Canad. Math. Bull. Vol. **51** (2), 2008 pp. 249–260

On the Inner Radius of a Nodal Domain

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Abstract. Let M be a closed Riemannian manifold. We consider the inner radius of a nodal domain for a large eigenvalue λ. We give upper and lower bounds on the inner radius of the type $C/\lambda^{\alpha}(\log\lambda)^{\beta}$. Our proof is based on a local behavior of eigenfunctions discovered by Donnelly and Fefferman and a Poincaré type inequality proved by Maz'ya. Sharp lower bounds are known only in dimension two. We give an account of this case too.

1 Introduction and Main Results

Let (M, g) be a closed Riemannian manifold of dimension n. Let Δ be the Laplace–Beltrami operator on M. Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues of Δ. Let φ_λ be an eigenfunction of Δ with eigenvalue λ. A *nodal domain* is a connected component of $\{\varphi_\lambda \neq 0\}$.

We are interested in the asymptotic geometry of the nodal domains. In particular, in this paper we consider the inner radius of nodal domains.

Let r_λ be the inner radius of the λ-nodal domain U_λ. Let $C_1, C_2, \ldots$ denote constants which depend only on (M, g). We prove

Theorem 1.1 *Let M be a closed Riemannian manifold of dimension $n \geq 3$. Then*

$$\frac{C_1}{\sqrt{\lambda}} \geq r_\lambda \geq \frac{C_2}{\lambda^{k(n)}(\log\lambda)^{2n-4}},$$

where $k(n) = n^2 - 15n/8 + 1/4$.

In dimension two we have the following sharp bound.

Theorem 1.2 *Let Σ be a closed Riemannian surface. Then*

$$\frac{C_3}{\sqrt{\lambda}} \geq r_\lambda \geq \frac{C_4}{\sqrt{\lambda}}.$$

1.1 Upper Bound

We remark that the upper bound is more or less standard and has been used in the literature [8]. However, we explain it here also.

We observe that $\lambda = \lambda_1(U_\lambda)$. This is true since the λ-eigenfunction does not vanish in U_λ [7, Ch. I.5]. Therefore, the existence of the upper bound in Theorems 1.1 and 1.2 follows from the following general upper bound on λ_1 of domains $\Omega \subseteq M$.

Received by the editors January 16, 2006.
AMS subject classification: Primary: 58J50; secondary: 35P15, 35P20.
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https://doi.org/10.4153/CMB-2008-026-2 Published online by Cambridge University Press

Theorem 1.3

$$\lambda_1(\Omega) \le \frac{C_5}{\mathrm{inrad}(\Omega)^2}.$$

The proof of this theorem is given in §5.

1.2 Lower Bound

For the lower bound on the inner radius in dimensions ≥ 3, we give a proof in §2.1 which is based on a local behavior of eigenfunctions discovered by H. Donnelly and C. Fefferman (Theorem 2.1). The same proof gives in dimension two the bound $C/\sqrt{\lambda \log \lambda}$.

In order to get rid of the factor $\sqrt{\log \lambda}$ in dimension two, we treat this case separately in §2.2. The proof for this case can basically be found in [10], and we bring it here for the sake of clarity and completeness.

For the dimension two case we also bring a new proof in §3. Moreover, this proof shows that a big inscribed ball can be taken to be with center at a maximal point of the eigenfunction in the nodal domain. This proof is due to F. Nazarov, L. Polterovich and M. Sodin and is based on complex analytic methods.

1.3 A Short Background

Related to the problem discussed in this paper is the problem of estimating the $(n-1)$-Hausdorff measure $H_{n-1}(\lambda)$ of the nodal set, *i.e.*, the set where an eigenfunction vanishes. J. Brüning and D. Gromes [3, 4] proved sharp lower estimates in dimension two. Namely, they showed $H_1(\lambda) \ge C\sqrt{\lambda}$. An estimate of the constant C is given in [21]. Later, S. T. Yau conjectured that in any dimension $C_1\sqrt{\lambda} \ge H_{n-1}(\lambda) \ge C_2\sqrt{\lambda}$. This was proved in the case of analytic metrics by H. Donnelly and C. Fefferman [8].

