X^{\mathsf{C}}$. From here, it is clear that

$$\mathfrak{M}, w \models^+ \sigma \text{ iff } \mathfrak{M}^{\mathsf{BDS5}}_\Gamma, w^{\mathsf{C}} \models^+ \sigma \qquad \mathfrak{M}, w \models^- \sigma \text{ iff } \mathfrak{M}^{\mathsf{BDS5}}_\Gamma, w^{\mathsf{C}} \models^- \sigma$$

holds for every σ when $w\mathscr{Z}w^{\mathsf{C}}$.

Now, recall from Definition 69 that $\mathfrak{M}^{\mathsf{BDS5}}_\Gamma$ is the disjoint union of all pairwise non-bisimilar clusters. Thus, we can define π^{C} as follows:

$$\pi^{\mathsf{C}}(\{w^{\mathsf{C}}\}) = \sum_{\substack{w\mathscr{Z}w^{\mathsf{C}} \\ w \in W}} \pi(\{w\})$$

It is now easy to see that $\pi^{\mathsf{C}}(|\sigma|^+) = \pi(|\sigma|^+)$ and $\pi^{\mathsf{C}}(|\sigma|^-) = \pi(|\sigma|^-)$ for every $\sigma \in \Gamma$, as required. Moreover, since $\mathfrak{M}$ w.l.o.g. contains only clusters with at most $2m + 1$ states, no cluster in $\mathfrak{M}^{\mathsf{BDS5}}_\Gamma$ with a greater number of states has a positive measure assignment under π^{C}.

The proof of (2) is similar. First, we take the *mono-relational reducts* of $\mathfrak{M}$, that is, the following two models: $\mathfrak{M}_{R_1} = \langle W, R_1, v^+, v^-, \pi\rangle$ and $\mathfrak{M}_{R_2} = \langle W, R_2, v^+, v^-, \pi\rangle$. Since every formula in Γ contains only one modality and thus depends only on R_1 or only on R_2, it is clear that the following equivalences hold for every $w \in W$, $\sigma \in \Gamma_1$, and $\tau \in \Gamma_2$:

$$\mathfrak{M}, w \models^+ \sigma \text{ iff } \mathfrak{M}_{R_1}, w \models^+ \sigma \qquad \mathfrak{M}, w \models^- \sigma \text{ iff } \mathfrak{M}_{R_1}, w \models^- \sigma$$
$$\mathfrak{M}, w \models^+ \tau \text{ iff } \mathfrak{M}_{R_2}, w \models^+ \tau \qquad \mathfrak{M}, w \models^- \tau \text{ iff } \mathfrak{M}_{R_2}, w \models^- \tau$$

Now, it is clear that $\mathfrak{M}_{R_1}$ and $\mathfrak{M}_{R_2}$ are disjoint unions of clusters (w.r.t. R_1 and R_2, respectively). We can again use Lemma 70 and assume w.l.o.g. that clusters w.r.t. R_1 and R_2 contain at most $2m + 1$ states. Hence:

(a) for every cluster $X \subseteq \mathfrak{M}_{R_1}$, there is a cluster $X^{\mathsf{C}} \subseteq \mathfrak{M}^{\mathsf{BDS5}}_\Gamma$ s.t. $X \simeq X^{\mathsf{C}}$;

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(b) for every cluster $Y \subseteq \mathfrak{M}_{R_2}$, there is a cluster $Y^{\mathsf{C}} \subseteq \mathfrak{M}^{\mathsf{BDS5}}_{\Gamma}$ s.t. $Y \simeq Y^{\mathsf{C}}$.

We use $\mathscr{Z}_1$ and $\mathscr{Z}_2$ to denote bisimilarity relations that witness (a) and (b), respectively. Now, we define π_1^{C} and π_2^{C} as below.

$$\pi_1^{\mathsf{C}}(\{w^{\mathsf{C}}\}) = \sum_{\substack{w\mathscr{Z}_1 w^{\mathsf{C}} \\ w\in W}} \pi_1(\{w\}) \qquad\qquad \pi_2^{\mathsf{C}}(\{w^{\mathsf{C}}\}) = \sum_{\substack{w\mathscr{Z}_2 w^{\mathsf{C}} \\ w\in W}} \pi_2(\{w\})$$

Again, it is straightforward to check that $\pi_1^{\mathsf{C}}(|\sigma|^+) = \pi_1(|\sigma|^+)$, $\pi_1^{\mathsf{C}}(|\sigma|^-) = \pi_1(|\sigma|^-)$, $\pi_2^{\mathsf{C}}(|\tau|^+) = \pi_2(|\tau|^+)$, and $\pi_2^{\mathsf{C}}(|\tau|^-) = \pi_2(|\tau|^-)$ for every $\sigma \in \Gamma_1$ and $\tau \in \Gamma_2$. □

The decision algorithms for $\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}$ and $\mathsf{Pr}^{\mathsf{NŁ}}_{\mathbf{S5}}$ are based upon the one presented in Theorem 42. Thus, we need a tableaux calculus for NŁ. It was given by Bílková et al. (2021), and we recall it here. All notions – open and closed branches, interpretations of entries, and satisfying valuations of branches – are the same as in $\mathscr{T}\left(\text{Ł}^2_{(\triangle,\to)}\right)$ (cf. Definitions 34 and 36), so we only list the rules in the next definition.

