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# APPLICATIONS OF DECOMPOSITION THEOREMS TO TRIVIALIZING $h$-COBORDISMS

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ABSTRACT. A geometric proof is presented that, under certain restrictions, the product of an $h$-cobordism with a closed manifold of Euler characteristic zero is a product cobordism. The results utilize open book decompositions and round handle decompositions of manifolds.

We wish to give a geometric approach to the following theorem.

PRODUCT THEOREM FOR $h$-COBORDISMS. *Suppose* $(W, M, M')$ *is an* $h$-*cobordism and* $P$ *is an orientable closed manifold with Euler characteristic* $\chi(P)=0$. *Then, if* $\dim W+\dim P\geq 6$, $(W, M, M')\times P\simeq(M\times I, M\times 0, M\times 1)\times P$.

One may prove the theorem using the product theorem for Whitehead torsion [7] and the $s$-cobordism theorem. Independent of the product theorem for Whitehead torsion, Kervaire [6] presented a geometric proof due to deRham that $(W, M, M')\times S^1\simeq(M\times I, M\times 0, M\times 1)\times S^1$. This proof (as given in [6]) was incomplete, but the gap was filled by Siebenmann [10]. Morton Brown independently has a proof that taking a product with a circle trivializes an $h$-cobordism (cf. [4]); still different proofs appear in [11], [8]. In [10], Siebenmann extended his geometric proof to $P=S^{2k+1}$ or $L^3$, where $L^3$ denotes a 3-dimensional lens space. His proof is based on the following lemma.

LEMMA S. ([10, Prop IV]). *Suppose* $P=P_1\cup P_2$ *where* $P_0=P_1\cap P_2$ *is collared in* $P_1$ *and* $c\times(P_i, P_0)$ *are product cobordisms,* $i=1, 2,$ *for any invertible cobordism* $c=(W, M, M')$. *Then* $c\times P$ *is a product.*

Recall that for $\dim W\geq 5$ any $h$-cobordism is invertible; invertible cobordisms are always $h$-cobordisms.

In this note we show how to apply the open book decomposition theorem and the round handle decomposition theorem to extend Siebenmann's proof to a wide class of manifolds $P$ with $\chi(P)=0$. We work throughout in the differentiable category; $\simeq$ denotes diffeomorphism.

We first look at the open book decomposition theorem as given by Alexander [2], A'Campo [1], and Winkelnkemper [12]. These papers express $P$ as $V_h\cup D^2\times N$, where $V_h$ denotes the mapping torus of a diffeomorphism $h$, $V$ is

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a compact manifold with boundary $N$, and $h \mid N$ is the identity, under the hypotheses:

(a) $P$ is any oriented closed 3-manifold [2].

(b) $P$ is any simply connected closed 5-manifold [1]. Here $N \simeq S^3$.

(c) $P$ is any simply connected closed $(2k+1)$-manifold, $k>2$ [12]. Here $N$ is also simply connected.

THEOREM 1. *If $c=(W, M, M')$ is an invertible cobordism and $P$ is any simply connected odd dimensional closed manifold, then $c \times P$ is a product cobordism.*

The proof is an easy induction argument using (a), (b), (c) above, Lemma S and the following lemma.

LEMMA 1. *If $c$ is an invertible cobordism, then $c \times (V_h, N \times S^1)$ is a product cobordism (as pair).*

**Proof.** Let $c=(W, M, M')$. Then $c \times V_h = ((W \times V)_{1\times h}, (M \times V)_{1\times h}, (M' \times V)_{1\times h})$. Now $c \times V$ is itself invertibly cobordant to $(M \times I, M \times 0, M \times 1) \times V$ (cf [10, Theorem I′]) and the map $1 \times h: W \times V \to W \times V$ extends to a diffeomorphism of this cobordism which restricts to $1 \times h:(M \times I) \times V \to (M \times I) \times V$ on the other end. Now by [9, Theorem 1] this implies that $(W \times V)_{1\times h} \simeq (M \times I \times V)_{1\times h}$. The argument is purely formal and respects $N \subset V$, thus giving a product structure to $c \times (V_h, N \times S^1)$.