Regarding the inner radius of nodal domains, we would like to mention the recent work of B. Xu [23], in which he obtains a sharp lower bound on the inner radius for at least two nodal domains, and the work of V. Maz'ya and M. Shubin [17], in which they give sharp bounds on the inner capacity radius of a nodal domain.

2 The Lower Bound on the Inner Radius

In this section we prove the existence of the lower bounds on the inner radius given in Theorems 1.1 and 1.2.

2.1 Lower Bound in Dimension $\ge$ Three

In this section we prove the existence of the lower bound in Theorem 1.1. The proof also gives a bound in the case where $\dim M = 2$, namely $r_\lambda \ge C/\sqrt{\lambda \log \lambda}$, but in the next section we treat this case separately to get rid of the $\sqrt{\log \lambda}$ factor.

Let $\{\sigma_i\}$ be a finite cellulation of M by cubes, such that for each i we can put a Euclidean metric e_i on σ_i, which satisfies $e_i/4 \le g \le 4e_i$. Let $r_{\lambda,i}$ be the inner radius

https://doi.org/10.4153/CMB-2008-026-2 Published online by Cambridge University Press

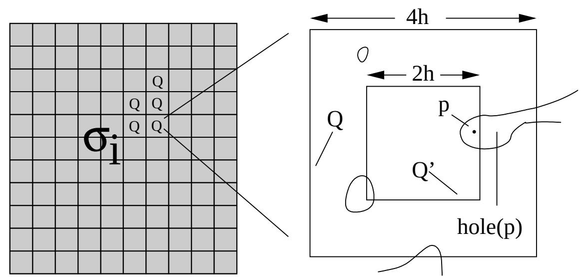

Figure 1: Proof of lower bound on inner radius.

of $U_{\lambda,i} = U_\lambda \cap \sigma_i$, and $r_{\lambda,i,e}$ the Euclidean inner radius of $U_{\lambda,i}$. Notice that

$$r_{\lambda,i,e} \leq 2r_{\lambda,i} \leq 2r_\lambda.$$

Step 1. (See Figure 1). We consider σ_i as a compact cube in $\mathbb{R}^n$, with edges parallel to the axes' directions. We cover σ_i by non-overlapping small cubes with edges of size $4h$, where $r_{\lambda,i,e} < h < 2r_{\lambda,i,e}$. Let Q be a copy of one of these small cubes. Let Q' be a concentric cube with parallel edges of size $2h$.

Step 2. We note that each copy of Q' contains a point $p \in \sigma_i \setminus U_\lambda$. Otherwise, we would have $r_{\lambda,i,e} \geq h$, which would contradict the definition of h.

Step 3. Denote by $\text{hole}(p)$ the connected component of $Q \setminus (\sigma_i \cap U_\lambda)$ which contains p. We claim

$$\frac{\text{Vol}_e(\text{hole}(p))}{\text{Vol}_e(Q)} \geq \frac{C_1}{\lambda^{\alpha(n)}(\log\lambda)^{4n}}, \tag{2.1}$$

where $\alpha(n) = 2n^2 + n/4$, and Vol_e denotes the Euclidean volume. We will denote the right-hand side term of (2.1) by $\gamma(\lambda)$.

Indeed, $\text{hole}(p)$ is a connected component of $U'_\lambda \cap Q$ for some λ-nodal domain U'_λ. Hence, we can apply the following Local Courant's Nodal Domain Theorem.