Definition 72 ($\mathscr{T}$(NŁ) – the tableau calculus for NŁ). *The rules are as follows.*

$$\neg{\leqslant_1}\dfrac{\neg\phi\leqslant_1 i}{\phi\leqslant_2 i}\qquad \neg{\leqslant_2}\dfrac{\neg\phi\leqslant_2 i}{\phi\leqslant_1 i}\qquad \neg{\geqslant_1}\dfrac{\neg\phi\geqslant_1 i}{\phi\geqslant_2 i}\qquad \neg{\geqslant_2}\dfrac{\neg\phi\geqslant_2 i}{\phi\geqslant_1 i}$$

$$\sim_{\mathsf{N}}{\leqslant_1}\dfrac{\sim_{\mathsf{N}}\phi\leqslant_1 i}{\phi\geqslant_1 1-i}\qquad \sim_{\mathsf{N}}{\leqslant_2}\dfrac{\sim_{\mathsf{N}}\phi\leqslant_2 i}{\phi\leqslant_1 i}\qquad \sim_{\mathsf{N}}{\geqslant_1}\dfrac{\sim_{\mathsf{N}}\phi\geqslant_1 i}{\phi\leqslant_1 1-i}\qquad \sim_{\mathsf{N}}{\geqslant_2}\dfrac{\sim_{\mathsf{N}}\phi\geqslant_2 i}{\phi\geqslant_2 i}$$

$$\rightharpoonup{\leqslant_1}\dfrac{\phi_1\rightharpoonup\phi_2\leqslant_1 i}{i\geq 1\;\left|\;\begin{array}{c}\phi_1\geqslant_1 1-i+j\\ \phi_2\leqslant_1 j\\ j\leq i\end{array}\right.}\qquad \rightharpoonup{\leqslant_2}\dfrac{\phi_1\rightharpoonup\phi_2\leqslant_2 i}{\begin{array}{c}\phi_1\leqslant_2 i+j\\ \phi_2\leqslant_1 1-j\end{array}}\qquad \rightharpoonup{\geqslant_1}\dfrac{\phi_1\rightharpoonup\phi_2\geqslant_1 i}{\begin{array}{c}\phi_1\leqslant_1 1-i+j\\ \phi_2\geqslant_1 j\end{array}}\qquad \rightharpoonup{\geqslant_2}\dfrac{\phi_1\rightharpoonup\phi_2\geqslant_2 i}{i\leq 0\;\left|\;\begin{array}{c}\phi_1\geqslant_2 i+j\\ \phi_2\geqslant_1 1-j\\ j\leq 1-i\end{array}\right.}$$

$$\wedge{\leqslant_1}\dfrac{\phi_1\wedge\phi_2\leqslant_1 i}{\phi_1\leqslant_1 i\mid\phi_2\leqslant_1 i}\qquad \wedge{\leqslant_2}\dfrac{\phi_1\wedge\phi_2\leqslant_2 i}{\begin{array}{c}\phi_1\leqslant_2 i\\ \phi_2\leqslant_2 i\end{array}}\qquad \wedge{\geqslant_1}\dfrac{\phi_1\wedge\phi_2\geqslant_1 i}{\begin{array}{c}\phi_1\geqslant_1 i\\ \phi_2\geqslant_1 i\end{array}}\qquad \wedge{\geqslant_2}\dfrac{\phi_1\wedge\phi_2\geqslant_2 i}{\phi_1\geqslant_2 i\mid\phi_2\geqslant_2 i}$$

Remark 73. *Note that all connectives in* NŁ *are continuous and thus all rules of* $\mathscr{T}$(NŁ) *can be linearised. This will require the introduction of another sort of variables that range over* $\{0, 1\}$. *To make the presentation of both tableaux calculi uniform, however, we decided against the linear version of the tableau rules.*

Theorem 74. *Validity in* $\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}$ *and* $\mathsf{Pr}^{\mathsf{NŁ}}_{\mathbf{S5}}$ *is* coNP*-complete.*

Proof. As $\text{Ł}^2_{(\triangle,\to)}$ and NŁ are coNP-complete (cf. Bílková et al. 2021), we only show the upper bound. The proof is similar to that of Theorem 42, so we address the main differences.

