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ON SEMIGROUPS IN $R^n \times L^p$ CORRESPONDING TO DIFFERENTIAL EQUATIONS WITH DELAYS

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1. Introduction. In this paper we study some properties of the semigroups associated with the linear retarded functional differential equations (FDE) in the setting of Banach spaces $R^n \times L^p(-h, 0, R^n)$, $1 < p < \infty$. Earlier investigations of these equations via semigroups defined on the customary space $C([-h, 0], R^n)$ played an important role in problems of stability, oscillations, bifurcation, asymptotic behavior etc. [**15**]. More recently, developments in control theory have indicated some distinct advantages of representing retarded FDE's as abstract evolution equations in the spaces $R^n \times L^p(-h, 0, R^n)$ (especially for $p = 2$). These spaces have been previously used in studies of FDE by N. N. Krasovski [**19**], Coleman and Mizel [**8**], Borisovic and Turbabin [**5**], Delfour and Mitter [**13**] and by several other authors. Borisovic and Turbabin were apparently first to state some basic properties of the semigroups in $R^n \times L^p$ corresponding to linear retarded FDE's (their paper contains no proofs, however). These or similar ideas were subsequently employed in other papers or reports [**2**; **10**; **26**; **27**] and, were also applied to some problems of control theory (e.g. [**2**; **21**; **23**]; for more references on related works using other types of evolution equations see e.g. [**9**]).

In this paper we consider four possible semigroups associated with the autonomous linear retarded FDE: the usual semigroup $\{S(t), t \geqq 0\}$ corresponding directly to the equation, the semigroup $\{S^+(t), t \geqq 0\}$ corresponding to the equation with transposed matrices, the semigroup $\{\tilde{S}(t), t \leqq 0\}$ corresponding to the differential adjoint equation, and the dual semigroup $S^*(t)$. An important point of this paper is the introduction of a certain bounded linear operator F, related to the equation, which turns out to play a key role in the semigroups listed above. By using this operator we explain the relation between the general linear functional defined on $R^n \times L^p(-h, 0; R^n)$ and the known special bilinear form associated with the FDE, and we prove a few new relationships, holding in the dual space, involving semigroups mentioned above and their infinitesimal generators. In addition, it is shown that the property of commutativity of the semigroup operator $S(t)$ with its infinitesimal generator has as its consequence the commutativity of the fundamental matrix with the matrices defining the

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FDE. Finally, in the Appendix we collect proofs of some basic properties of the semigroup $S(t)$ which are not readily available in the published literature.

2. Preliminaries. We consider the linear retarded function differential equation

$$(2.1) \qquad \dot{x}(t) = \sum_{i=0}^{N} A_i x(t - h_i) + \int_{-h}^{0} A_{01}(\theta) x(t + \theta) d\theta \quad \text{a.e. for } t \geqq 0$$

$$(2.2) \qquad x(0) = \phi^0, x(\theta) = \phi^1(\theta), \quad \theta \in (-h, 0),$$

where $x \in R^n$, $\phi^0 \in R^n$, $\phi^1 \in L^p(-h, 0; R^n)$; A_i, $i = 0, \ldots, N$, are $n \times n$ real matrices; the $n \times n$ real matrix valued function $\theta \rightarrow A_{01}(\theta)$ is bounded measurable; and $0 = h_0 < h_1 < \ldots < h_N = h$. We adopt the customary notation $x_t = x_t(\theta) = x(t + \theta), \theta \in [-h, 0]$.

The solutions of this equation will be treated as elements of the space $X = R^n \times L^p(-h, 0; R^n)$, that is $(x(t), x_t) \in X$, or $x(t) \in R^n$, $x_t \in L^p(-h, 0; R^n)$. Elements $\psi \in X$ will be denoted (ψ^0, ψ^1), where $\psi^0 \in R^n$, $\psi^1 \in L^p(-h, 0; R^n)$. X is a Banach space with the norm

$$||(\psi^0, \psi^1)||_X = ||\psi^0||_{R^n} + ||\psi^1||_{L^p(-h,0;R^n)}.$$

The dual space of X will be denoted by X^*. The scalar product between $\psi \in X^*$ and $\phi \in X$ will be denoted by $\langle \psi, \phi \rangle$. Occasionally, the scalar product between $\psi^1 \in L^q(-h, 0; R^n)$ and $\phi^1 \in L^p(-h, 0; R^n)$, $1/p + 1/q = 1$, will be used and denoted by $\langle \psi^1, \phi^1 \rangle_{L^p}$. $W^{1,p}(a, b; R^n)$ will denote the Sobolev space of $x \in L^p(a, b; R^n)$ with derivative $\dot{x}$ in $L^p(a, b; R^n)$. Abbreviations A.C. and B.V. will denote absolutely continuous and bounded variation functions, respectively. Superscript T will denote transposition of a vector in R^n.

Let $x(t; \phi)$ and $x_t(\cdot, \phi)$ denote the solution of (2.1) (2.2) in R^n and $L^p(-h, 0; R^n)$ respectively. Existence and uniqueness of solution $x(t, \phi)$ with the initial conditions (2.2) ($\phi \in X$) has been asserted in [**13**]; moreover one has that $t \rightarrow x(t, \phi)$ is in $W^{1,p}(0, \tau; R^n)$ for all fixed $\tau > 0$, and

$$(2.3) \qquad ||x(\cdot, \phi)||_{W^{1,p}} \leqq c||\phi||_X$$

where c depends on τ (see Appendix).

In order to keep formulas concise we will use the Stieltjes integral notation. Extend $A_{01}(\cdot)$ to $(-\infty, \infty)$ by putting $A_{01}(s) \equiv 0$ for $s \notin [-h, 0]$, and define

$$(2.4) \qquad G(\theta) = -\sum_{i=1}^{N} A_i \chi_{(-\infty, -h_i]}(\theta) - \int_{\theta}^{0} A_{01}(s) ds$$

where $\chi_{(a,b)}$ denotes the characteristic function of the interval (a, b). Define

$$(2.5) \qquad N(\theta) = -A_0 \chi_{(-\infty, 0)}(\theta) + G(\theta).$$

Both $G(\cdot)$ and $N(\cdot)$ are B.V. functions; more specifically, they are piecewise

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A.C. with finite number of jumps. For $\phi \in C([-h, 0]; R^n)$ one can define $L: C \to R^n$

$$(2.6) \qquad L(\phi) = \int_{-h}^{0} dN(\theta)\phi(\theta).$$

For $t \geqq 0$ define the operator $S(t) : X \to X$ by

$$(2.7) \qquad S(t)\phi = (x(t;\phi), x_t(\cdot, \phi)) \quad \phi \in X.$$

The following proposition summarizes the known properties of $S(t)$:

PROPOSITION 2.1. (i) *For all* $t \geqq 0$, $S(t)$ *is a linear bounded operator;*
(ii) *the family* $\{S(t),\ t \geqq 0\}$ *is a strongly continuous semigroup of operators;*
(iii) *for all* $t \geqq h$, $S(t)$ *is compact;*
(iv) *the infinitesimal generator* A *of* $\{S(t), t \geqq 0\}$ *is given by*

$$(2.9) \qquad \mathscr{D}(A) = \{\phi \in X | \phi^1 \in W^{1,p}(-h, 0; R^n), \phi^1(0) = \phi^0\}$$

$$(2.10) \qquad A\phi = (L(\phi^1), \dot{\phi}^1) \quad \textit{for } \phi \in \mathscr{D}(A).$$

Proof. See Appendix.