Theorem 2.1 ([5, 9, 13]) *Let $B \subseteq M$ be a metric ball. Let B' be a concentric ball of half the radius of B. Let U_λ be a λ-nodal domain. Let B_λ be a connected component of $B \cap U_\lambda$ which intersects B'. Then*

$$\text{Vol}(B_\lambda)/\,\text{Vol}(B) \geq \frac{C_2}{\lambda^{\alpha(n)}(\log\lambda)^{4n}}, \tag{2.2}$$

where $\alpha(n) = 2n^2 + n/4$.

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We remark that in our case, (2.2) is true also for the quotient of Euclidean volumes, since the Euclidean metric on σ_i is comparable with the metric coming from M.

Step 4. We let $\tilde{\varphi}_\lambda = \chi(U_\lambda)\varphi_\lambda$, where $\chi(U_\lambda)$ is the characteristic function of U_λ, and similarly, $\tilde{\varphi}_{\lambda,i} = \chi(U_\lambda \cap \sigma_i)\varphi_\lambda$. Then we have the inequality

$$\int_Q |\tilde{\varphi}_{\lambda,i}|^2 \, \mathrm{d(vol)} \leq \beta(\lambda) h^2 \int_Q |\nabla \tilde{\varphi}_{\lambda,i}|^2 \, \mathrm{d(vol)}, \tag{2.3}$$

where

$$\beta(\lambda) = \begin{cases} C_3 \log(1/\gamma(\lambda)), & n = 2, \\ C_4/(\gamma(\lambda))^{(n-2)/n} & n \geq 3. \end{cases}$$

To see this, observe that $\tilde{\varphi}_{\lambda,i}$ vanishes on hole(p). We will use the following Poincaré type inequality due to Maz'ya. We discuss it in §4.1. A general version of this inequality, with weights instead of Lebesgue measure, is proved in [6].

Theorem 2.2 *Let $Q \subset \mathbb{R}^n$ be a cube whose edge is of length a. Let $0 < \gamma < 1$. Then*

$$\int_Q |u|^2 \, \mathrm{d(vol)} \leq \beta a^2 \int_Q |\nabla u|^2 \, \mathrm{d(vol)}$$

for all Lipschitz functions u on Q, which vanish on a set of measure $\geq \gamma a^n$, and where

$$\beta = \begin{cases} C_5 \log(1/\gamma) & n = 2, \\ C_6/\gamma^{(n-2)/n} & n \geq 3. \end{cases}$$

From (2.1) and Theorem 2.2 applied to $\tilde{\varphi}_{\lambda,i}$, it follows that

$$\int_Q |\tilde{\varphi}_{\lambda,i}|^2 \, \mathrm{d(vol}_e) \leq \beta(\lambda) h^2 \int_Q |\nabla_e \tilde{\varphi}_{\lambda,i}|_e^2 \, \mathrm{d(vol}_e).$$

Since the metric on σ_i is comparable to the Euclidean metric, we also have inequality (2.3).

Step 5.

$$\int_{\sigma_i} |\tilde{\varphi}_\lambda|^2 \, \mathrm{d(vol)} \leq 16\beta(\lambda) r_\lambda^2 \int_{\sigma_i} |\nabla \tilde{\varphi}_\lambda|^2 \, \mathrm{d(vol)}. \tag{2.4}$$

This is obtained by summing up inequalities (2.3) over all cubes Q which cover σ_i, and recalling that $h < 2r_{\lambda,i,e} \leq 4r_\lambda$.

