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THE BOREL STRUCTURE OF ITERATES OF CONTINUOUS FUNCTIONS

by PAUL D. HUMKE* and M. LACZKOVICH*

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0. Notation

Let $\mathscr{C}[0,1]$ be the Banach space of continuous functions defined on $[0,1]$ and let $\mathscr{C}$ be the set of functions $f\in\mathscr{C}[0,1]$ mapping $[0,1]$ into itself. If $f\in\mathscr{C}$, f^k will denote the kth iterate of f and we put $\mathscr{C}^k=\{f^k: f\in\mathscr{C}\}$. The set of increasing ($\equiv$ nondecreasing) and decreasing ($\equiv$ nonincreasing) functions in $\mathscr{C}$ will be denoted by $\mathscr{I}$ and $\mathscr{D}$, respectively. If a function f is defined on an interval I, we let $C(f)$ denote the set of points at which f is locally constant, i.e.

$$C(f)=\{x\in I:\text{there is a } \delta>0 \text{ such that } f \text{ is constant on } (x-\delta,x+\delta)\cap I\}.$$

We let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}^{\mathbb{N}}$ denote the Baire space of sequences of positive integers.

1. Increasing iterates

In this section we prove that the sets $\mathscr{C}^k$ and $\mathscr{C}^k\cap\mathscr{I}$ are analytic and non-Borel subsets of $\mathscr{C}[0,1]$ for every $k\geqq 2$. The fact that $\mathscr{C}^k$ is analytic follows directly from the continuity of the mapping $f\mapsto f^k$ ($f\in\mathscr{C}$). As $\mathscr{I}$ is closed in $\mathscr{C}[0,1]$, the set $\mathscr{C}^k\cap\mathscr{I}$ is also analytic. The goal of the next series of lemmas is to show that for each $k\geqq 2$ and for each Borel subset $B\subset\mathbb{N}^{\mathbb{N}}$ there is a continuous map $F:\mathbb{N}^{\mathbb{N}}\to\mathscr{I}$ such that $F^{-1}(\mathscr{C}^k)=B$. From this it easily follows that neither of the sets $\mathscr{C}^k$ nor $\mathscr{C}^k\cap\mathscr{I}$ is Borel. Indeed suppose $\mathscr{C}^k$ or $\mathscr{C}^k\cap\mathscr{I}$ is Borel and is of Borel class $\alpha<\omega_1$. We can choose a Borel set $B\subset\mathbb{N}^{\mathbb{N}}$ of class higher than α and construct a map F as above. Since F is continuous and maps into $\mathscr{I}$, $F^{-1}(\mathscr{C}^k)=F^{-1}(\mathscr{C}^k\cap\mathscr{I})=B$ is of class α which is contrary to the choice of B.

In order to construct this mapping, F, we introduce the following subclasses of $\mathscr{I}$. For any choice of numbers $0<a<b<c<1$ we let $\mathscr{N}_{abc}$ denote the set of functions $f\in\mathscr{I}$ satisfying the following conditions.

1. $f(0)=0$ and $f(1)=1$.

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2. $f(a)=b$ and $f(b)=c$.
3. f is linear on each of the intervals $[0,a]$ and $[b,1]$.

Our initial aim is to characterize the functions belonging to $\mathcal{N}_{abc}\cap\mathcal{C}^k$ in terms of the set $C(f)$. Throughout the remainder of this section we consider the numbers a, b, and c fixed and simply use $\mathcal{N}$ to denote $\mathcal{N}_{abc}$.

We begin with the following simple lemma whose proof is omitted.

Lemma 1. *Let h_1 and h_2 be increasing and continuous functions defined on the closed interval $[x,y]$ such that $C(h_1)=C(h_2)$. Then there is a strictly increasing continuous function j defined on $[h_1(x),h_1(y)]$ such that $h_2=j\circ h_1$.*

Lemma 2. *Let $f\in\mathcal{N}\cap\mathcal{C}^k$, $k\geqq 2$. Then there exists a $g\in\mathcal{I}$ and points $a=x_0<x_1<\cdots<x_k=b$ such that $f=g^k$, $g(0)=0$ and $g(1)=1$, $g(x_i)=x_{i+1}$, $i=0,1,\ldots,k-1$ and g is strictly increasing on each of the intervals $[0,x_{k-1}]$ and $[b,1]$.*

