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CHARACTERIZATIONS OF THE SPHERE BY THE MEAN II-CURVATURE

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The notion of "mean II-curvature" of a C^4-surface (without parabolic points) in the three-dimensional Euclidean space has been introduced by Ekkehart Glässner. The aim of this note is to give some global characterizations of the sphere related to the above notion.

In the three-dimensional Euclidean space E^3 we consider a sufficiently smooth *ovaloid* S (closed convex surface) with Gaussian curvature $K > 0$. The ovaloid S possesses a positive definite second fundamental form II , if appropriately oriented. During the last years several authors have been concerned with the problem of characterizations of the sphere by the curvature of the second fundamental form of S . In this paper we give some characterizations of the sphere using the concept of the *mean* II-*curvature* H_{II} (of S), defined by Ekkehart Glässner.

Let H be the mean curvature of S and let Δ_{II} denote the second Beltrami operator with respect to II . Then, by definition [1, p. 194], we have

(1) $$H_{II} := H + \tfrac{1}{2}\Delta_{II} \ln\sqrt{K} .$$

We shall prove the following

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THEOREM 1. *On an ovaloid* S *either* $H_{II} - H$ *changes sign or* S *is a sphere.*

Proof. Let do_{II} denote the area element of S with respect to II. Then, integrating over all of S, we deduce, from (1),

$$\int H_{II} do_{II} = \int H do_{II} + \tfrac{1}{2} \int \Delta_{II} \ln\sqrt{K}\, do_{II} . \tag{2}$$

The second integral in the right-hand side of (2) is equal to zero, since S is closed. Hence, the equalty (2) yields

$$\int \left(H_{II} - H\right) do_{II} = 0 . \tag{3}$$

Therefore, it suffices to show that $H_{II} = H$ is true only if S is a sphere. In that case we obtain, from (1),

$$\Delta_{II} \ln\sqrt{K} = 0 .$$

This means that the function $\ln\sqrt{K}$ is harmonic. Note that a harmonic function defined on the compact surface S must be constant. Thus we have $K = \text{const.}$ and so S is a sphere.

LEMMA. *Let* S *be an ovaloid in* E^3 *and* P_0 (*respectively* P_1) *be a point on* S *where* K *attains its maximum* (*respectively minimum*). *Then*

$$H_{II}\left(P_0\right) \le H\left(P_0\right) \quad \left(\textit{respectively} \;\; H_{II}\left(P_1\right) \ge H\left(P_1\right)\right).$$

Proof. We shall prove the first inequality of the lemma. The proof of the second one is essentially the same. Let $\left(u^1, u^2\right)$ be local coordinates and let $\Gamma^m_{ij}(\mathrm{II})$ and ∇^{II} denote the Christoffel symbols of the second kind and covariant differentiation with respect to the second fundamental form II of S, respectively. If b_{ij} are the tensor components of II and b^{ij} are the components of the inverse tensor of b_{ij}, then we have ([3])

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$$(4)\quad \begin{cases} \Delta_{II} \ln\sqrt{K} = \sum_{i,j=1}^{2} b^{ij} \nabla_j^{II} \left(\frac{\partial(\ln\sqrt{K})}{\partial u^i} \right) \\ \qquad = \sum_{i,j=1}^{2} b^{ij} \left(\frac{\partial^2(\ln\sqrt{K})}{\partial u^j \partial u^i} - \sum_{m=1}^{2} \Gamma_{ij}^{m}(II) \frac{\partial(\ln\sqrt{K})}{\partial u^m} \right) \\ \qquad = \sum_{i,j=1}^{2} b^{ij} \frac{\partial^2(\ln\sqrt{K})}{\partial u^j \partial u^i} - \sum_{i,j,m=1}^{2} b^{ij} \Gamma_{ij}^{m}(II) \frac{\partial(\ln\sqrt{K})}{\partial u^m} . \end{cases}$$

We observe that the right-hand side of (4) is a second-order, linear partial differential expression of elliptic type, since

$$\det\left(b^{ij}\right) > 0 \quad \text{and} \quad b^{11} > 0 .$$

We assume that

$$(5) \qquad \Delta_{II} \ln\sqrt{K\left(P_0\right)} > 0 .$$

Then, because of the continuity of the function $\Delta_{II} \ln\sqrt{K}$, there is a neighbourhood T of P_0 such that

$$\Delta_{II} \ln\sqrt{K(P)} > 0$$

for every $P \in T$. On the other hand the function $\ln\sqrt{K}$ attains its maximum at the point $P_0 \in T$. Then, using a result by Hopf [2, p. 147], we conclude that $\ln\sqrt{K} = \text{const.}$ in T , from which we obtain that

$$\Delta_{II} \ln\sqrt{K(P)} = 0$$

for each $P \in T$. This is a contradiction to our assumption (5). Hence we have

$$\Delta_{II} \ln\sqrt{K\left(P_0\right)} \le 0 ,$$

and from (1) it follows that

$$H_{II}\left(P_0\right) \le H\left(P_0\right) .$$

Next, using the above results, we can prove

THEOREM 2. *Let S be an ovaloid in E^3 . Then each of the assumptions*

(i) $H_{II} = Hf(K)$,

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(ii) $H_{II} = H + f(K)$,

where f *is an increasing function, implies that* S *is a sphere.*

Proof. *(i)* We have

$$H_{II}(P_0) = H(P_0)f(K(P_0)) \le H(P_0) ,$$

from which we obtain that

$$f(K(P)) \le f(K(P_0)) \le 1$$

for every $P \in S$. Then we get

$$H_{II}(P) = H(P)f(K(P)) \le H(P)$$

for every $P \in S$. Hence, by Theorem 1, S is a sphere.

The case *(ii)* can be proved in a similar way.

Finally, we prove the following

THEOREM 3. *Let* S *be an ovaloid in* E^3 *and let* K_{II} *denote the curvature of the second fundamental form of* S *. If on the ovaloid* S *we have identically* $H_{II} = K_{II}$ *, then* S *is a sphere.*

Proof. If we denote by do the area element of S with respect to the first fundamental form I , it is obvious that $do_{II} = \sqrt{K}\, do$, since K equals the quotient of the determinants of II and I . Then by the Gauss-Bonnet Theorem (applied to the Riemannian spaces (S, I) and (S, II)) we have

$$4\pi = \int Kdo = \int K_{II}do_{II} ,$$

from which we obtain that

$$\int \left(K_{II}-\sqrt{K}\right) do_{II} = 0 . \tag{6}$$

Thus from (3) and (6) it follows that

$$\int \left(H_{II}-K_{II}\right) do_{II} = \int (H-\sqrt{K}) do_{II} . \tag{7}$$

If $H_{II} = K_{II}$, and because $H \ge \sqrt{K}$, from (7) we have $H = \sqrt{K}$. This means that the ovaloid S consists entirely of umbilic points and S must be a sphere.

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