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COEFFICIENTS OF AN ANALYTIC FUNCTION SUBORDINATION CLASS DETERMINED BY ROTATIONS

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Abstract

Let $\mathscr{A}$ denote the set of all functions analytic in $U = \{z : |z| < 1\}$ equipped with the topology of uniform convergence on compact subsets of U. For $F \in \mathscr{A}$ define

$$s(F) = \{F \circ \phi : \phi \in \mathscr{A} \text{ and } |\phi(z)| \le |z|\}.$$

Let $\overline{\text{co}}\, s(F)$ and $\mathscr{E}\overline{\text{co}}\, s(F)$ denote the closed convex hull of $s(F)$ and the set of extreme points of $\overline{\text{co}}\, s(F)$, respectively.

Let $\mathscr{R}$ denote the class of all $F \in \mathscr{A}$ such that $\mathscr{E}\overline{\text{co}}\, s(F) = \{F_x : |x| = 1\}$ where $F_x(z) = F(xz)$.

We prove that $|A_N| \le |A_{MN}|$ for all positive integers M and N, and $(2\sqrt{2}/3)|A_2| \le |A_3|$ for $F(z) = \sum_{N=0}^{\infty} A_N z^N \in \mathscr{R}$. We also prove that if $F(z) = \sum_{N=0}^{\infty} A_N z^N \in \mathscr{R}$ and $|A_1| = |A_2|$, then F is a univalent halfplane mapping.

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1. Introduction

Let $\mathscr{A}$ denote the set of all functions analytic in $U = \{z : |z| < 1\}$. $\mathscr{A}$ is a linear topological space with respect to the topology of uniform convergence on compact subsets of U. Let $F \in \mathscr{A}$ and let $s(F)$ denote the set of all $f \in \mathscr{A}$ such that f is subordinate to F. A function f in $\mathscr{A}$ is *subordinate* to F (written $f \prec F$) if there exists $\phi \in \mathscr{B}_0 = \{\phi \in \mathscr{A} : |\phi(z)| \le |z| \text{ for all } z \in U\}$ such that $f = F \circ \phi$. Let $\overline{\text{co}}\, s(F)$ and $\mathscr{E}\overline{\text{co}}\, s(F)$ denote the closed convex hull of $s(F)$ and the set of extreme points of $\overline{\text{co}}\, s(F)$, respectively.

Let $\mathscr{F}$ be a compact subset of $\mathscr{A}$. A function $f \in \mathscr{F}$ is called a support point of $\mathscr{F}$ if there is a continuous linear functional J on $\mathscr{A}$ such that f maximizes Re J over $\mathscr{F}$

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and Re J is nonconstant on $\mathcal{F}$, that is Re $J(f) = \max\{\text{Re } J(g) : g \in \mathcal{F}\}$ and Re J is nonconstant on $\mathcal{F}$. We use $\Sigma\mathcal{F}$ to denote the set of support points of $\mathcal{F}$.

Let $\mathcal{R}$ denote the class of all $F \in \mathcal{A}$ such that $\overline{\text{co}}\, s(F) = \{\int_\Gamma F(xz)\, d\mu(x) : \mu \in \Lambda\}$ where Λ denotes the set of all probability measures on $\Gamma = \{z : |z| = 1\}$. It is worthy of note that $F \in \mathcal{R}$ if and only if $\mathcal{E}\overline{\text{co}}\, s(F) = \{F_x : |x| = 1\}$ where $F_x(z) = F(xz)$. We will show this in Lemma 1.

The problem of finding the general conditions for F to be in $\mathcal{R}$ was posed by T. Sheil-Small. Many examples were shown to be in $\mathcal{R}$ by various authors ([2, 3, 4, 6, 9, 10]).

The aim of this paper is to find coefficient conditions for $F(z) = \sum_{N=0}^{\infty} A_N z^N$ to be in $\mathcal{R}$. In [8], D. J. Hallenbeck, S. Perera and D. R. Wilken proved that if $F(z) = \sum_{N=0}^{\infty} A_N z^N \in \mathcal{R}$ and if $A_N \neq 0$, where $N \geq 1$, then $A_M \neq 0$ for every $M \geq N$. Here we prove that $|A_N| \leq |A_{MN}|$ for all positive integers M and N, and $2\sqrt{2}/3|A_2| \leq |A_3|$ for $F(z) = \sum_{N=0}^{\infty} A_N z^N \in \mathcal{R}$. We also prove that if $F(z) = \sum_{N=0}^{\infty} A_N z^N \in \mathcal{R}$ and $|A_1| = |A_2|$, then F is a univalent halfplane mapping.