3. Operator F, its dual F^* and its relation to semigroup $S(t)$. We now introduce the operator $F : X \to X$ which will play an important role in this paper, as well as in other related developments [**21**]. First, let $\phi^1 \in L^p(-h, 0; R^n)$ and define the operator $H : L^p(-h, 0; R^n) \to L^p(-h, 0; R^n)$ by the formula

$$(3.1) \qquad (H\phi^1)(\theta) \triangleq \int_{-h}^{\theta} dG(s)\phi^1(s - \theta)$$

or, more explicitly

$$(3.2) \qquad = \sum_{i=1}^{N} A_i\chi_i(\theta)\phi^1(-h_i - \theta) + \int_{-h}^{\theta} A_{01}(s)\phi^1(s - \theta)ds$$

where $\chi_i(\cdot)$ is the characteristic function of the interval $[-h_i, 0]$. We note that H is related to the strictly retarded part of equation (2.1), that is H does not depend on A_0. The operator H will enable us to represent in a concise notation the contribution of the initial function ϕ^1 to the solution $x(t)$ of (2.1). We also note that $H\phi^1$ is a *convolution* of ϕ^1 with the real measure $G(\cdot)$. From the explicit form of $H\phi^1$ given by (3.2) it follows immediately that H is a linear bounded operator. More importantly, its dual H^* is an operator of the same type, as shown by the following result.

PROPOSITION 3.1. *Let* $\psi^1 \in L^q(-h, 0; R^n)$, $1/p + 1/q = 1$. *Then*

$$(3.3) \qquad (H^*\psi^1)(\theta) = \int_{-h}^{\theta} dG^T(s)\psi^1(s - \theta).$$

Proof. Take arbitrary $\phi^1 \in L^p(-h, 0; R^n)$ and $\psi^1 \in L^q(-h, 0; R^n)$, and compute

$$\langle \psi^1, H\phi^1 \rangle_{L^p} = \int_{-h}^{0} \psi^{1^T}(\theta)\left[\int_{-h}^{\theta} dG(s)\phi^1(s-\theta)\right]d\theta$$

(3.4)
$$= \int_{-h}^{0} \psi^{1^T}(\theta)\left[\sum_{i=1}^{N} A_i\chi_i(\theta)\phi^1(-h_i-\theta)\right]d\theta + \int_{-h}^{0} \psi^{1^T}(\theta)\int_{-h}^{\theta} A_{01}(s)\phi^1(s-\theta)dsd\theta.$$

Since $A_{01}(\xi) \equiv 0$ for $\xi < -h$, the second term can be rewritten as

(3.5)
$$\int_{-h}^{0} \psi^{1^T}(\theta)\int_{-h}^{0} A_{01}(s+\theta)\phi^1(s)dsd\theta.$$

We now show that the order of integration can be changed by using the Fubini theorem; in fact, this is a part of a standard result concerning the convolution of functions in L^1, [**18**, 21.31; **14**, VIII.1.24]. By virtue of arguments given in [**18**, p. 395-396], the function $(s, \theta) \to |\psi^{1^T}(\theta)A_{01}(s+\theta)\phi^1(s)|$ is measurable with respect to the completion of Lebesgue's product σ-algebra on $[-h, 0] \times [-h, 0]$. We then compute

$$\int_{-h}^{0}\int_{-h}^{0} |\psi^{1^T}(\theta)A_{01}(s+\theta)\phi^1(s)|dsd\theta$$
$$\leqq \int_{-h}^{0} ||\psi^{1^T}(\theta)||_{R^n}\int_{-h}^{0} ||A_{01}(s+\theta)||_{R^{n^2}}||\phi^1(s)||_{R^n}dsd\theta$$
$$\leqq ||A_{01}||_{L^\infty}\cdot||\phi^1||_{L^1}\cdot||\psi^1||_{L^1}.$$

Hence we may apply Fubini's theorem [**18**, 21.13 and 21.17], obtaining that (3.5) is equal to

(3.6)
$$\int_{-h}^{0}\int_{-h}^{0} \psi^{1^T}(\theta)A_{01}(s+\theta)d\theta\phi^1(s)ds = \int_{-h}^{0}\left[\int_{-h}^{s} A_{01}{}^T(\theta)\psi^1(\theta-\mathrm{s})\mathrm{d}\theta\right]^T\phi^1(s)ds.$$

The first term of (3.4) can be transformed without difficulty. Adding it to (3.6) we obtain that (3.4) is equal to

$$\int_{-h}^{0}\left[\int_{-h}^{\alpha} dG^T(s)\psi^1(s-\alpha)\right]^T\phi^1(\alpha)d\alpha \equiv \langle H^*\psi^1, \phi^1\rangle_{L^p}.$$

If $\phi^1 \in$ A.C. $([-h, 0], R^n)$, then the explicit expression for H (3.2) indicates that the function $\theta \to (H\phi^1)(\theta)$ is A.C. in $(-h_{i+1}, -h_i)$, with possible jumps at $\theta = -h_i$, $i = 1, \ldots, N$.

Define $F: X \to X$ by $F = \begin{bmatrix} I & 0 \\ 0 & H \end{bmatrix}$, where I is the identity operator on $\mathbf{R}^n$.

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That is, for any $\phi \in X$

$$[F\phi]^0 = \phi^0, \quad [F\phi]^1 = H\phi^1.$$

We note again that F is linear bounded, and that

$$(3.7) \qquad F^* = \begin{bmatrix} I & 0 \\ 0 & H^* \end{bmatrix}.$$

Let us now give some introductory motivation for the operator F. We show below that F is closely related to the well known bilinear form (α, ϕ) described by Hale [**15**, § 7.3, Eq. (3.1)] and that for $t \geqq h$ F is a part of $S(t)$.

The bilinear form (α, ϕ) is given in [**15**] as a mapping $C([0, h], R^{n*}) \times C([-h, 0], R^n) \to R$ where R^{n*} is the space of transposed (i.e. row) vectors. For $\alpha \in C([0, h], R^{n*})$, $\phi \in C([-h, 0], R^n)$, (α, ϕ) is defined by

$$(3.8) \qquad (\alpha, \phi) = \alpha(0)\phi(0) - \int_{-h}^{0} \int_{0}^{s} \alpha(\xi - s) dN(s) \phi(\xi) d\xi$$

where the inner integration is with respect to ξ and the outer one with respect to s; (the order of α, N, ϕ cannot be changed due to the matrix notation). The form (α, ϕ) is important in the theory for several reasons [**15**, Chapters 6, 7]. In particular the spectral projections on eigenmanifolds can be expressed as

$$\sum_{j=1}^{m} (\psi_{\lambda_j}, \phi) \phi_{\lambda_j}$$

where ϕ_{λ_j}, ψ_{λ_j} are, respectively, the eigenfunctions associated with Eq. (2.1) and with its adjoint equation, corresponding to some eigenvalues λ_j, $j = 1, \ldots, m$ ([**15**, Lemma 3.4]; see also [**3**]). We now show that (3.8) is closely related to F.