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Step 6. We sum up (2.4) over all cubical cells σ_i to obtain a global inequality.

$$\begin{aligned}(2.5)\qquad \int_{U_\lambda} |\varphi_\lambda|^2 \,\mathrm{d(vol)} &= \int_M |\tilde{\varphi}_\lambda|^2 \,\mathrm{d(vol)} = \sum_i \int_{\sigma_i} |\tilde{\varphi}_\lambda|^2 \,\mathrm{d(vol)} \\ &\le 16\beta(\lambda) r_\lambda^2 \sum_i \int_{\sigma_i} |\nabla \tilde{\varphi}_\lambda|^2 \,\mathrm{d(vol)} \\ &= 16\beta(\lambda) r_\lambda^2 \int_M |\nabla \tilde{\varphi}_\lambda|^2 \,\mathrm{d(vol)} \\ &= 16\beta(\lambda) r_\lambda^2 \int_{U_\lambda} |\nabla \varphi_\lambda|^2 \,\mathrm{d(vol)}.\end{aligned}$$

Step 7.

$$r_\lambda \ge \begin{cases} C_7/\sqrt{\lambda \log \lambda} & n = 2, \\ C_8/\lambda^{n^2-15n/8+1/4} (\log \lambda)^{2n-4} & n \ge 3. \end{cases}$$

Indeed, by (2.5)

$$\lambda = \frac{\int_{U_\lambda} |\nabla \varphi_\lambda|^2 \,\mathrm{d(vol)}}{\int_{U_\lambda} |\varphi_\lambda|^2 \,\mathrm{d(vol)}} \ge \frac{1}{16\beta(\lambda) r_\lambda^2}\,.$$

Thus,

$$\begin{aligned} r_\lambda \ge \frac{1}{4\sqrt{\lambda \beta(\lambda)}} &= \begin{cases} C_9/\sqrt{\lambda \log(1/\gamma(\lambda))} & n = 2, \\ C_{10}\gamma(\lambda)^{(n-2)/2n}/\sqrt{\lambda} & n \ge 3. \end{cases} \\ &\ge \begin{cases} C_{11}/\sqrt{\lambda \log \lambda} & n = 2, \\ C_{12}/\big(\lambda^{n^2-15n/8+1/4} (\log \lambda)^{2n-4}\big) & n \ge 3. \end{cases}\end{aligned}$$

This completes the proof of the lower bound in Theorem 1.1. ∎

2.2 Lower Bound in Dimension = Two

We prove the existence of the lower bound on the inner radius in Theorem 1.2. The arguments below can basically be found in [10, Ch. 7].

We begin the proof of Theorem 1.2 with Step 1 and Step 2 of §2.1. We proceed as follows.

Step 3′. If $\mathrm{hole}(p)$ does not touch ∂Q,

$$\frac{\mathrm{Area}_e(\mathrm{hole}(p))}{\mathrm{Area}_e(Q)} \ge C_1\,,$$

where Area_e denotes the Euclidean area.

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Proof We recall the Faber–Krahn inequality in $\mathbb{R}^n$.

Theorem 2.3 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. Then $\lambda_1(\Omega) \geq C_2/\operatorname{Vol}(\Omega)^{2/n}$.*

We apply Theorem 2.3 with $\Omega = \operatorname{hole}(p)$. We emphasize that $\lambda_1(\operatorname{hole}(p), g) \geq C_3\lambda_1(\operatorname{hole}(p), e)$, since the two metrics are comparable.

Thus, we obtain

$$\lambda = \lambda_1(\operatorname{hole}(p), g) \geq \frac{C_4}{\operatorname{Area}_e(\operatorname{hole}(p))},$$

or, written differently, $\operatorname{Area}_e(\operatorname{hole}(p)) \geq C_4/\lambda$. On the other hand, $\operatorname{Area}_e(Q) = (4h)^2 \leq 64r_\lambda^2 \leq 64C_5/\lambda$, where the last inequality is the upper bound on the inner radius in Theorem 1.2. So take $C_1 = C_4/(64C_5)$. ∎

Step 4′(a). There exists an edge of Q on which the orthogonal projection of $\operatorname{hole}(p)$ is of Euclidean size $\geq \gamma \cdot 4h$, where $0 < \gamma < 1$ is independent of λ.