From the definition of $\mathcal{R}$ we have the following.

FACT 1. $F \in \mathcal{R}$ if and only if $aF + b \in \mathcal{R}$ for all numbers $a,\ b \in \mathbf{C}$.

FACT 2. $F \in \mathcal{R}$ if and only if $F_x \in \mathcal{R}$, $|x| = 1$.

So, $F \in \mathcal{R}$ if and only if $e^{i\eta} F(e^{i\theta} z) \in \mathcal{R}$ for all real η, θ.

LEMMA 1. *A nonconstant $F \in \mathcal{A}$ is in $\mathcal{R}$ if and only if $\mathcal{E}\overline{\text{co}}\, s(F) = \{F_x : |x| = 1\}$.*

PROOF. The sufficiency is obtained by Theorem 1 of [5] and Theorem 5.5 of [7]. Next, we have (with $\mathcal{F} = s(F)$ in [7, p.92])

$$\overline{\text{co}}\,(\Sigma s(F) \cap \mathcal{E}\overline{\text{co}}\, s(F)) = \overline{\text{co}}\, s(F).$$

To show $F \in \mathcal{R}$, it is enough to show

$$\Sigma s(F) \cap \mathcal{E}\overline{\text{co}}\, s(F) \subset \left\{ \int_\Gamma F(xz) d\mu(x) : \mu \in \Lambda \right\}.$$

If $f \in \Sigma s(F)$, $f = F \circ B$ with B a finite Blaschke product (in [7, p.166]) and $f = F \circ B \in \mathcal{E}\overline{\text{co}}\, s(F) = \{F_x : |x| = 1\}$ implies $f = F_x$ and $f \in \{\int_\Gamma F(xz) d\mu(x) : \mu \in \Lambda\}$.

LEMMA 2. *Let $F \in \mathcal{A}$. If there is a continuous linear functional J and $\varphi \in \mathcal{B}_0$ such that Re $J(F(\varphi)) >$ Re $J(F_x)$ for all $|x| = 1$, then $f \notin \mathcal{R}$.*

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PROOF. If $F \in \mathscr{R}$, then

$$\overline{\text{co}}\, s(F) = \left\{ \int_\Gamma F(xz)d\mu(x) : \mu \in \Lambda \right\}.$$

So, for any $\varphi \in \mathscr{B}_0$, we have

$$F \circ \varphi \in \left\{ \int_\Gamma F(xz)d\mu(x) : \mu \in \Lambda \right\}.$$

Thus, for any continuous linear functional J on $\mathscr{A}$, we have

$$\begin{aligned}\text{Re } J(F \circ \varphi) &\leq \max_{\mu\in\Lambda} \text{Re } J\left(\int_\Gamma F(xz)d\mu(x) \right) \\ &= \max_{\mu\in\Lambda} \int_\Gamma \text{Re } J(F(xz))d\mu(x) = \max_{|x|=1} \text{Re } J(F_x).\end{aligned}$$

2. Coefficients of elements of the class $\mathscr{R}$

In this section, we show that, if $F(z) = \sum_{N=0}^{\infty} A_N z^N \in \mathscr{R}$, $|A_N| \leq |A_{MN}|$ for every $M, N = 1, 2, 3, \ldots$ and we have that if $|c+1| < 1$ then $\exp c((1+z)/(1-z)) \notin \mathscr{R}$ as a corollary. We also show that $(2\sqrt{2}/3)|A_2| \leq |A_3|$.

LEMMA 3. *If* $F(z) = \sum_{N=1}^{\infty} A_N z^N \in \mathscr{R}$, *then*

$$|A_N| \leq |A_{MN}|, \qquad M, N = 1, 2, 3, \ldots.$$

PROOF. Since $F \in \mathscr{R}$, for every $\varphi \in \mathscr{B}_0$, there is $\mu \in \Lambda$ such that

$$F(\varphi(z)) = \int_\Gamma F(xz)d\mu(x).$$

Take $\varphi(z) = z^M$. Then

$$F(z^M) = \int_\Gamma F(xz)d\mu(x) \quad \text{for some} \quad \mu \in \Lambda, \quad \text{that is}$$

$$\sum_{N=1}^{\infty} A_N z^{MN} = \int_\Gamma \left(\sum_{N=1}^{\infty} A_N x^N z^N \right) d\mu(x) = \sum_{N=1}^{\infty} A_N \left(\int_\Gamma x^N d\mu(x) \right) \cdot z^N.$$