Define a special bilinear form associated with F. Let $\langle\langle \cdot , \cdot \rangle\rangle : X^* \times X \to R$ be given by

$$(3.9) \qquad \langle\langle \psi, \phi \rangle\rangle \triangleq \langle \psi, F\phi \rangle \equiv \langle F^*\psi, \phi \rangle.$$

Let $\phi^1, \psi^1 \in C([-h, 0], R^n)$ and $\phi^0 = \phi^1(0)$, $\psi^0 = \psi^1(0)$. Then by using previous calculations we obtain (the superscripts ^{1} are dropped)

$$\langle\langle \psi, \phi \rangle\rangle = \psi^T(0)\phi(0) + \int_{-h}^{0} \int_{s}^{0} \underline{\psi^T(s - \theta)} dG(s) \underline{\phi(\theta)} \underline{d\theta}$$

where the underlined elements belong to the inner integral (The order of ψ, G, ϕ cannot be changed due to the matrix notation.). Since for $s = 0$ this integral is equal to zero, one can replace $dG(s)$ in the outer integral by $dN(s)$ (which differs from $dG(s)$ only by the jump $-A_0$ at $s = 0$). Therefore

$$(3.10) \qquad \langle\langle \psi, \phi \rangle\rangle = \psi^T(0)\phi(0) + \int_{-h}^{0} \int_{s}^{0} \psi^T(s - \theta) dN(s) \phi(\theta) d\theta$$

which coincides with (3.8) if we let $\psi^T(\theta) = \alpha(-\theta)$. Therefore (3.9) is an analog of (3.8).

By making use of the known variation of constants formula (see e.g. [**15**, p. 150], or [**13**, Theorem 5.2]) one can now obtain an explicit expression for $S(t)$. Let $X(t)$ denote the fundamental matrix of Eq. (2.1), i.e. $X(t) \equiv 0$ for $t < 0$, $X(0) = I$, $\dot{X}(t) = L(X_t)$ a.e. where $X_t(\cdot)$ denotes $X_t(\theta) = X(t+\theta)$, $\theta \in [-h, 0]$, [**15**, Theorem 2.5]. Columns of X_t can be regarded as elements of $L^p(-h, 0; R^n)$. Taking the variation of constants formula, given by [**15**, § 6.3, Eq. (3.10)], one obtains the following expression for $x(t, \phi)$:

$$(3.11)\quad x(t, \phi) = X(t)\phi^0 + \int_{-h}^{0} U_t(\theta)\phi^1(\theta)d\theta$$

where

$$(3.12)\quad U_t(\theta) = \sum_{i=1}^{N} X(t-\theta-h_i)A_i\chi_i(\theta) + \int_{-h}^{\theta} X(t-\theta+s)A_{01}(s)ds.$$

Let us introduce the following notation. Let $\xi^j(t)$, ξ_t^j, u_t^j denote the j-th column of $X^T(t), X_t^T, U_t^T$ respectively. Let $\hat{\xi}_t^j = (\xi^j(t), \xi_t^j)$ and $\hat{X}_t^T = [\hat{\xi}_t^1, \ldots \hat{\xi}_t^n]$. By the rules of matrix multiplication, if a^j, b^j denote the columns of $n \times n$ matrices A, B, the statement $A = CB$, where C is an $n \times n$ matrix, is equivalent to $a^j = Cb^j$, $j = 1, \ldots, n$. The symbol $\langle \hat{X}_t^T, \phi \rangle$ will be understood as a row vector of scalar products $\langle \hat{\xi}_t^j, \phi \rangle$, $j = 1, \ldots, n$. Let $x_j(t, \phi)$ denote the j-th component of $x(t, \phi)$. Then, by transposing (3.12) we obtain

$$u_t^j = H^*\xi_t^j$$

and

$$\begin{aligned}(3.13)\quad x_j(t, \phi) &= \langle \xi^j(t), \phi^0 \rangle_{R^n} + \langle u_t^j, \phi^1 \rangle_{L^p} \\ &= \langle \xi_t^j(0), \phi^0 \rangle_{R^n} + \langle H^*\xi_t^j, \phi^1 \rangle_{L^p} \\ &= \langle F^*\hat{\xi}_t^j, \phi \rangle \\ &= \langle \hat{\xi}_t^j, F\phi \rangle.\end{aligned}$$

Hence

$$(3.14)\quad x(t, \phi) = \langle \hat{X}_t^T, F\phi \rangle^T, \quad t \geqq 0.$$

From (2.7) we have now

$$(3.15)\quad [S(t)\phi]^0 = \langle \hat{X}_t^T, F\phi \rangle^T$$

$$(3.16)\quad [S(t)\phi]^1(\theta) = \langle \hat{X}_{t+\theta}^T, F\phi \rangle^T + \phi^1(t+\theta)\chi_{[-h,-t)}(\theta).$$

Note that for $t \geqq h$ the last term on the right hand side disappears. Define the following linear operator $\mathscr{G}_t$ from $R^n \times L^p(-h, 0; R^n)$ into itself:

$$(3.17)\quad \begin{cases} [\mathscr{G}_t\psi]^1(\theta) = X(t+\theta)\psi^0 + \int_{-h}^{0} X(t+s+\theta)\psi^1(s)ds \\ [\mathscr{G}_t\psi]^0 = [\mathscr{G}_t\psi]^1(0). \end{cases}$$

Then (3.15) and (3.16) can be rewritten as

(3.18) $S(t)\phi = \mathscr{G}_t F\phi + s(t)\phi$

where $[s(t)\phi]^0 = 0$ and $[s(t)\phi]^1(\theta) = \phi^1(t+\theta)\chi_{[-h,-t)}(\theta)$, i.e. $s(t) \equiv 0$ for $t \geqq h$. As a conclusion we obtain that for $t \geqq h$ the operator $S(t)$ is a composition of operators $\mathscr{G}_t$ and F. Consequently

(3.19) $\operatorname{Ker} F \subset \operatorname{Ker} S(t), \quad t \geqq h.$

The operator $\mathscr{G}_t$ for $t = h$ along with the F play an important role in the problem of completeness of eigenfunctions—see [**22**].

4. A consequence of commutativity of $S(t)$ and A. It is well-known (e.g. [**25**]) that

(4.1) $S(t)A\phi = AS(t)\phi$ for all $\phi \in \mathscr{D}(A)$ and for all $t \geqq 0$.