Proof. As $f\in\mathcal{C}^k$, there is a $g\in\mathcal{C}$ such that $f=g^k$. Since f has no fixed point in $(0,1)$, neither does g. Consequently, either $g(x)<x$ holds for every $x\in(0,1)$ or $g(x)>x$ holds for every $x\in(0,1)$. The former entails that $f(x)=g^k(x)\leqq g^{k-1}(x)\leqq\cdots\leqq g(x)<x$ for every $x\in(0,1)$ which is not the case. Then $g(x)>x$ for each $x\in(0,1)$ and this fact implies that $g(1)=1$; as $f(0)=0$ we also deduce that $g(0)=0$. Also, as $f(x)<1$ for $x<1$ it follows that $g(x)<1$ for $x<1$. We have

$$x<g(x)<\cdots<g^k(x)=f(x)\quad\text{for}\quad x\in(0,1). \tag{2.1}$$

Define $x_i=g^i(a)$, $i=0,1,\ldots,k$. By (2.1) and the fact that $f(a)=b$ we have $a=x_0<x_1<\cdots<x_k=b$. Now, $g^{k-1}(0)=0$ and $g^{k-1}(a)=x_{k-1}$ so that $[0,x_{k-1}]\subset g^{k-1}([0,a])$. But $f(x)=g^k(x)=g(g^{k-1}(x))$ and f is injective on $[0,a]$. Hence, g is injective on $[0,x_{k-1}]$ and as $g(0)=0$, g is strictly increasing there. Similarly, as $f=g^k$ is injective on $[b,1]$, g is strictly increasing on $[b,1]$. What remains is to prove that g is increasing on $[x_{k-1},b]$. As $g^{k-1}(a)=x_{k-1}$ and $g^{k-1}(x_1)=b$ it follows that $g^{k-1}([a,x_1])=[x_{k-1},b]$. But then the result follows by noting that g^{k-1} is strictly increasing on $[a,x_1]$ and $f=g(g^{k-1})$ is increasing on $[a,x_1]$.

Lemma 3. *For every $f\in\mathcal{N}$ and $k\geqq 2$, $f\in\mathcal{C}^k$ if and only if there are points $a=x_0<x_1<\cdots<x_k=b$ and a function ϕ defined on $[x_0,x_{k-1}]$ such that for each $i=1,2,\ldots,k-1$, ϕ is an increasing homeomorphism mapping $[x_{i-1},x_i]$ onto $[x_i,x_{i+1}]$ satisfying*

$$\phi(C(f\,|\,[x_{i-1},x_i]))=C(f\,|\,[x_i,x_{i+1}]). \tag{3.1}$$

Proof. If $f\in\mathcal{N}\cap\mathcal{C}^k$ then there are points $a=x_0<x_1<\cdots<x_k=b$ and a function $g\in\mathcal{C}$ which satisfy the conclusion of Lemma 2. Let $\phi=g\,|\,[x_0,x_{k-1}]$. It follows directly

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from Lemma 2 that for each $i=1,2,\dots,k-1$, ϕ is an increasing homeomorphism mapping $[x_{i-1},x_i]$ onto $[x_i,x_{i+1}]$. As g is strictly increasing on each of the intervals $[x_{i-1},x_i]$ $i=1,2,\dots,k-1$ and on $[b,1]$ we have

$$\phi(C(f\,|\,[x_{i-1},x_i]))=\phi(C(g\circ f\,|\,[x_{i-1},x_i]))=\phi(C(f\circ\phi\,|\,[x_{i-1},x_i]))=C(f\,|\,[x_i,x_{i+1}]).$$

This completes the proof of the necessity and we now turn to the sufficiency proof.

Suppose that the numbers x_i, $i=0,1,\dots,k$ and the function ϕ are given and satisfy the conditions of the lemma. We prove that ϕ can be extended to a continuous function g defined on $[0,1]$ such that $f=g^k$. First note that $f(x_{i-1})<f(x_i)$ $(i=1,\dots,k)$. Indeed, if $f(x_{i-1})=f(x_i)$ then $C(f\,|\,[x_{i-1},x_i])=[x_{i-1},x_i]$. This implies, by (3.1) that f is constant on the entire interval $[a,b]$. This, of course, contradicts the fact that $f(a)=b<c=f(b)$.

Next, we extend the sequence $\{x_0,x_1,\dots,x_k\}$ by defining $x_n=f(x_{n-k})$ for $n>k$ and $x_n=f^{-1}(x_{n+k})$ for $n<0$. Since f is strictly increasing on each of the intervals $[0,a]$ and $[b,1]$, and $x_k<x_{k+1}<\cdots<x_{2k-1}$ (our prior remark) it is easy to verify that $x_n<x_{n+1}$ for every integer n. If $v=\lim_{n\to\infty}x_n$ then $f(v)=v$ and as $v>0$, $v=1$. Similarly, $\lim_{n\to\infty}x_{-n}=0$.