By considering the coefficient of z^{MN}, we have

$$A_N = A_{MN} \int_\Gamma x^{MN} d\mu(x).$$

Hence

$$|A_N| \leq |A_{MN}| \int_\Gamma \left| x^{MN} \right| d\mu(x) = |A_{MN}|.$$

COROLLARY 1. *If* $F = \sum_{N=0}^{\infty} A_N z^N \in \mathscr{R}$, *then* $|A_1| \leq |A_M|$ *for all* $M = 1, 2, 3, \ldots$.

PROOF. Let $N = 1$ in Lemma 3.

Although the following lemma was proved in [6], we give a shorter proof by using the closedness of $\mathscr{R}$.

LEMMA 4. *If* $\operatorname{Re} c \geq 0$, *then* $\exp(c(1+z)/(1-z)) \in \mathscr{R}$.

PROOF. Note $(1 + w/N)^N$ converges uniformly on compact subsets of U to $\exp w$ as N goes to ∞. Let $\operatorname{Re} c \geq 0$. By a simple calculation we see $\exp(c(1+z)/(1-z))$ is the limit of

$$f_N = \left(\frac{1 + ((c-N)/(c+N))z}{1-z}\right)^N \cdot \left(\frac{c+N}{N}\right)^N, \quad N = 1, 2, 3, \ldots$$

each of which is in $\mathscr{R}$ ([4]), since $|(c-N)/(c+N)| \leq 1$. Since $\mathscr{R}$ is closed ([8]), the limit function $\exp(c(1+z)/(1-z))$ is in $\mathscr{R}$.

If $c < 0$, then $\exp(c(1+z)/(1-z)) \in H^1$ so that $\exp(c(1+z)/(1-z)) \notin \mathscr{R}$ ([1]). So we conjecture:

$$\exp\left(c\frac{1+z}{1-z}\right) \notin \mathscr{R} \quad \text{if} \quad \operatorname{Re} c < 0.$$

Corollary 2 is a partial solution for this.

COROLLARY 2. *If* $|c+1| < 1$, *then* $\exp(c(1+z)/(1-z)) \notin \mathscr{R}$.

PROOF. Suppose $F(z) = \exp(c(1+z)/(1-z)) = \sum_{N=0}^{\infty} A_N z^N \in \mathscr{R}$. Then we have

$$A_1 = 2c \exp c \quad \text{and} \quad A_2 = \frac{1}{2} \cdot (4c^2 + 4c) \exp c$$

by simple calculation. By Corollary 1,

$$|A_1| \leq |A_2|, \quad \text{that is} \quad 2|c||\exp c| \leq \frac{1}{2}\left|4c^2 + 4c\right| |\exp c| \quad \text{or} \quad 1 \leq |c+1|.$$

This proves the corollary.

To show $(2\sqrt{2}/3)|A_2| \leq |A_3|$, we need a technical lemma;

LEMMA 5. *If* $r\cos\Phi > \frac{1}{2}$, $0 < r < 1$, *then there exists* θ *such that*

$$2\cos\theta - r\cos(\Phi + 2\theta) \geq \sqrt{2}.$$

PROOF. First, we assume $0 \leq \Phi < \pi/3$. Let

$$\begin{aligned} f(\theta) &= 2\cos\theta - r\cos(\Phi + 2\theta) \\ &= 2\cos\theta - 2r\cos\Phi \cdot \cos^2\theta + r\cos\Phi + 2r\sin\Phi\sin\theta\cos\theta. \end{aligned}$$

Let $\theta = \cos^{-1}(1/2r\cos\Phi)$ with $0 < \theta < \pi/2$. Then $\cos\theta = 1/2r\cos\Phi$ and $\sin\theta > 0$. Hence

$$\begin{aligned} f(\theta) &\geq \frac{1}{r\cos\Phi} - \frac{1}{2r\cos\Phi} + r\cos\Phi \\ &= \left(\frac{1}{\sqrt{2r\cos\Phi}} - \sqrt{r\cos\Phi}\right)^2 + \frac{2}{\sqrt{2}} \geq \frac{2}{\sqrt{2}} = \sqrt{2}. \end{aligned}$$

Similarly, we can choose θ with $-\pi/2 < \theta < 0$ for the case $-\pi/3 < \Phi < 0$.