A direct consequence of this identity is

LEMMA 4.1. *The following identity holds:*

(4.2) $$\int_{-h}^{0} dN(\theta)X(t+\theta) = \int_{-h}^{0} X(t+\theta)dN(\theta) \quad \textit{for all } t \geqq 0,$$

where $N(\cdot)$ is given by (2.5), *and $X(t)$ is the fundamental matrix of Equation* (2.1).

Note that for a system with one delay, ($N = 1$) and $A_{01}(\theta) \equiv 0$, (4.2) takes the form

$$A_0X(t) + A_1X(t-h) = X(t)A_0 + X(t-h)A_1 \quad t \geqq 0.$$

Proof. Consider the R^n-components of identity (4.1). For $\phi \in \mathscr{D}(A)$ one has, by virtue of (2.10) and (2.6)

$$[AS(t)\phi]^0 = L([S(t)\phi]^1) = \int_{-h}^{0} dN(\theta)[S(t)\phi]^1(\theta),$$

and, by using (3.16)

(4.3) $$\begin{aligned}[AS(t)\phi]^0 = &\int_{-h}^{0} dN(\theta)X(t+\theta)\phi^0 \\ &+ \int_{-h}^{0} dN(\theta)\{\langle H^*X^T_{t+\theta}, \phi^1\rangle_{L^p}^T + \phi^1(t+\theta)\chi_{[-h,-t)}(\theta)\}.\end{aligned}$$

The integral in the second term can be switched with $\langle\ ,\ \rangle$ by using the same reasoning as in the proof of Proposition 3.1. Using the fact that $dG(\theta) = dN(\theta)$ for $\theta \in [-h, -t)$, $t > 0$, we obtain,

(4.4) $$\begin{aligned}[AS(t)\phi]^0 = &\int_{-h}^{0} dN(\theta)X(t+\theta)\phi^0 \\ &+ \int_{-h}^{0} g_1(s)\phi^1(s)ds + \int_{-h}^{-t} dG(\theta)\phi^1(t+\theta)\end{aligned}$$

where the last term is present only for $t < h$, and

$$(4.5)\qquad g_1(s) = \int_{-h}^{0} dN(\theta)[(H^* X_{t+\theta}^T)^T(s)]$$

On the other hand, for $\phi \in \mathscr{D}(A)$, $A\phi = (L(\phi^1), \dot{\phi}^1)$ and, by using (3.11), one has

$$(4.6)\qquad [S(t)A\phi]^0 = X(t)L(\phi^1) + \int_{-h}^{0} U_t(\theta)\dot{\phi}^1(\theta)d\theta.$$

We now claim that $\theta \to U_t(\theta)$ is B.V. on $[-h, 0]$. Indeed, by inspecting formula (3.12) one finds that the first term on the right hand side of that formula is piece-wise A.C. with finite number of jumps at $\theta = -h_i$, $i = 1, \ldots, N-1$, and $\theta = t - h_i$ (if $t - h_i \in [-h, 0)$), $i = 1, \ldots, N$; denoting these jumps by $\Delta U_t(\theta) = U_t(\theta^+) - U_t(\theta^-)$ we have

$$(4.7)\qquad \Delta U_t(-h_i) = X(t)A_i; \quad \Delta U_t(t - h_i) = -A_i\chi_{[-h,0)}(t - h_i).$$

The second term on the right hand side of (3.12) is Lipschitz, hence A.C.. Let $\eta(\cdot)$ denote the A.C. part of $U_t(\cdot)$. From (3.12)

$$(4.8)\qquad \dot{\eta}(\theta) = X(t)A_{01}(\theta) - A_{01}(\theta - t) - g_2(\theta)$$

where the presence of $A_{01}(\theta - t)$ follows from the fact that the lower limit of integration in (3.12) is, for $\theta - t \geqq -h$, actually equal to $\theta - t$ (because $X(\alpha) \equiv 0$ for $\alpha < 0$); the last term equals

$$g_2(\theta) = \int \dot{X}_t(s - \theta)dG(s),$$

where the integration is taken on the interval $[\sigma, \theta]$, $\sigma = \max\{-h, \theta - t\}$.

Since $U_t(\cdot)$ is B.V., one can use for (4.6) the following integration by parts formula [**24**]

$$(4.9)\qquad \int_{-h}^{0} U_t(\theta)\dot{\phi}^1(\theta)d\theta = U_t(\theta)\phi^1(\theta)\Big|_{-h}^{0} - \int_{-h}^{0} [d_\theta U_t(\theta)]\phi^1(\theta).$$

Using (3.12) (4.7) and (4.8) to compute the right hand side of (4.9) and rearranging terms one obtains

$$(4.10)\qquad \int_{-h}^{0} U_t(\theta)\dot{\phi}^1(\theta)d\theta = \int_{-h}^{0} X(t + s)dG(s)\phi^0 - X(t)\int_{-h}^{0} dG(s)\phi^1(s) + \int_{-h}^{-t} dG(\theta)\phi^1(t + \theta) + \int_{-h}^{0} g_2(\theta)\phi^1(\theta)d\theta.$$

Substituting (4.10) into (4.6) and using (2.5) we finally obtain

$$[S(t)A\phi]^0 = \int_{-h}^{0} X(t + s)dN(s)\phi^0 + \int_{-h}^{-t} dG(\theta)\phi^1(t + \theta) + \int_{-h}^{0} g_2(\theta)\phi^1(\theta)d\theta.$$

Consequently

$$0 \equiv [AS(t)\phi]^0 - [S(t)A\phi]^0$$
$$= \left\langle \left[\int_{-h}^{0} dN(\theta)X(t+\theta) - \int_{-h}^{0} X(t+\theta)dN(\theta), g_1 - g_2 \right]^T, (\phi^0, \phi^1) \right\rangle$$

for all $(\phi^0, \phi^1) \in \mathscr{D}(A)$. Since $\mathscr{D}(A)$ is dense in X, the conclusion follows.

Remark 4.1. Formula (4.2) can also be derived by other methods, see e.g. [**20**, Remark 4], or, for the special case of differential-difference equations, [**4**]. What we want to emphasize in the proof above is the connection of this formula with the commutativity of $S(t)$ with A. The formula itself has some applications in studies of controllability and observability—see e.g. [**23**, Sec. 8].

5. The semigroups $S^+(t)$, $\tilde{S}(t)$, and the dual semigroup $S^*(t)$. In this section we consider three other semigroups associated with Eq. (2.1). The dual semigroup $\{S^*(t), t \geqq 0\}$ plays an important role in many problems, especially in control theory (see e.g. [**11**; **12**]). In qualitative theory of functional differential equations one often uses the so-called adjoint equation [**15**, § 6.3] which gives rise to a semigroup denoted by $\{\tilde{S}(t), t \leqq 0\}$. We now show that both $S^*(t)$ and $\tilde{S}(t)$ are closely related to $S^+(t)$, where $\{S^+(t), t \leqq 0\}$ is the semigroup corresponding to the equation with transposed matrices. The details follow below.

