ON THE ENUMERATION OF ROOTED NON-SEPARABLE PLANAR MAPS

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1. Introduction. It has been shown elsewhere **(1, 4)** that the number of rooted non-separable planar maps with n edges is

$$\frac{2(3n-3)!}{n!(2n-1)!}.$$

In the present paper we improve upon this result by finding the number $f_{i,j}$ of rooted non-separable planar maps with $i+1$ vertices and $j+1$ faces. We use the definitions of **(1)**.

Among the non-separable planar maps only the loop-map and the link-map have $i = 0$ or $j = 0$. We therefore confine our attention to the case in which i and j are both positive.

2. An empirical formula. A catalogue of rooted non-separable planar maps was constructed, listing all such maps with $i + j \leqslant 8$. It was observed that the formula

$$f_{i,j} = \frac{(2i+j-2)!\,(2j+i-2)!}{i!j!\,(2i-1)!\,(2j-1)!} \tag{2.1}$$

was valid in this range.

This empirical formula was presented by one of us at the International Congress of Mathematicians in Stockholm (1962).

It is natural to suppose that (2.1) is valid for all positive integers i and j. On this assumption it is possible to interpret the series

$$f(x, y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f_{i,j}\, x^i y^j \tag{2.2}$$

as a Lagrangian expansion involving two variables. This approach leads to the conclusion that $f(x, y)$ is the Taylor expansion, in powers of x and y, of the analytic function

$$uv(1-u-v),$$

where u and v are the functions of x and y, analytic in a sufficiently small neighbourhood of $(0, 0)$, determined parametrically by the following equations

$$\begin{cases} x = u(1-v)^2, \\ y = v(1-u)^2, \\ (u, v) = (0, 0) \quad \text{if } (x, y) = (0, 0); \end{cases} \tag{2.3}$$

see **(3**, I, §188 and II, Part I, §104**)**.

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We prove this parametric formula for $f(x, y)$, which so far is based only on observation and conjecture, in §3. We do this by establishing a functional equation for a power series $h(x, y, z)$ in three variables with the property that $h(x, y, 1) = f(x, y)$. A situation is then encountered which, in our experience, is typical of the enumerative theory of planar maps. There is no evident direct method of solving the functional equation, but if the function $f(x, y)$ can be guessed correctly, then the equation can be used to verify the guess and to determine $h(x, y, z)$. In §3 we thus prove that

$$f(x, y) = uv(1 - u - v).$$

In a forthcoming paper by one of us a more direct method of solving such functional equations will be described.

In §4 we use (2.3) to determine the coefficients $f_{i,j}$, thereby establishing the general validity of (2.1).

3. Proof of the conjecture. Let M be any rooted non-separable planar map. We refer to the face incident on the left with the root as the *external* face of M. We say that M is of type $\{n, m, j\}$ if it has n edges and $j + 1$ faces, and if the valency of the external face is m. Let $w_{n,m,j}$ be the number of such maps. We define as formal power series

$$w_{.m.}(x, z) = \sum_{n=2}^{\infty} \sum_{j=1}^{\infty} w_{n,m,j}\, x^n z^j, \tag{3.1}$$

$$w(x, y, z) = \sum_{m=2}^{\infty} w_{.m.}(x, z) y^m. \tag{3.2}$$

We now apply the argument of **(1**, §3**)** in a slightly more general form. As in that discussion we observe that M decomposes uniquely into its root and a sequence of $s + 1 \geqslant 1$ rooted non-separable planar maps M_α $(0 \leqslant \alpha \leqslant s)$. We write the type of M_α, as $\{n_\alpha, m_\alpha, j_\alpha\}$. The *link-map*, or map of one edge with distinct ends, considered to be of type $\{1, 2, 0\}$, may appear in this sequence; cf. **(1**, Figure 1**)**.

With each member M_α of the sequence we associate an index r_α. This is the number of edges which M_α contributes to the boundary of the external face of M.

We observe that the following conditions are satisfied:

$$\begin{aligned} &0 < r_\alpha < m_\alpha, \\ &1 - \delta_{m_\alpha,2} < n_\alpha, \qquad \alpha = 0, 1, \ldots, s \\ &\;- \delta_{m_\alpha,2} < j_\alpha, \\ &\sum_{\alpha=0}^{s} r_\alpha = m - 1, \\ &\sum_{\alpha=0}^{s} n_\alpha = n - 1, \\ &\sum_{\alpha=0}^{s} j_\alpha = j - 1. \end{aligned} \tag{3.3}$$

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Conversely, suppose we are given a sequence $(M_0, M_1, \ldots, M_s)$ of $s + 1 \geqslant 1$ rooted non-separable planar maps, possibly including link-maps.

Suppose also that M_α $(0 \leqslant \alpha \leqslant s)$, is of type $\{n_\alpha, m_\alpha, j_\alpha\}$, that M_α has an associated index r_α, and that conditions (3.3) are satisfied. Then the sequence of maps M_α, together with the indices r_α, determines a rooted non-separable map of type $\{n, m, j\}$.