We now look at the round handle decomposition theorem of Asimov [3]. It states that if $P$ is a closed manifold of dimension $\neq 3$ with $\chi(P)=0$, then $P$ has a round handle decomposition. A round handle decomposition of $P$ expresses $P$ as $R_0^1 + \cdots + R_0^{k_0} + \cdots + R_{m-1}^1 + \cdots + R_{m-1}^{k_{m-1}}$, where $R_i^j$ is $S^1 \times D^i \times D^{m-i-1}$ and is attached to the boundary of the round handlebody to the left of it via an embedding of $S^1 \times S^{i-1} \times D^{m-i-1}$. If further $\dim P \geq 6$, we may decompose $P$ as $A \cup B$, where $A$, $B$ are round handlebodies where the core $(S^1 \times S^{i-1} \times 0)$ of the attaching maps are embeddings of codimension $\geq 3$ and $\partial A = \partial B$. We are now ready for our second product theorem.

THEOREM 2. *Suppose $(W, M, M')$ is an invertible cobordism $P$ is a closed $p$-dimensional manifold, $p \geq 6$, with $\chi(P)=0$. Then $W \times P \simeq M \times I \times P$.*

**Proof.** Write $P = A \cup B$ as above. The result now will follow from Lemma S if we can show $W \times (A, \partial A)$ and $W \times (B, \partial B)$ are products. The significant fact about $A$ (or $B$) is that it is a round handlebody where the cores of the attaching maps have codimension $\geq 3$. We now show $W \times (A, \partial A)$ is a product using induction on the number of round handles in the decomposition of $A$. If this is one, then $A = S^1 \times D^n$ and the result follows from the result for $S^1$.

For the induction step, we write $A = C \cup R$, where $C$ is a round handlebody with $W \times (C, \partial C) \simeq M \times I \times (C, \partial C)$ and $R$ is a round handle (so $W \times (R, \partial R) \simeq M \times I \times (R, \partial R)$, again using the $S^1$ factor). Now look at the composition

$M\times I\times\partial R \xrightarrow{\approx} W\times\partial R\subset W\times\partial C\xrightarrow{\approx} M\times I\times\partial C$. This is a concordance from the inclusion of $M\times 0\times\partial R$ in $M\times I\times\partial C$. Now the core of $\partial R$ is $S^1\times S^{k-1}\times 0$ and is of codimension $\geq 3$ in $\partial C$. Then Hudson's concordance implies isotopy theorem [5] together with the isotopy extension theorem implies that $M\times I\times S^1\times S^{k-1}\times 0\subset M\times I\times\partial R\to M\times I\times\partial C$ extends to a diffeomorphism of $M\times I\times\partial C$ which is the identity on $M\times 0\times\partial C$. Moreover, this concordance is concordant to the identity (cf. [10]) and thus extends to a diffeomorphism of $M\times I\times C$. Composing the trivialization $W\times C\to M\times I\times C$ with inverse of this diffeomorphism then readjusts the map $M\times I\times\partial R\to M\times I\times\partial C$ so that it preserves $M\times I\times S^1\times S^{k-1}\times 0$. By the tubular neighborhood theorem we may then assume that the new trivialization $W\times C\to M\times I\times C$ restricts to a diffeomorphism between $W\times\partial R$ and $M\times I\times\partial R$. One now applies Lemma $S$.

REMARKS. Theorems 1 and 2 take care of the product theorem except for $P$ of dimension 2, 4, or 5. Dimension 2 is easily handled since the only closed 2-manifolds with Euler characteristic zero are the torus and Klein bottle. Theorem 1 covers dimension 5 in the simply connected case. There are no simply connected closed 4-manifolds with Euler characteristic zero so the result is proved for all simply connected manifolds $P$ with $\chi(P)=0$.

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