Let us denote by $|\operatorname{pr}(\operatorname{hole}(p))|$ the maximal size of the projections of $\operatorname{hole}(p)$ on one of the edges of Q. If $\operatorname{hole}(p)$ touches ∂Q, then $|\operatorname{pr}(\operatorname{hole}(p))| \geq 4h/4 = h$, and we can take $\gamma = 1/4$. Otherwise, by Step 3′

$$|\operatorname{pr}(\operatorname{hole}(p))| \geq \sqrt{\operatorname{Area}_e(\operatorname{hole}(p))} \geq \sqrt{C_1(4h)^2} = 4\sqrt{C_1}h\,.$$

So, we can take $\gamma = \sqrt{C_1}$.

Step 4′(b).

$$\int_Q |\tilde{\varphi}_{\lambda,i}|^2 \,\mathrm{d}\operatorname{vol}_e \leq C_6h^2 \int_Q |\nabla\tilde{\varphi}_{\lambda,i}|^2 \,\mathrm{d}\operatorname{vol}_e\,.$$

Notice that $\tilde{\varphi}_{\lambda,i}$ vanishes on $\operatorname{hole}(p)$. Hence, Step 4′(a) permits us to apply the following Poincaré type inequality to $\tilde{\varphi}_{\lambda,i}$. Its proof is given in §4.2. An inequality in the same spirit can be found in [22].

Theorem 2.4 (**[10, Ch. 7]**) *Let $Q \subseteq \mathbb{R}^2$ be a cube whose edge is of length a. Let u be a Lipschitz function on Q which vanishes on a curve whose projection on one of the edges is of size $\geq \gamma a$. Then*

$$\int_Q |u|^2 \,\mathrm{d}x \leq C(\gamma)a^2 \int_Q |\nabla u|^2 \,\mathrm{d}x.$$

Steps 5′–7′. We continue in the same way as in Steps 5–7 of §2.1. This completes the proof of Theorem 1.2. ∎

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3 A New Proof in Dimension Two

This section is due to L. Polterovich, M. Sodin and F. Nazarov. In dimension two we give a proof based on the harmonic measure and the fact due to Nadirashvili that an eigenfunction on the scale comparable to the wavelength is almost harmonic in a sense to be defined below. This proof also gives information about the location of a big ball inscribed in the nodal domain U_λ. Namely, we show that if $\phi_\lambda(x_0) = \max_{U_\lambda} |\phi_\lambda|$, then one can find a ball of radius $C/\sqrt{\lambda}$ centered at x_0 and inscribed in U_λ.

Let $D_p \subseteq \Sigma$ be a metric disk centered at p. Let f be a function defined on D_p. Let $\mathbb{D}$ denote the unit disk in $\mathbb{C}$.

Definition 3.1 We say that f is (K, δ)-*quasiharmonic* if there exists a K-quasiconformal homeomorphism $h\colon D_p \to \mathbb{D}$, a harmonic function u on $\mathbb{D}$, and a function v on $\mathbb{D}$ with $1 - \delta \le v \le 1$, such that $f = (v \cdot u) \circ h$.

Remark We will assume without loss of generality that $h(p) = 0$.

Theorem 3.2 **([18, 19])** *There exist $K, \varepsilon, \delta > 0$ such that for every eigenvalue λ and disk $D \subseteq \Sigma$ of radius $\le \varepsilon/\sqrt{\lambda}$, $\varphi_\lambda|_D$ is (K, δ)-quasiharmonic.*

We now choose a preferred system of conformal coordinates on (Σ, g).

Lemma 3.3 *There exist positive constants q_+, q_-, ρ such that for each point $p \in M$, there exists a disk $D_{p,\rho}$ centered at p of radius ρ, a conformal map $\Psi_p\colon \mathbb{D} \to D_{p,\rho}$ with $\Psi_p(0) = p$, and a positive function $q(z)$ on $\mathbb{D}$ such that $\Psi_p^*(g) = q(z)|dz|^2$, with $q_- < q < q_+$.*

Let us take a point p, where $|\varphi_\lambda|$ admits its maximum on U_λ. Let $R = \varepsilon/\sqrt{\lambda q_+}$. Let $D_{p,R\sqrt{q_+}} \subseteq D_{p,\rho}$ be a disk of radius $R\sqrt{q_+}$ centered at p.