We deduce that

$$w_{n,m,j} = \sum_{s=0}^{\infty} \sum \prod_{\alpha=0}^{s} (w_{n_\alpha,m_\alpha,j_\alpha} + \delta_{n_\alpha,1}\,\delta_{m_\alpha,2}\,\delta_{j_\alpha,0}),$$

where the second summation is taken over all ordered sets of integers

$$(n_0, n_1, \ldots, n_s; m_0, m_1, \ldots, m_s; j_0, j_1, \ldots, j_s; r_0, r_1, \ldots, r_s)$$

satisfying (3.3), for each s. This formula may be rewritten as

$$w(x, y, z) = xyz \sum_{s=0}^{\infty} \left\{ \sum_{m=2}^{\infty} w_{.m.}(x, z)[y + y^2 + \ldots + y^{m-1}] + xy \right\}^{s+1}.$$

This equation can be simplified by the method used for **(1,** (3.3)**)**. We then obtain

$$(3.4)\quad w^2(x, y, z) + [(1 - y)(1 - xy) + xyz - yw(x, 1, z)]\, w(x, y, z) - xy^2z[x(1 - y) + w(x, 1, z)] = 0.$$

Let $h_{i,j,m}$ be the number of rooted non-separable planar maps, excluding link-maps, such that the number of vertices is $i + 1$, the number of faces is $j + 1$, and the valency of the external face is m. By the Euler polyhedron formula we have

$$(3.5)\qquad h_{i,j,m} = w_{i+j,m,j}.$$

We define the following formal power series:

$$(3.6)\qquad h_{.j.}(x, z) = \sum_{i=1}^{\infty} \sum_{m=2}^{\infty} h_{i,j,m}\, x^i z^m,$$

$$(3.7)\qquad h(x, y, z) = \sum_{j=1}^{\infty} h_{.j.}(x, z) y^j.$$

Thus $h(x, y, z) = w(x, z, x^{-1}y)$, by (3.5). Applying this result to (3.4) we obtain

$$(3.8)\quad h^2(x, y, z) + [(1 - z)(1 - xz) + yz - zh(x, y, 1)]\, h(x, y, z) - yz^2[x(1 - z) + h(x, y, 1)] = 0.$$

We note that $h(x, y, 1) = f(x, y)$.

Now equation (3.8) can be rewritten as

$$(3.9)\quad (1 - z)(1 - xz)\, h(x, y, z) = -h^2(x, y, z) + [-yz + zh(x, y, 1)]\, h(x, y, z) + yz^2[x(1 - z) + h(x, y, 1)].$$

If the functions $h_{.j.}(x, z)$ are known from $j = 0$ to $j = q \geqslant 0$, then we can find the corresponding function for $j = q + 1$ by equating powers of y^{q+1} in (3.9). Hence, the functions $h_{.j.}(x, z)$ are uniquely determined by (3.8) and the single extra condition

$$h_{.0.}(x, z) = 0, \tag{3.10}$$

which is an immediate consequence of the definition of $h_{i,j,m}$.

We deduce that the only solution-pair $\{\sigma(x, y, z), \tau(x, y)\}$, in non-negative powers of the indeterminates, of the functional equation

$$\sigma^2(x, y, z) + [(1 - z)(1 - xz) + yz - z\tau(x, y)]\, \sigma(x, y, z) \\ - yz^2[x(1 - z) + \tau(x, y)] = 0 \tag{3.11}$$

which satisfies the conditions

(3.12) $\sigma(x, y, 1)$ *is well defined and equal to* $\tau(x, y)$

and

$$\sigma(x, 0, z) = 0 \tag{3.13}$$

is $\{h(x, y, z), f(x, y)\}$.

For any function $\tau(x, y)$ which is analytic at $(0, 0)$ we can solve (3.11) as a quadratic equation in $\sigma(x, y, z)$. If one of the solutions satisfies (3.12) and (3.13), then its Taylor expansion about $(x, y, z) = (0, 0, 0)$, if any, must be $h(x, y, z)$.

Let us take

$$\tau(x, y) = uv(1 - u - v), \tag{3.14}$$

where u and v are defined parametrically by (2.3). Write

$$B = (1 - z)(1 - xz) + yz - z\tau(x, y), \tag{3.15}$$

$$C = yz^2[x(1 - z) + \tau(x, y)]. \tag{3.16}$$

On substituting the appropriate expressions in u and v for x, y and $\tau(x, y)$ in (3.15) and (3.16), we obtain

$$B = 1 - (1 + u - v + uv - 2u^2v)z + u(1 - v)^2 z^2, \tag{3.17}$$

$$C = uv(1 - u)^2 \{(1 - v - uv)z^2 - (1 + v)^2 z^3\}. \tag{3.18}$$

Hence, we may verify that

$$B^2 + 4C = \{1 - (1 - v)z\}^2\{1 - 2u(1 + v - 2uv)z + u^2(1 - v)^2 z^2\}. \tag{3.19}$$

We may now write one solution of (3.11) as

$$\sigma(x, y, z) \\ = \tfrac{1}{2}\{-B + (1 - (1 - v)z)\sqrt{1 - 2u(1 + v - 2uv)z + u^2(1 - v)^2z^2}\}, \tag{3.20}$$

where the square root is chosen to have the value $+1$ when $u = 0$.