Semigroup $S^+(t)$, $t \geqq 0$. Consider the system analogous to (2.1), but with all the matrices A_i transposed,

$$\text{(5.1)} \qquad \dot{y}(t) = \int_{-h}^{0} dN^T(\theta)y(t+\theta) \quad \text{a.e. } t \geqq 0, y \in R^n,$$

$$\text{(5.2)} \qquad (y(0), y_0) = \psi \quad \text{for some } \psi \in X^*.$$

With this system we associate the semigroup $S^+(t)$: $X^* \to X^*$, $t \geqq 0$, defined by

$$\text{(5.3)} \qquad S^+(t)\psi = (y(t, \psi), y_t(\cdot, \psi))$$

where $y(\cdot, \psi)$ is the solution of (5.1) with the initial data (5.2). Clearly, $S^+(t)$ has all the properties given by Proposition 2.1, with the $\mathscr{D}(A^+)$ now given by

$$\text{(5.4)} \qquad \mathscr{D}(A^+) = \{\psi \in X^* | \psi^1 \in W^{1,q}(-h, 0; R^n), \psi^1(0) = \psi^0\}$$

and

$$\text{(5.5)} \qquad A^+\psi = (L^+(\psi^1), \dot{\psi}^1)$$

where L^+ is given by (2.6) with $N(\cdot)$ replaced by $N^T(\cdot)$.

Semigroup $\tilde{S}(t)$, $t \leqq 0$. Consider the following equation

$$\text{(5.6)} \qquad \dot{z}(t) = -\int_{-h}^{0} dN^T(\theta)z(t-\theta) \quad \text{a.e. for } t \leqq 0$$

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with the initial data

(5.7) $\quad (z(0), z_h(\cdot)) = \xi \in X^*$

where $z_t(\cdot)$ is still defined by $z_t(\theta) = z(t+\theta)$, $\theta \in [-h, 0]$. Equation (5.6) has been used in the control theory literature as the so-called "adjoint equation". For $t \leqq 0$ define

(5.8) $\quad \tilde{S}(t)\xi = (z(t, \xi), z_{t+h}(\cdot, \xi)) \quad t \leqq 0$

where $z(\cdot, \xi)$ is the solution of (5.6) on $(-\infty, 0]$ with the initial data (5.7); this solution is unique, so that indeed $\tilde{S}(t)$ is a well defined semigroup operator. We define $\tilde{A}$ by $\lim_{t\to 0^-} (\tilde{S}(t) - I)/t$.

For any $r \geqq 1$ define the operator $\mathscr{J}^1$ from $L^r(-h, 0; R^n)$ into itself by $(\mathscr{J}^1\psi)(\theta) = \psi(-h-\theta)$, $\theta \in (-h, 0)$, and (for $r = q$) define $\mathscr{J}: X^* \to X^*$ by $\mathscr{J} = (I, \mathscr{J}^1)$. Obviously $\mathscr{J}$ is one-to-one, bounded and invertible, and $\mathscr{J}^2$ is an identity on X^*.

We now have the following relations.

PROPOSITION 5.1. (i) $\tilde{S}(-t) = \mathscr{J} S^+(t) \mathscr{J}$, *for all* $t \geqq 0$
(ii) $\mathscr{D}(\tilde{A}) = \mathscr{J}\mathscr{D}(A^+)$
(iii) $\tilde{A} = \mathscr{J} A^+ \mathscr{J}$.

Proof. Part (i) follows by defining $\eta: [-h, \infty) \to R^n$ by the formula $\eta(t) = z(-t, \xi)$, $t \in [-h, \infty)$. Now $\eta(0) = z(0) = \xi^0$, $\eta(\theta) = z_h(-h-\theta, \xi)$ for $\theta \in [-h, 0)$, i.e. $\eta(\theta) = (\mathscr{J}^1 z_h)(\theta) = \xi^1(\theta)$, and, by a simple calculation one sees that for $t \geqq 0$, $\eta(t)$ satisfies the differential equation (5.1). If now $\xi = \mathscr{J}\psi$, one has $\eta(\theta) = \psi^1(\theta)$, $\theta \in [-h, 0)$, $\eta(0) = \psi^0$, and by uniqueness, $\eta(t) \equiv y(t)$ for $t \geqq 0$. Using (5.3) and (5.8) one obtains $\tilde{S}(-t)\mathscr{J}\psi = \mathscr{J}S^+(t)\psi$, for all $\psi \in X^*$, which gives (i). From this and the properties of $\mathscr{J}$, (ii) and (iii) follow trivially.

This proposition shows that $S^+(t)$ contains all the information about the solutions of the "adjoint equation" (5.6). We now turn to the dual semigroup $S^*(t)$ and show that it is also closely related to $S^+(t)$.

Since X is a reflexive Banach space $(1 < p < \infty)$ one has that the dual semigroup $S^*(t)$ is defined on all of X^*, is strongly continuous in the topology of X^*, and its infinitesimal generator A^* is the dual of A, with $\mathscr{D}(A^*)$ strongly dense in X^* [**7**, § 1.4].

To describe A^*, define, for $\psi = (\psi^0, \psi^1) \in X^*$

$$g(\theta) = \int_{-h}^{\theta} dG^T(s)\psi^0 - \psi^1(\theta).$$

Then, performing some standard manipulations similar to those reported in [**26**] one can prove that

(5.9) $\quad \mathscr{D}(A^*) = \{\psi \in X^* | g \in W^{1,q}(-h, 0; R^n), g(-h) = 0\}$

and, for $\psi \in \mathscr{D}(A^*)$

$$(5.10) \quad [A^*\psi]^0 = \psi^1(0) + A_0^T\psi^0, \; [A^*\psi]^1(\theta) = \frac{d}{d\theta}g(\theta).$$

Our next result exhibits the role played by F^*, and the related bilinear form $\langle\langle\,,\rangle\rangle$.

PROPOSITION 5.2. *The following relations are true:*
(i) $F^*\mathscr{D}(A^+) \subset \mathscr{D}(A^*)$
(ii) $A^*F^* = F^*A^+$ *on* $\mathscr{D}(A^+)$
(iii) $\langle\langle\psi, A\phi\rangle\rangle = \langle\langle A^+\psi, \phi\rangle\rangle$ *for all* $\phi \in \mathscr{D}(A)$, $\psi \in \mathscr{D}(A^+)$.

Proof. (i) Taking an arbitrary $\phi = (\phi^0, \phi^1) \in \mathscr{D}(A^+)$ one has that $\psi = F^*\phi \in F^*\mathscr{D}(A^+)$ is given by

$$(5.11) \quad \psi^0 = \phi^0, \quad \psi^1(\theta) = (H^*\phi^1)(\theta).$$

Now

$$(5.12) \quad g(\theta) = \int_{-h}^{\theta} dG^T(s)\psi^0 - \psi^1(\theta) = \int_{-h}^{\theta} dG^T(s)[\phi^0 - \phi^1(s-\theta)].$$

One verifies without difficulty that $g(\cdot)$ is absolutely continuous, and

$$(5.13) \quad \frac{d}{d\theta}g(\theta) = \int_{-h}^{\theta} dG^T(s)\dot{\phi}^1(s-\theta).$$

Writing the right hand side in detail, one observes that (5.13) belongs to $L^q(-h, 0; R^n)$, hence $g(\cdot) \in W^{1,q}(-h, 0; R^n)$. Furthermore

$$g(-h) = A_N^T[\phi^0 - \phi^1(0)] = 0$$

so that $\psi \in \mathscr{D}(A^*)$ as claimed.