We now take the functions u, v defined on $\mathbb{D}$ which correspond to $\varphi_\lambda|_{D_{p,R\sqrt{q_+}}}$ in Theorem 3.2. We observe that $\varphi_\lambda(p) = u(0)v(0) \ge u(z)v(z) \ge u(z)(1 - \delta)$, for all $z \in \mathbb{D}$. Hence

$$u(0) \ge (1 - \delta) \max_{\mathbb{D}} u. \tag{3.1}$$

Now we apply the harmonic measure technique. Let $U_\lambda^0 \subseteq \mathbb{D}$ be the connected component of $\{u > 0\}$, which contains 0. Let $E = \mathbb{D} \setminus U_\lambda^0$. Let ω be the harmonic measure of E in $\mathbb{D}$. Then ω is a bounded harmonic function on U_λ^0, which tends to 1 on $\partial U_\lambda^0 \cap \operatorname{Int}(\mathbb{D})$ and to 0 on the interior points of $\partial U_\lambda^0 \cap \partial\mathbb{D}$. Let $r_0 = \inf\{|z| : z \in E\}$.

By the Beurling–Nevanlinna theorem [2, §3-3],

$$\omega(0) \ge 1 - C_1\sqrt{r_0}. \tag{3.2}$$

By the majorization principle

$$u(0)/\max u \le 1 - \omega(0). \tag{3.3}$$

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Combining inequalities (3.1), (3.2), and (3.3) gives us

$$r_0 \geq C_2. \tag{3.4}$$

In the final step we apply a distortion theorem proved by Mori for quasiconformal maps. Denote by $\mathbb{D}_r \subseteq \mathbb{C}$ the disk $\{|z| < r\}$. Observe that $\Psi_p(\mathbb{D}_R) \subseteq D_{p,R\sqrt{q_+}}$. Hence, we can compose $\tilde{h} = h \circ \Psi_p \colon \mathbb{D}_R \to \mathbb{D}$. Then $\tilde{h}$ is a K-quasiconformal map. By Mori's theorem [1, Ch. III.C], it is $\frac{1}{K}$-Hölder. Moreover, it satisfies an inequality

$$|\tilde{h}(z_1) - \tilde{h}(z_2)| \leq M\Big(\frac{|z_1 - z_2|}{R}\Big)^{1/K}, \tag{3.5}$$

with M depending only on K. Inequalities (3.4) and (3.5) imply that

$$\frac{\operatorname{dist}(p, \partial(U_\lambda \cap D_{p,R}))}{R} \geq \Big(\frac{C_2}{M}\Big)^K \sqrt{q_-}.$$

Hence, $\operatorname{inrad}(U_\lambda) \geq (\frac{C_2}{M})^K \sqrt{q_-} R = C_3/\sqrt{\lambda}$, as desired.

4 A Review of Poincaré Type Inequalities

We give an overview of several Poincaré type inequalities. In particular, we prove Theorem 2.2 and Theorem 2.4.

4.1 Poincaré Inequality and Capacity

Theorem 2.2 is a direct corollary of the following two inequalities proved by Maz'ya.

Theorem 4.1 **([14]; [15, §10.1.2])** *Let $Q \subseteq \mathbb{R}^n$ be a cube whose edge is of length a. Let $F \subseteq Q$. Then*

$$\int_Q |u|^2 \, \mathrm{d(vol)} \leq \frac{C_1 a^n}{\operatorname{cap}(F, 2Q)} \int_Q |\nabla u|^2 \, \mathrm{d(vol)}$$

for all Lipschitz functions u on Q which vanish on F.