Putting $z = 1$ in (3.17) and (3.20), we obtain

$$(3.21)\qquad \begin{aligned}\sigma(x, y, 1) &= \tfrac{1}{2}\{-v + 3uv - 2u^2v - uv^2 + v(1 - u - uv)\}\\ &= uv(1 - u - v)\\ &= \tau(x, y).\end{aligned}$$

Hence the solution (3.20) satisfies (3.12).

Let us put $y = 0$ in (2.3). Then, in any sufficiently small neighbourhood of $(0, 0)$, we have $u = x$ and $v = 0$. Substituting these values of u and v, we find that the solution (3.20) satisfies (3.13).

It is clear that $\sigma(x, y, z)$, as given by (3.20), is analytic in a sufficiently small neighbourhood of $(0, 0, 0)$. We may now assert that its Taylor expansion about $(0, 0, 0)$ is $h(x, y, z)$. Hence the Taylor expansion of $\tau(x, y)$ about $(0, 0)$ is $f(x, y)$.

4. The coefficients $f_{i,j}$. We note that

$$\begin{aligned}uv(1 - u - v) &= uv\{(1 - u) + (1 - v) - 1\}\\ &= xy\left\{\frac{1}{(1-u)(1-v)^2} + \frac{1}{(1-u)^2(1-v)} - \frac{1}{(1-u)^2(1-v)^2}\right\},\end{aligned}$$

by (2.3).

Let r and s be positive integers. Then, by (2.3) and (**2**, Theorem 12),

$$\frac{1}{(1-u)^r(1-v)^s} = \sum_{\substack{i\geqslant 0\\ j\geqslant 0}} \frac{x^iy^j}{i!j!}\left[\frac{\partial^{i+j}}{\partial a^i\partial b^j} f^ig^j \frac{1}{(1-a)^r(1-b)^s}\Delta\right]_{\substack{a=0\\ b=0}},$$

where

$$f = \frac{1}{(1-b)^2}, \qquad g = \frac{1}{(1-a)^2},$$

and

$$\Delta = \begin{vmatrix} 1 & -x\dfrac{df}{db}\\ -y\dfrac{dg}{da} & 1\end{vmatrix} = 1 - \frac{4xy}{(1-a)^3(1-b)^3}.$$

Hence,

$$\frac{1}{(1-u)^r(1-v)^s} = \sum_{\substack{i\geqslant 0\\ j\geqslant 0}} \frac{x^iy^j}{i!j!}\left[\frac{\partial^{i+j}}{\partial a^i\partial b^j}\left(\frac{1}{1-a}\right)^{2j+r}\left(\frac{1}{1-b}\right)^{2i+s}\Delta\right]_{\substack{a=0\\ b=0}}$$

$$= \sum_{\substack{i\geqslant 0\\ j\geqslant 0}} \frac{x^i y^j}{i!j!}\left[\frac{(2j+i+r-1)!(2i+j+s-1)!}{(2j+r-1)!(2i+s-1)!} - 4xy\frac{(2j+i+r+2)!(2i+j+s+2)!}{(2j+r+2)!(2i+s+2)!}\right]$$

$$= \sum_{\substack{i\geqslant 0\\ j\geqslant 0}} \frac{x^i y^j}{i!j!}\frac{(2j+i+r-1)!(2i+j+s-1)!}{(2j+r)!(2i+s)!}\{2js+2ir+rs\}.$$

We may now write

$$f(x,y) = uv(1-u-v)$$
$$= \sum_{\substack{i\geqslant 0\\ j\geqslant 0}} \frac{x^{i+1}y^{j+1}(2j+i)!(2i+j)!}{i!j!(2j+2)!(2i+2)!}X,$$

where

$$X = (2j+2)(2i+j+1)(4j+2i+2) + (2i+2)(2j+i+1)(4i+2j+2) - (2i+j+1)(2j+i+1)(4i+4j+4)$$
$$= 4(2i+j+1)(2j+i+1).$$

Hence,

$$f(x,y) = \sum_{\substack{i\geqslant 0\\ j\geqslant 0}} \frac{x^{i+1}y^{j+1}(2j+i+1)!(2i+j+1)!}{(i+1)!(j+1)!(2i+1)!(2j+1)!}$$
$$= \sum_{\substack{i\geqslant 1\\ j\geqslant 1}} \frac{x^i y^j(2j+i-2)!(2i+j-2)!}{i!j!(2i-1)!(2j-1)!}.$$

Thus, formula (2.1) is valid for all positive integers i and j.

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