(ii) By (4.10) one has that, for $\phi \in \mathscr{D}(A^+)$, $\eta \triangleq A^*F^*\phi$ is given by

$$(5.14) \quad \eta^0 = A_0^T\phi^0 + (H^*\phi^1)(0) = A_0^T\phi^1(0) + \int_{-h}^{0} dG^T(s)\phi^1(s) = L^+(\phi^1)$$

and, by using (5.10), (5.13) and (2.11)

$$(5.15) \quad \eta^1(\theta) = \frac{d}{d\theta}g(\theta) = \int_{-h}^{\theta} dG^T(s)\dot{\phi}^1(s-\theta) \equiv (H^*\dot{\phi}^1)(\theta)$$

for $\theta \in (-h, 0)$. Therefore, by using (5.14), (5.15), (2.12)

$$(5.16) \quad \begin{aligned} A^*F^*\phi = \eta &= (L^+(\phi^1), H^*\dot{\phi}^1) = F^*(L^+(\phi^1), \dot{\phi}^1) \\ &= F^*A^+\phi \quad \text{for all } \phi \in \mathscr{D}(A^+). \end{aligned}$$

(iii) This follows now easily from (i) and (ii), namely, for $\psi \in \mathscr{D}(A^+)$, $\phi \in \mathscr{D}(A)$

$$(5.17) \quad \begin{aligned} \langle\langle\psi, A\phi\rangle\rangle = \langle\psi, FA\phi\rangle = \langle F^*\psi, A\phi\rangle = \langle A^*F^*\psi, \phi\rangle &= \langle F^*A^+\psi, \phi\rangle \\ &= \langle\langle A^+\psi, \phi\rangle\rangle. \end{aligned}$$

Remark 5.1. The property (iii) is analogous to (3.2) on p. 173 of [**15**], and indicates that A^+ is a "dual" operator of A with respect to the bilinear form $\langle\langle\, ,\, \rangle\rangle$. The difference between the present approach and that of [**15**] is that in [**15**] one has $C = C([-h, 0], R^n)$ and $C^* = C([0, h], R^{n*})$, with A^* of [**15**] defined on $\mathscr{D}(A^*) \subset C^*$, where C^* is not a dual space of C; in the present approach, both A^+ and A^* have their domains in X^*, and both X and X^* involve the interval $[-h, 0]$.

In order to prove similar properties for the semigroups generated by A^* and A^+ we need the following result.

LEMMA 5.3. *Let Y be a Banach space, $K \in \mathscr{L}(Y)$ and B_1, B_2 the infinitesimal generators of the strongly continuous semigroups $\{T_1(t)\}_{t \geqq 0}$ and $\{T_2(t)\}_{t \geqq 0}$ on Y respectively. Then*

$$K\mathscr{D}(B_1) \subset \mathscr{D}(B_2) \quad \textit{and} \quad B_2K = KB_1 \quad \textit{on } \mathscr{D}(B_1)$$

if and only if

$$T_2(t)K = KT_1(t) \quad \textit{for all } t \geqq 0.$$

Proof. To prove this result, we use the following fact [**25**, Theorem 5.5]: if $\{T(t)\}_{t \geqq 0}$ is a strongly continuous semigroup on a Banach space Y, A its infinitesimal generator and A^λ the Yosida approximation of A defined by $A^\lambda \triangleq \lambda^2 R(\lambda; A) - \lambda I$ for $\lambda \in \rho(A)$ (the resolvent set of A), where $R(\lambda; A)$ is the resolvent operator of A, then for all $\xi \in Y$,

$$T(t)\xi = \lim_{\lambda \to +\infty} e^{tA^\lambda}\xi.$$

In fact, since K is a bounded operator this last result gives us immediately that

$$KT_1(t)\psi = \lim_{\lambda \to +\infty} K\, e^{tB_1{}^\lambda}\psi \quad \text{and} \quad T_2(t)K\psi = \lim_{\lambda \to +\infty} e^{tB_2{}^\lambda}K\psi \quad \text{for } t \geqq 0, \; \psi \in Y.$$

Since the resolvent sets $\rho(B_1)$, $\rho(B_2)$ contain (ω, ∞) for some ω [**25**, Corollary 5.3], we have that for $\lambda > \omega$, $KB_1 = B_2K$ on $\mathscr{D}(B_1)$ implies $K(\lambda I - B_1)^{-1} = (\lambda I - B_2)^{-1}K$ on a dense subset of Y, hence on all of Y. This in turn yields $KR^n(\lambda; B_1) = R^n(\lambda; B_2)K$ for $\lambda > \omega$, $n = 1, 2, 3, \ldots$. Using this last identity we obtain, by a series expansion, that $e^{R(\lambda; B_2)}K = Ke^{R(\lambda; B_1)}$. Since $e^{U+V} = e^Ue^V$ if U and V are bounded linear operators such that $UV = VU$, we have

$$e^{tB_2{}^\lambda}K = Ke^{tB_1{}^\lambda}, \; t \geqq 0, \; \lambda > \omega.$$

Hence for all $\psi \in Y$,

$$KT_1(t)\psi = \lim_{\lambda \to \infty} Ke^{tB_1{}^\lambda}\psi = \lim_{\lambda \to \infty} e^{tB_2{}^\lambda}K\psi = T_2(t)K\psi,$$

i.e. $KT_1(t) = T_2(t)K$, for all $t \geqq 0$.

The reverse implication of the lemma follows easily by using only the definition of the infinitesimal generator.

We now present the relations between $S^*(t)$, $S^+(t)$ and $\tilde{S}(-t)$, $t \geqq 0$.

THEOREM 5.4. *The following relations hold:*
(i) $S^*(t)F^* = F^*S^+(t)$ *for all* $t \geqq 0$
(ii) $\langle\langle S^+(t)\psi, \phi\rangle\rangle = \langle\langle \psi, S(t)\phi\rangle\rangle$ *for all* $(\phi, \psi) \in X \times X^*$, *for all* $t \geqq 0$
(iii) $F^*[\mathscr{J}\tilde{S}(-t)\mathscr{J}] = S^*(t)F^*$ *for all* $t \geqq 0$.

Proof. Part (i) follows directly from Lemma 5.3 by taking X^* for Y, $S^*(t)$ and A^* for $T_2(t)$ and B_2, $S^+(t)$, A^+ for $T_1(t)$ and B_1, and F^* for K. Part (ii) results directly from (i) and (3.9), while (iii) follows from (i) and Proposition (5.2).