We begin with $\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}$. We show how to check the falsifiability of $\alpha \in \mathscr{L}_{\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}}$ in nondeterministic polynomial time. Assume that α is built over $\mathsf{Pr}\sigma_1, \ldots, \mathsf{Pr}\sigma_n$ and that $|\mathsf{Lit}(\alpha)| = n$ (just as in Theorem 42, we can add superfluous modal atoms) and that α is outer-¬-free (this holds because $e(\neg\mathsf{Pr}\Box\phi) = e(\mathsf{Pr}\Diamond\neg\phi)$ and $e(\neg\mathsf{Pr}\Diamond\phi) = e(\mathsf{Pr}\Box\neg\phi)$ in every model).

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We construct a $\mathscr{T}\left(Ł^2_{(\triangle,\to)}\right)$ constraint tableau beginning with $\{\alpha^- \leqslant_1 c, c < 1\}$ where α^- is the result of replacement of every modal atom $\mathsf{Pr}\sigma_i$ with a fresh variable q_{σ_i} and check in nondeterministic polynomial time whether it is closed. If it has open branches, we guess one (say, $\mathscr{B}$) and consider its corresponding system of linear inequalities.

$$z_1 \triangledown t_1, \ldots, z_n \triangledown t_n, k_1 \le k'_1, \ldots, k_r \le k'_r, m_1 \ge 1, \ldots, m_s \ge 1, m'_1 \le 0, \ldots, m'_t \le 0 \qquad (\mathrm{LI}(1)^{\mathscr{B}}_{\Box})$$

Here, z_i's correspond to the values of q_{σ_i}'s in α^- and t_i's are linear polynomials that label q_{σ_i}'s. Numerical constraints give us k's, k''s, m's and m''s. Denote the number of inequalities and the number of variables in $(\mathrm{LI}(1)^{\mathscr{B}}_{\Box})$ with l_1 and l_2, respectively. It is clear that $l_1 = \mathscr{O}(\ell(\alpha^-))$ and $l_2 = \mathscr{O}(\ell(\alpha^-))$.

To check that z_i's are coherent as probabilities of σ_i's, we introduce $2^{2^n} - 1$ new variables of the form u_v with $v \in \{0, 1, ;\}^*$. Here, v encodes the valuation of literals of α in the corresponding cluster of $\mathfrak{M}^{\mathsf{BDS5}}_{\alpha}$ (cf. Definitions 66 and 69). For example, if $\mathsf{Lit}(\alpha) = \{p_1, \neg p_1, p_2, \neg p_2\}$, $u_{1000;1100;0101}$ encodes $\{\{p_1\}, \{p_1, \neg p_1\}, \{\neg p_1, \neg p_2\}\}$. Additionally, we put $a_{i,v} = 1$ if $\mathfrak{M}^{\mathsf{BDS5}}_{\alpha}, w \vDash^+ \sigma_i$ for every state w in the cluster corresponding to v and $a_{i,v} = 0$, otherwise. We then add the following equalities for every i.

$$\sum_v u_v = 1 \qquad \sum_v (a_{i,v} \cdot u_v) = z_i \qquad (\mathrm{LI}(2\exp)^{\mathscr{B}}_{\Box})$$

As in Theorem 42, we guess a list L of at most $l_1 + n + 1$ words v. By Theorem 71, we w.l.o.g. assume that only clusters with at most $2n + 1$ states have a positive measure assignment. Thus, L contains only u_v's whose length is not greater than $n \cdot (2n + 1)$, whence, the size of L is not greater than $n \cdot (2n + 1) \cdot (l_1 + n + 1)$. We compute the values of $a_{i,v}$'s for $i \le n$ and $v \in L$ which takes deterministic polynomial time w.r.t. $\ell(\alpha^-)$ because clusters contain at most $2n + 1$ states. This gives us the following system of inequalities.

$$\sum_{v \in L} u_v = 1 \qquad \sum_{v \in L} (a_{i,v} \cdot u_v) = z_i \qquad (\mathrm{LI}(2\mathrm{poly})^{\mathscr{B}}_{\Box})$$

One can see that the size of $(\mathrm{LI}(1)^{\mathscr{B}}_{\Box}) \cup (\mathrm{LI}(2\mathrm{poly})^{\mathscr{B}}_{\Box})$ is polynomial in $\ell(\alpha^-)$ as there are polynomially many (in)equalities containing polynomially many variables with linearly bounded labels. Hence, it can be solved in nondeterministic polynomial time. The rest of the proof follows that of Theorem 42.