A few remarks:

- $2Q$ denotes a cube concentric with Q, with parallel edges of size twice as large.
- If $\Omega \subseteq \mathbb{R}^n$ is an open set, and $\bar{F} \subseteq \Omega$, then $\operatorname{cap}(F, \Omega)$ denotes the L^2-capacity of F in Ω, namely
$$\operatorname{cap}(F, \Omega) = \inf_{u \in \mathcal{F}} \Big\{ \int_\Omega |\nabla u|^2 \, \mathrm{d}x \Big\},$$
where $\mathcal{F} = \{u \in C^\infty(\Omega),\ u \equiv 1 \text{ on } F,\ \operatorname{supp}(u) \subseteq \Omega\}$.
- By Rademacher's theorem [24], a Lipschitz function is differentiable almost everywhere, and thus the right-hand side has a meaning.
- A generalization of the inequality to a body which is starlike with respect to a ball is proved in [16].

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The next theorem is a capacity-volume inequality.

Theorem 4.2 ([**15, §2.2.3**])

$$\operatorname{cap}(F,\Omega) \geq \begin{cases} C_2/\log(\operatorname{Area}(\Omega)/\operatorname{Area}(F)) & n=2, \\ C_3/(\operatorname{Vol}(F)^{-(n-2)/n} - \operatorname{Vol}(\Omega)^{-(n-2)/n}) & n \geq 3. \end{cases}$$

In particular, for $n \geq 3$ we have $\operatorname{cap}(F,\Omega) \geq C_3 \operatorname{Vol}(F)^{(n-2)/n}$.

4.2 A Poincaré Inequality in Dimension Two

In this section we prove Theorem 2.4. The proof can be found in [10, Ch. 7]. We bring it here for the sake of clarity.

Proof of Theorem 2.4 Let the coordinates be such that $Q = \{0 \leq x_1, x_2 \leq a\}$. Let the given edge be $Q \cap \{x_1 = 0\}$, and let pr denote the projection from Q onto this edge. Set $E = \operatorname{pr}^{-1}(\operatorname{pr}(\operatorname{hole}(p)))$. We claim

$$\int_E |u|^2 \, dx \leq a^2 \int_Q |\nabla u|^2 \, dx. \tag{4.1}$$

Indeed, let $E_t := E \cap \{x_2 = t\}$. We recall the following Poincaré type inequality in dimension one whose proof is given below.

Lemma 4.3

$$\int_a^b |u|^2 \, dx \leq |b-a|^2 \int_a^b |u'|^2 \, dx \tag{4.2}$$

for all Lipschitz functions u on $[a,b]$ which vanish at a point of $[a,b]$.

By this lemma,

$$\int_{E_t} |u|^2 \, dx_1 \leq a^2 \int_{E_t} |\partial_1 u(x_1,t)|^2 \, dx_1.$$

Integrating over $t \in \operatorname{pr}(\operatorname{hole}(p))$ gives us (4.1).

Next we show

$$\int_Q |u|^2 \, dx \leq C_1 a^2 \int_Q |\nabla u|^2 \, dx. \tag{4.3}$$

By the mean value theorem $\exists t_0$ such that

$$\int_{E_{t_0}} |u|^2 \, dx_1 \leq \frac{1}{\gamma \cdot a} \int_E |u|^2 \, dx. \tag{4.4}$$

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In addition, we have

$$
\begin{aligned}
|u(x)|^2 &\leq 2|u(x_1,t_0)|^2 + 2|u(x) - u(x_1,t_0)|^2 \\
&\leq 2|u(x_1,t_0)|^2 + 2\Big(\int_{t_0}^{x_2} |\partial_2 u(x_1,s)|\, \mathrm{d}s\Big)^2 \\
&\leq 2|u(x_1,t_0)|^2 + 2\cdot a \int_0^a |\partial_2 u(x_1,s)|^2\, \mathrm{d}s.
\end{aligned}
$$