Remark 5.2. Part (i) of Theorem 5.4 can also be proved by computing directly the semigroups $S^*(t)$, $S^+(t)$ from (3.15), (3.16). In this case, Proposition 5.2 would become a consequence of Theorem 5.4 via the equivalence statement contained in Lemma 5.3.

The explicit characterization of $S^*(t)$, which can be obtained through some rather standard manipulations, is given below:

(5.18) $$S^*(t) = F^*\mathscr{G}_t^* + s^*(t)$$

where

(5.19) $$\begin{cases} [\mathscr{G}_t^*\psi]^1(\theta) = X^T(t+\theta)\psi^0 + \int_{-h}^0 X^T(t+s+\theta)\psi^1(s)ds \\ [\mathscr{G}_t^*\psi]^0 = [\mathscr{G}_t^*\psi]^1(0) \end{cases}$$

$$[s^*(t)\psi]^0 = 0, \quad [s^*(t)\psi]^1(\theta) = \psi^1(\theta - t)\chi_{[0,\theta+h)}(t).$$

Remark 5.3. One interesting consequence of the above theorem is that for $\psi \in \text{Im } F^*$ the semigroup $S^*(t)$ can be replaced by $F^*S^+(t)$, that is

$$\psi \in \text{Im } F^* \Rightarrow \text{there exists } \eta \in X^*, \ \psi = F^*\eta.$$

Then

$$S^*(t)\psi = S^*(t)F^*\eta = F^*S^+(t)\eta.$$

The advantage of this is that $S^+(t)\eta$ can be easily obtained by solving the differential equation (5.1), while $S^*(t)$ in general does not necessarily have that type of property.

In particular, in many optimal control problems involving *FDE* with targets in R^n, the boundary conditions on the adjoint equation (5.6) are of the type $\psi = (\psi^1, 0)$, and such a ψ obviously belongs to Im F^*.

There are several other interesting consequences of the relationships presented above, in particular for the spectral analysis of $S(t)$ in Hilbert space $R^n \times L^2(-h, 0; R^n)$, and for applications of $S(t)$ and $S^*(t)$ to control theory. Some of them are described in [**21**; **12**], while a more comprehensive treatment of spectral analysis will be given in a different paper co-authored by M. C. Delfour and one of the present authors (A.M.).

One can also observe some analogies between the results of this section and those of Henry [**17**], who developed the duality theory for retarded FDE's with $X = C([-h, 0], R^n)$ and $X^* = BV([-h, 0], R^{n*})$. In fact, if F^* is invertible, Theorem 5.4(i) asserts that $S^*(t)$ and $S^+(t)$ are related through a similarity transformation, analogously as in the case studied by Henry. A more interesting fact is that such a similarity transformation always exists, regardless of invertibility of F^*. For more details on this, see [**22**].

Appendix. We collect below proofs of the assertions (i)–(iv) of Proposition 2.1. As stated in the Introduction, the basic properties of $S(t)$ given by Proposition 2.1 were announced in [**5**] without proofs. Since then, several authors relied on those properties, even though the proofs are still not readily available in the published literature. Some proofs, e.g. (iv) for $p = 2$, were given in technical reports [**26**] and [**2**] (the proof given in [**26**] is, in our view, unduly complicated). By presenting our versions of the proofs we do not want, in this situation, to make any claims of novelty, but we merely want to include these proofs for the sake of both completeness and an easy reference for the reader.

The facts that $S(t)$ is a bounded linear operator and that it satisfies the semigroup property follow easily from the existence, uniqueness and continuous dependence of solutions of (2.1) on the initial data.

Strong continuity of $\{S(t)\}_{t \geqq 0}$. We show that

$$S(t)\phi \underset{t \downarrow 0}{\rightarrow} \phi \quad \text{for all } \phi \in X.$$

For $\phi = (\phi^0, \phi^1) \in X$, we have

$$\|S(t)\phi - \phi\|_X = \|x(t) - \phi^0\|_{R^n} + \|x_t - \phi^1\|_{L^p}.$$

But $x(0) = \phi^0$ and $x(\cdot)$ is absolutely continuous in $[0, \infty)$. Then

$$\|x(t) - \phi^0\|_{R^n} \underset{t \downarrow 0}{\rightarrow} 0$$

and

$$\int_{-h}^{0} |x_t(s) - \phi^1(s)|^p ds = \int_{-h}^{0} |x(t+s) - x(s)|^p ds \underset{t \downarrow 0}{\rightarrow} 0$$

where we used in the last step the well-known fact that

$$(A.1) \quad \lim_{\sigma \to 0} \int_a^b |f(s+\sigma) - f(s)|^p ds = 0$$

for $f \in L^p(a - \delta, b + \delta)$, $\delta > 0$ [**16**].

Compactness of $S(t)$, $t \geqq h$. Let $M = \{\phi \in X \mid \|\phi\|_X \leqq K\}$ be a bounded set in X. For fixed $t > 0$, $R(t) : \phi \to x(\cdot\,;\phi) : X \to C([0, t]; R^n)$ is continuous. So $R(t)M$ is bounded in $C([0, t]; R^n)$. Also, $x(\cdot\,;\phi) \in W^{1,p}_{[0,t]}$ with

$$\|x\|_{W^{1,p}} \leqq c(t)\|\phi\|_X$$

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(see (2.3) and its proof at the end of this Appendix), so for $s, s' \in [0, t]$,

$$||x(s') - x(s)||_{R^n} \leqq \int_s^{s'} ||\dot{x}(t)||dt \leqq |s - s'|^{1/q}c(t)K.$$

We thus have that $R(t)M$ is a bounded, equi-continuous family in $C([0, t]; R^n)$, and so $R(t)M$ is relatively compact by Ascoli's theorem. We next define, for $t \geqq h$, the operator $P(t)$: $C([0, t]; R^n) \to X$ given by $P(t)x = (x(t), x_t(\cdot))$, where $x = x(\cdot, \phi)$ is the solution to (2.1) defined and continuous on $[0, t]$. We claim that $P(t)$ is continuous. Indeed, if x, y are two elements of $C([0, t]; R^n)$, $t \geqq h$, such that

$$||x - y||_C < \min \left\{\frac{\epsilon}{2}, \frac{1}{h^{1/p}} \frac{\epsilon}{2}\right\}$$

one easily verifies that $||(x(t), x_t(\cdot)) - (y(t), y_t(\cdot))||_X < \epsilon$, proving that $P(t)$ is continuous. Hence $P(t)[R(t)M]$ is relatively compact and we see that, for $t \geqq h$, $S(t)$ is the composition $P(t)R(t)$.