The coNP-membership proof for $\mathsf{Pr}^{\mathsf{NŁ}}_{\mathbf{S5}}$ is similar, so, we provide only a sketch. The main differences are as follows. First, there are two modalities – Pr_1 and Pr_2; this is why, when transforming α into α^-, we replace $\mathsf{Pr}_1\sigma_i$'s with $q_{\sigma_{1_i}}$'s and $\mathsf{Pr}_2\sigma_i$'s with $q_{\sigma_{2_i}}$'s. Thus, we introduce two sorts of z's (z_{1_i}'s and z_{2_i}'s) in the system of inequalities corresponding to an open branch in the $\mathscr{T}(\mathsf{NŁ})$ tableau for α^-. Note, however, that since $\Box_1$ and $\Diamond_1$ do not occur in the same modal atom as $\Box_2$ or $\Diamond_2$, we can utilise $\mathfrak{M}^{\mathsf{BDS5}}_{\alpha^-}$ for both sets of modalities. However (recall Theorem 71), we will need two independent measures.

Second, we need two sorts of u_v's – u^1_v and u^2_v (v's are defined as in the case of $\mathsf{Pr}^{(\triangle,\to)}_{\mathbf{S5}}$) – to differentiate between two measures on $\mathfrak{M}^{\mathsf{BDS5}}_{\alpha}$. This means that $(\mathrm{LI}(2\exp)^{\mathscr{B}}_{\Box})$ will have the following form:

$$\sum_v u^1_v = 1 \qquad \sum_v (a_{i_1,v} \cdot u^1_v) = z_{i_1} \qquad \sum_v u^2_v = 1 \qquad \sum_v (a_{i_2,v} \cdot u^2_v) = z_{i_2}$$

Thus, we have $l_1 + 2n + 2$ inequalities in total. Hence, we can guess the list L containing $l_1 + 2n + 2$ variables. By Theorem 71, we need only v's of length at most $n \cdot (2n + 1)$. Therefore, the size of L is at most $n \cdot (2n + 1) \cdot (l_1 + 2n + 2)$, which means that it takes nondeterministic polynomial time to solve the system of inequalities obtained after guessing L. The result is as follows. □

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Corollary 75. *Validity in* $\mathsf{Bel}^{\text{Ł}^2}_{\triangle}$ *and* $\mathsf{Bel}^{\mathsf{NŁ}}$ *is* coNP*-complete.*

Proof. Immediately from Lemma 59 and Theorems 64 and 74. □

6. Conclusion

In this paper, we considered several two-layered logics for reasoning with probability measures and belief functions defined in the BD-framework. In particular, we established that the logics of **4**- and ±-probabilities can be faithfully embedded into one another (Theorems 24 and 25) and constructed a complete Hilbert axiomatisation for the logic of **4**-probabilities (Theorem 33). We also established coNP-completeness for logics of ±- and **4**-probabilities and coNP-completeness for logics of belief and plausibility functions in BD (Theorem 42 and Corollary 75) utilising the connection between belief assignments of propositional and modal formulas (Theorem 52) and a version of the small model property for the canonical model (Theorem 71). Still, several important questions remain open.

First, our belief and plausibility functions are close to ±-probabilities because they assign each statement ϕ two measures: the measure of its positive extension and the measure of its negative extension. It is thus instructive to define **4**-valued belief and plausibility functions that will assign measures to ϕ according to the pure belief, pure disbelief, conflict and uncertainty extensions (cf. Convention 5). This is not a trivial task since belief and plausibility functions are not additive, whence, the measures of these extensions cannot be directly obtained from $\mathtt{bel}(|\phi|^+)$ and $\mathtt{bel}(|\phi|^-)$.

Third, observe that axiomatisations of $\mathsf{Bel}^{\text{Ł}^2}_{\triangle}$ proposed by Bílková et al. (2023d) are *infinite*. In the case of belief functions over the classical logic, it is shown by Godo et al. (2001, 2003) how to produce a *finite* axiomatisation for the two-layered logics of belief functions in the classical case using the representation of $\mathtt{bel}(\|\phi\|)$ as $\pi(\|\Box\phi\|)$. Recall that a similar property (Theorem 52) holds for the BD as well. It makes sense to use it to obtain a finite axiomatisation of belief and plausibility functions over BD.

Finally, Bílková et al. (2023a) proposed two-layered logics for *qualitative reasoning* about different uncertainty measures. In particular, we constructed logics for qualitative reasoning with belief functions and probabilities. These logics utilised the bi-Gödel logic (biG) in the outer layer. Since biG is also NP-complete, the question arises whether we can apply the technique of Hájek and Tulipani (2001) to prove the NP-completeness of two-layered logics for qualitative reasoning about uncertainty.

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