REMARK. If $F(z) = A_1z + A_2z^2 + A_3z^3 + \cdots$ and $\varphi(z) = b_1z + b_2z^2 + b_3z^3 + \cdots$, then

$$\begin{aligned} F(\varphi(z)) &= A_1(b_1z + b_2z^2 + b_3z^3 + \cdots) + A_2(b_1z + b_2z^2 + b_3z^3 + \cdots)^2 \\ &\quad + A_3(b_1z + b_2z^2 + b_3z^3 + \cdots)^3 + \cdots \\ &= A_1b_1z + (A_1b_2 + A_2b_1^2)z^2 + (A_1b_3 + 2A_2b_1b_2 + A_3b_1^3)z^3 + \cdots. \end{aligned}$$

THEOREM 1. *If* $F(z) = \sum_{N=1}^{\infty} A_N z^N = A_1z + A_2z^2 + A_3z^3 + \cdots \in \mathscr{R}$, *then*

$$\frac{2\sqrt{2}}{3}|A_2| \leq |A_3|.$$

PROOF. We may assume $A_2 \neq 0$ so $A_3 \neq 0$ ([8]).

By the Facts 1 and 2 in §1, $F \in \mathscr{R}$ if and only if $aF(xz) \in \mathscr{R}$ for all $a \in \mathbf{C}$, $|x| = 1$. Take

$$a = \frac{\overline{A}_2^3}{|A_2|^4}\frac{A_3^2}{|A_3|^2} \quad \text{and} \quad x = \frac{A_2}{|A_2|} \cdot \frac{\overline{A}_3}{|A_3|},$$

then

$$\left[z^2\text{-coefficient of } aF(xz)\right] = aA_2x^2 = \frac{\overline{A}_2^3}{|A_2|^4}\frac{A_3^2}{|A_3|^2} \cdot A_2 \cdot \frac{A_2^2}{|A_2|^2}\frac{\overline{A}_3^2}{|A_3|^2} = 1$$

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and

$$\left[z^3\text{-coefficient of } aF(xz)\right] = aA_3x^3 = \frac{\overline{A}_2^3}{|A_2|^4} \cdot \frac{A_3^2}{|A_3|^2} \cdot A_3 \cdot \frac{A_2^3}{|A_2|^3} \cdot \frac{\overline{A}_3^3}{|A_3|^3}$$
$$= \frac{1}{|A_2|} \cdot |A_3| > 0.$$

Let $A_2 = 1$, $A_3 > 0$ and it suffices to show that $A_3 \geq 2\sqrt{2}/3$. Let $A_1 = re^{i\Phi}$ and suppose $A_3 < 2\sqrt{2}/3$. Then by Corollary 1 we have $r \leq A_3 < 2\sqrt{2}/3 < 1$. We define a continuous linear functional J on $\mathscr{A}$ by $J(f) = a_3/A_3$ where $f(z) = \sum_{N=0}^{\infty} a_N z^N \in \mathscr{A}$. Then

$$\max_{|x|=1} \operatorname{Re} J(F_x) = \max_{|x|=1} \operatorname{Re} \frac{1}{A_3} \cdot A_3x^3 = 1 .$$

We will see that there is a $\varphi \in \mathscr{B}_0$ such that

$$\operatorname{Re} J(F \circ \varphi) > 1 \quad \text{if} \quad A_3 < \frac{2\sqrt{2}}{3},$$

which will prove $F \notin \mathscr{R}$ by Lemma 2.

We have two cases

(i) $\operatorname{Re} A_1 \leq 1/2$.

(ii) $\operatorname{Re} A_1 > 1/2$, that is $1/2 < r\cos\Phi$ and $1/2 < r < A_3 < 2\sqrt{2}/3$.

Case (i) Consider

$$\varphi(z) = \sum_{n=1}^{\infty} b_n z^n = z\frac{z+\alpha}{1+\overline{\alpha}z} = \alpha z + \left(1 - |\alpha|^2\right)z^2 + \overline{\alpha}\left(|\alpha|^2 - 1\right)z^3 + \cdots$$

Let $\alpha = 1 - \varepsilon,\ 0 < \varepsilon < 1$, then

$$b_1 = 1 - \varepsilon, \qquad b_2 = 2\varepsilon - \varepsilon^2, \qquad b_3 = -2\varepsilon + 3\varepsilon^2 - \varepsilon^3.$$