The infinitesimal generator A of $\{S(t)\}$. We recall that A is defined by

$$A\phi = \lim_{t\downarrow 0} \frac{S(t)\phi - \phi}{t} \quad \text{for } \phi \in \mathscr{D}(A),$$

where $\mathscr{D}(A)$ is precisely the set of points where this last limit exist. We want to show:

$$\mathscr{D}(A) = \{\phi = (\phi^0, \phi^1) \in X | \phi^1 \in W^{1,p}([-h, 0]), \phi^1(0) = \phi^0\}.$$

Let ϕ be in this last set. We first note that $x \in$ A.C. $[-h, T]$ for all $T \geqq 0$; in fact x is absolutely continuous in $[-h, 0]$, and also on $[0, T]$ for all $T > 0$ and $\phi^1(0) = \phi^0 = x(0)$. In particular, x is uniformly continuous on $[-h, T]$. This gives us easily that $s \to x_s(\cdot)$: $[0, T] \to C[-h, 0]$ is continuous. Moreover we know that L, defined in (2.6), is continuous. So $s \to L(x_s)$ is continuous and we have:

$$\frac{x(t) - x(0)}{t} = \frac{1}{t}\int_0^t \dot{x}(s)ds = \frac{1}{t}\int_0^t L(x_s)ds \underset{t\downarrow 0}{\to} L(x_0),$$

i.e. $\lim_{t\downarrow 0} (x(t) - x(0))/t$ exist. Also, since $\dot{x} \in L^p[0, T]$ for all $T > 0$, we have

$$\begin{aligned}\left\|\frac{x_t - \phi^1}{t} - \dot{\phi}^1\right\|_{L^q}^p &= \int_{-h}^0 \left\|\frac{x(t+\theta) - x(\theta)}{t} - \dot{x}(\theta)\right\|_{R^n}^p d\theta \\ &= \int_{-h}^0 \left\|\int_0^t \frac{\dot{x}(s+\theta) - \dot{x}(\theta)}{t} ds\right\|_{R^n}^p d\theta \\ &\leqq \frac{1}{t}\int_{-h}^0 \left[\int_0^t ||\dot{x}(s+\theta) - \dot{x}(\theta)||_{R^n}^p ds\right] d\theta \\ &= \frac{1}{t}\int_0^t \left[\int_{-h}^0 ||\dot{x}(s+\theta) - \dot{x}(\theta)||_{R^n}^p d\theta\right] ds \\ &\leqq \epsilon \quad \text{for } t \text{ sufficiently small}\end{aligned}$$

where we used successively Hölder's inequality, Fubini's theorem and relation (A.1). Hence $\phi \in \mathscr{D}(A)$. Conversely if $\phi \in \mathscr{D}(A)$, there exists $(z^0, z^1) \in X$ such that

$$\left\| \frac{x(t) - \phi^0}{t} - z^0 \right\|_{R^n} \underset{t \downarrow 0}{\longrightarrow} 0$$

and

$$\left\| \frac{x_t - \phi^1}{t} - z^1 \right\|_{L^p} \underset{t \downarrow 0}{\longrightarrow} 0.$$

Hence, for $\alpha, \beta \in [-h, 0]$,

$$\frac{1}{t}\int_\beta^{\beta+t} x(s)ds - \frac{1}{t}\int_\alpha^{\alpha+t} x(s)ds = \frac{1}{t}\int_{\alpha+t}^{\beta+t} x(s)ds - \frac{1}{t}\int_\alpha^\beta x(s)ds$$
$$= \int_\alpha^\beta \left[\frac{x(s+t) - x(s)}{t}\right] ds \underset{t \downarrow 0}{\longrightarrow} \int_\alpha^\beta z^1(s)ds.$$

And we know that for almost all $\alpha \in [-h, 0]$,

$$\frac{1}{t}\int_\alpha^{\alpha+t} x(s)ds \underset{t \downarrow 0}{\longrightarrow} x(\alpha) \quad [\mathbf{16}].$$

So if we take such an $\alpha \in [-r, 0]$, we have for almost all $\beta \in [-h, 0]$ that

$$x(\beta) - x(\alpha) = \int_\alpha^\beta z^1(s)ds;$$

Let us redefine $x(t)$ on a set of null measure to obtain:

$$x(t) \equiv \int_\alpha^t z^1(s)ds + x(\alpha).$$

In this manner $x(\cdot) \in$ A.C. $[-h, 0]$ and has a.e. a derivative equal to $z^1 \in L^p[-h, 0]$. We next check that $\phi^1(0) = \phi^0$. We have already seen that for $\alpha, \beta \in [-r, 0]$,

$$\frac{1}{t}\int_\beta^{\beta+t} x(\theta)d\theta - \frac{1}{t}\int_\alpha^{\alpha+t} x(\theta)d\theta$$
$$= \int_\alpha^\beta \frac{x(t+\theta) - x(\theta)}{t} d\theta \underset{t \downarrow 0}{\longrightarrow} \int_\alpha^\beta \dot{\phi}^1(\theta)d\theta.$$

So if $\beta = 0$ and $\alpha \in [-h, 0]$ we obtain

$$\frac{1}{t}\int_0^t x(\theta)d\theta - \frac{1}{t}\int_\alpha^{\alpha+t} x(s)ds \underset{t \downarrow 0}{\longrightarrow} \phi^1(0) - \phi^1(\alpha).$$

But

$$\frac{1}{t}\int_0^t x(\theta)d\theta \underset{t \downarrow 0}{\longrightarrow} x(0) = \phi^0$$

since x is continuous for $t \geqq 0$. And

$$\frac{1}{t}\int_{\alpha}^{\alpha+t} x(s)ds \underset{t\downarrow 0}{\rightarrow} \phi^1(\alpha)$$

since $\phi^1 \in$ A.C. $[-h, 0]$. So $\phi^1(0) = \phi^0$, and $\phi^1 \in W^{1,p}[-h, 0]$. Moreover, we have seen in this demonstration that for $\phi \in \mathscr{D}(A)$,

$$A\phi = (\dot{x}(0^+), \dot{\phi}^1) = (L(\phi^1), \dot{\phi}^1).$$

Proof of estimate (2.3). The estimate was given in [**13**, Theorem 3.1 ii]. We supply below a brief proof of (2.3).

From the variation of constants formula (3.11) it is obvious that the mapping $\phi \rightarrow x(\cdot\,;\phi)$, $X \rightarrow C(0, T; R^n)$ is continuous. That is there exists a constant c_1 (depending on $T, A_0, \ldots, A_N, A_{01}(\cdot)$) such that $||x(\cdot\,,\phi)||_C \leqq c_1||\phi||_X$. Since $\phi^1 \in L^p(0, T; R^n)$, by splitting the right hand side of (2.1) into two parts, one depending on $x(t)$ for $t \geqq 0$, and one depending on ϕ, we observe that the function $t \rightarrow \dot{x}(t)$, $t \in [0, T]$, is in $L^p(0, T; R^n)$ and there exist constants c_2, c_3 (again depending on $T, A_0, \ldots, A_N, A_{01}(\cdot)$) such that

$$||\dot{x}(\cdot\,;\phi)||_{L^p} \leqq c_2||\phi||_X + c_3||x(\cdot\,;\phi)||_C \leqq c_4||\phi||_X.$$

where $c_4 = c_2 + c_1c_3$. Now (2.3) follows with $c = \max(c_1, c_4)$.

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