Integrating the last inequality over Q gives us

$$
\int_Q |u|^2\, \mathrm{d}x \leq 2\cdot a \int_{E_{t_0}} |u(x_1,t_0)|^2\, \mathrm{d}x_1 + 2a^2 \int_Q |\partial_2 u|^2\, \mathrm{d}x\,. \tag{4.5}
$$

Finally, we combine (4.1), (4.4), and (4.5) to get (4.3).

$$
\int_Q |u|^2\, \mathrm{d}x \leq 2\cdot a \frac{1}{\gamma a} \int_E |u|^2\, \mathrm{d}x + 2\cdot a^2 \int_Q |\nabla u|^2\, \mathrm{d}x \leq C_1 a^2 \int_Q |\nabla u|^2\, \mathrm{d}x. \qquad \blacksquare
$$

4.3 A Poincaré Inequality in Dimension One

Proof of Lemma 4.3 By scaling, it is enough to prove (4.2) for the segment $[0,1]$. Suppose $u(x_0) = 0$. Since a Lipschitz function is absolutely continuous, we have

$$
|u(x)|^2 = \Big|\int_{x_0}^{x} u'(t)\, \mathrm{d}t\Big|^2 \leq \int_0^1 |u'(t)|^2\, \mathrm{d}t.
$$

We integrate over $[0,1]$ to get the desired inequality. ■

5 λ_1 and Inner Radius

We prove Theorem 1.3, which relates the inner radius to λ_1.

Proof Let $\{V_i\}$ be a finite open cover of M, such that for each i one can put a Euclidean metric e_i on V_i, which satisfies $e_i/4 \leq g \leq 4e_i$. Let α be the Lebesgue number of the covering.

Let $r = \min(\operatorname{inrad}(\Omega), \alpha)$. Let $B \subseteq \Omega$ be a ball of radius r. We can assume that $B \subseteq V_1$. Let $B_e \subseteq B$ be a Euclidean ball of radius $r/2$. By monotonicity of λ_1, we know that $\lambda_1(\Omega, g) \leq \lambda_1(B, g) \leq \lambda_1(B_e, g)$, but since the Riemannian metric on B_e is comparable to the Euclidean metric on it, it follows from the variational principle that

$$
\lambda_1(B_e, g) \leq C_1 \lambda_1(B_e, e_1) = C_2/r^2 \leq C_3/\operatorname{inrad}(\Omega)^2,
$$

where in the last inequality we used the fact that $\operatorname{inrad}(\Omega) \leq C_4\alpha$. ■

Remark We would like to emphasize that in general there is no lower bound on λ_1 in terms of the inner radius. However, in dimension two, as pointed out to us by Daniel Grieser and Mikhail Shubin, there exists a lower bound on λ_1 in terms of the inner radius and the connectivity of Ω. This was proved in [11, 20]. For a more detailed account of the subject one can consult [12].

https://doi.org/10.4153/CMB-2008-026-2 Published online by Cambridge University Press

Acknowledgements I am grateful to Leonid Polterovich for introducing the problem to me and for fruitful discussions. I would like to thank Joseph Bernstein, Lavi Karp and Mikhail Sodin for enlightening discussions. I am thankful to Sven Gnutzmann for showing me the nice nodal domains pictures he generated with his computer program and for nice discussions. I owe my gratitude also to Moshe Marcus, Yehuda Pinchover and Itai Shafrir for explaining to me the subtleties of Sobolev spaces.

I would like to thank Leonid Polterovich, Mikhail Sodin and Fëdor Nazarov for explaining their proof in dimension two to me, and for letting me publish it in §3.

Special thanks are sent to Sagun Chanillo, Daniel Grieser, Mikhail Shubin, Bin Xu and the anonymous referee for their comments and corrections on the first manuscript of this paper.

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https://doi.org/10.4153/CMB-2008-026-2 Published online by Cambridge University Press