From the remark before the Theorem 1, we have

$$J(F(\varphi)) = \frac{1}{A_3}\left(A_1b_3 + 2b_1b_2 + A_3b_1^3\right) = b_1^3 + \frac{2}{A_3}b_1b_2 + \frac{A_1}{A_3}b_3$$
$$= (1-\varepsilon)^3 + \frac{2}{A_3}(1-\varepsilon)\left(2\varepsilon - \varepsilon^2\right) + \frac{A_1}{A_3}\left(-2\varepsilon + 3\varepsilon^2 - \varepsilon^3\right)$$

So,

$$\operatorname{Re} J(F(\varphi)) - 1 = -3\varepsilon + \frac{2\varepsilon}{A_3}\left[2 - \operatorname{Re} A_1\right] + \mathscr{O}(\varepsilon^2)$$

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where $\mathscr{O}(\varepsilon^2)$ is such that

$$\lim_{\varepsilon\to O^+} \frac{\mathscr{O}(\varepsilon^2)}{\varepsilon^2}$$

is finite.

If Re $A_1 \leq 1/2$, since $A_3 < 1$, there is $\varepsilon > 0$ such that

$$\operatorname{Re} J\,(F(\varphi)) > 1 = \max_{|x|=1} J\,(F_x(z))\,.$$

Case (ii) Consider $\varphi_1(z) = e^{-i\theta}\varphi(e^{i\theta z})$. Let $\alpha = 1-\varepsilon,\ 0<\varepsilon<1$, then

$$b_1 = 1-\varepsilon, \qquad b_2 = (2\varepsilon-\varepsilon^2)e^{i\theta}, \qquad b_3 = -(2\varepsilon - 3\varepsilon^2+\varepsilon^3)e^{i\cdot 2\theta}.$$

Again from the remark we have

$$\begin{aligned} J(F(\varphi_1)) &= b_1^3 + \frac{2}{A_3}b_1b_2 + \frac{A_1}{A_3}b_3 \\ &= (1-\varepsilon)^3 + \frac{2}{A_3}(2\varepsilon-3\varepsilon^2+\varepsilon^3)e^{i\theta} - \frac{A_1}{A_3}(2\varepsilon-3\varepsilon^2+\varepsilon^3)e^{i2\theta}. \end{aligned}$$

So,

$$\begin{aligned} \operatorname{Re} J(F(\varphi_1)) - 1 &= -3\varepsilon + \frac{4\varepsilon}{A_3}\operatorname{Re} e^{i\theta} - \frac{2\varepsilon}{A_3}\operatorname{Re} A_1e^{i2\theta} + \mathscr{O}(\varepsilon^2) \\ &= \varepsilon\left[-3+\frac{2}{A_3}\,(2\cos\theta - r\cos(\Phi+2\theta))\right] + \mathscr{O}(\varepsilon^2). \end{aligned}$$

By Lemma 5, there exist $\varepsilon > 0$ and θ such that

$$\operatorname{Re} J(F(\varphi_1)) - 1 > 0,$$

which proves $F \notin \mathscr{R}$ in case (ii).

3. Univalent halfplane mapping

In this section we prove that if $F(z) = \sum_{N=0}^{\infty} A_N z^N \in \mathscr{R}$ satisfies $|A_1| = |A_2|$, then F is a univalent halfplane mapping. By the facts in §1 we may assume $A_0 = 0$ and $A_1 = A_2 = 1$ without loss of generality. We will show $A_N = 1$ for all $N = 3, 4, 5, \ldots$.

By the definition of $\mathscr{R}$, for every $\varphi \in \mathscr{B}_0$, there corresponds a $\mu \in \Lambda$ such that $F(\varphi(z)) = \int_\Gamma F(xz)d\mu(x)$. For $F(z) = z + z^2 + \sum_{N=3}^{\infty} A_N z^N \in \mathscr{R}$, the probability measure μ which corresponds to $\varphi(z) = z(z+\varepsilon)/(1+\varepsilon z)$, $-1<\varepsilon<1$, is given as in the following lemma.

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LEMMA 6. *If* $F(z) = z + z^2 + \sum_{N=3}^{\infty} A_N z^N \in \mathscr{R}$, *the probability measure* μ *which corresponds to* $\varphi(z) = z(z+\varepsilon)/(1+\varepsilon z)$, $-1 < \varepsilon < 1$, *is*

$$\mu = \left(\frac{1+\varepsilon}{2}\right)\delta_1 + \left(\frac{1-\varepsilon}{2}\right)\delta_{-1},$$

where δ_x *is point mass at* x.

PROOF. Let $F(z) = z + z^2 + \sum_{N=3}^{\infty} A_N z^N \in \mathscr{R}$. If μ is the probability measure corresponding to $\varphi(z) = z(z+\varepsilon)/(1+\varepsilon z) \in \mathscr{B}_0$, then

$$F\left(z\frac{z+\varepsilon}{1+\varepsilon z}\right) = \int_\Gamma (xz + x^2z^2 + \sum_{N=3}^{\infty} A_N x^N z^N) d\mu(x) \quad \text{that is}$$

$$z\frac{z+\varepsilon}{1+\varepsilon z} + \left(z\frac{z+\varepsilon}{1+\varepsilon z}\right)^2 + \cdots = \int_\Gamma x d\mu(x)\cdot z + \int_\Gamma x^2 d\mu(x)\cdot z^2 + \cdots .$$

By comparing the coefficients of the $z-$ve and z^2-ve terms, we have

$$\int_\Gamma x d\mu(x) = \varepsilon \quad \text{and} \quad \int_\Gamma x^2 d\mu(x) = 1.$$

Let $A = \{1, -1\}$, $B = \Gamma \backslash A$. Suppose $0 < \mu(B) \le 1$. Then there are a positive number η and a subset B_0 of B such that $0 < \mu(B_0)$ and $B_0 = \{x \in \Gamma : |\mathrm{Im} x| \ge \sin\eta\}$. (Note : $0 < \eta < \pi/2$). Then

$$\begin{aligned} 1 &= \mathrm{Re}\int_\Gamma x^2 d\mu(x) = \mathrm{Re}\int_{B_0} x^2 d\mu(x) + \mathrm{Re}\int_{\Gamma\backslash B_0} x^2 d\mu(x) \\ &\le \left[\max_{x\in B_0}\mathrm{Re}\, x^2\right]\mu(B_0) + \mu(\Gamma\backslash B_0) \le \sqrt{1-\sin^2\frac{\eta}{2}}\,\mu(B_0) + \mu(\Gamma\backslash B_0) \\ &< \mu(B_0) + \mu(\Gamma\backslash B_0) = 1. \end{aligned}$$

This contradiction gives $\mu(B) = 0$ and $\mu(A) = 1$. Thus $\mu = \lambda\mu_1 + (1-\lambda)\mu_{-1}$ with $0 \le \lambda \le 1$.

Now, $\int_\Gamma x d\mu(x) = \varepsilon$ gives $\lambda = (1+\varepsilon)/2$, which implies the lemma.

THEOREM 2. *If* $F(z) = z + z^2 + \sum_{N=3}^{\infty} A_N z^N \in \mathscr{R}$, *then* $A_N = 1$ *for all* $N = 3, 4, 5, \ldots$, *so that* F *is a univalent halfplane mapping.*

PROOF. By Lemma 6, every $F(z) = z + z^2 + \sum_{N=3}^{\infty} A_N z^N$ in $\mathscr{R}$ satisfies

$$(*) \qquad F\left(z\frac{z+\varepsilon}{1+\varepsilon z}\right) = F(z) + \frac{1-\varepsilon}{2}\{F(-z) - F(z)\}.$$

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From (*) we have, by differentiating twice with respect to ε,

$$F''\left(z\frac{z+\varepsilon}{1+\varepsilon z}\right)\frac{1-z^2}{1+\varepsilon z}-2F'\left(z\frac{z+\varepsilon}{1+\varepsilon z}\right)=0. \tag{**}$$

Continuing differentiation, we have

$$F^{(N+1)}\left(z\frac{z+\varepsilon}{1+\varepsilon z}\right)\frac{1-z^2}{1+\varepsilon z}-(N+1)F^{(N)}\left(z\frac{z+\varepsilon}{1+\varepsilon z}\right)=0.$$

Let $z=0$. Then we have

$$F^{(N+1)}(0)-(N+1)F^{(N)}(0)=0,$$

which implies $A_N=1$ for all $N=3,4,5,\ldots$.

COROLLARY 3. *If* $F(z)=\sum_{N=0}^{\infty}A_Nz^N\in\mathscr{R}$ *and* $|A_1|=|A_{2N}|=1$ *for some positive integer* N, F *is a univalent halfplane mapping.*

PROOF. By Lemma 3, we have $|A_1|=|A_2|=1$.

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