We inductively define a function ϕ_n on the interval $[x_{n-1},x_n]$ such that

A_n. ϕ_n is increasing and continuous on $[x_{n-1},x_n]$.

B_n. ϕ_n maps $[x_{n-1},x_n]$ onto $[x_n,x_{n+1}]$.

C_n. If $n\neq k$, then ϕ_n is strictly increasing.

We begin by defining $\phi_n=\phi\,|\,[x_{n-1},x_n]$ for $n=1,2,\dots,k-1$. By hypothesis, A_n, B_n, and C_n are true for these n. Next we define $\phi_k=f\circ\phi_1^{-1}\circ\phi_2^{-1}\circ\cdots\circ\phi_{k-1}^{-1}$ and note that A_k and B_k are satisfied. Suppose now that $n\geqq 0$ and that for each $i=1,2,\dots,n+k$, ϕ_i has been defined and satisfies A_i, B_i, and C_i. We prove that

$$C(\phi_{n+k}\circ\phi_{n+k-1}\circ\cdots\circ\phi_{n+2})=C(f\,|\,[x_{n+1},x_{n+2}]). \tag{3.2}$$

There are two cases. First suppose that $n\leqq k-2$. Then, as the functions $\phi_{k+1},\phi_{k+2},\dots,\phi_{k+n}$ are strictly increasing (property C_i), the left hand side of (3.2) reduces to $C(\phi_k\circ\phi_{k-1}\circ\cdots\circ\phi_{n+2})$. Using (3.1) and the definition of ϕ_k it is easy to check that $C(\phi_k\circ\phi_{k-1}\circ\cdots\circ\phi_{n+2})=C(f\,|\,[x_{n+1},x_{n+2}])$. If $n>k-2$ then all of the functions extant in (3.2) are strictly increasing so that both sides of (3.2) are empty. We apply Lemma 1 with $h_1=\phi_{n+k}\circ\phi_{n+k-1}\circ\cdots\circ\phi_{n+2}$, $h_2=f\,|\,[x_{n+1},x_{n+2}]$. Thus we obtain a strictly increasing continuous function, ϕ_{n+k+1}, defined on $h_1([x_{n+1},x_{n+2}])=[x_{n+k},x_{n+k+1}]$ such that

$$\phi_{n+k+1}\circ\phi_{n+k}\circ\cdots\circ\phi_{n+2}=f\,|\,[x_{n+1},x_{n+2}]. \tag{3.3}$$

Again, conditions A_{n+k+1}, B_{n+k+1}, and C_{n+k+1} are satisfied. Hence ϕ_n has been defined for every $n>0$ and we now turn to the case when $n\leqq 0$.

Suppose $n\leqq 0$ and that for each $i>n$, ϕ_i has been defined and satisfies the conditions A_i, B_i, and C_i. We put

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$$\phi_n = \phi_{n+1}^{-1} \circ \phi_{n+2}^{-1} \circ \cdots \circ \phi_{n+k-1}^{-1} \circ (f \,|\, [x_{n-1}, x_n]). \tag{3.4}$$

As in the previous cases, A_n, B_n, and C_n are transparent. In this way, ϕ_n has been defined for every integer n and we define

$$g(x) = \begin{cases} \phi_n(x) & \text{if } x \in [x_{n-1}, x_n],\, n \in \mathbb{Z} \\ 0 \text{ if } x = 0 & \text{and } 1 \text{ if } x = 1. \end{cases}$$

It follows from the conditions A_n and B_n that g is increasing and continuous on $[0, 1]$. Now, if $x \in (0, 1)$ then there is an integer n such that $x \in [x_{n-1}, x_n]$. If $n \leqq 0$ then $f(x) = g^k(x)$ by (3.4); if $n \geqq 2$ then $f(x) = g^k(x)$ by (3.3). Then sole remaining case is that when $n = 1$ and the fact that $f(x) = g^k(x)$ for $x \in [x_0, x_1]$ follows from the definition of ϕ_k. The proof of Lemma 3 is completed by noting that 0 and 1 are fixed points of both g and f.

A family of subsets of $\mathbb{R}$, $\{I_\gamma : \gamma \in \Gamma\}$, is said to be *discrete* if there is a family of pairwise disjoint open sets $\{U_\gamma : \gamma \in \Gamma\}$ such that $\bar{I}_\gamma \subseteq U_\gamma$ ($\bar{A} \equiv A$ closure) for every $\gamma \in \Gamma$. A family of pairwise disjoint intervals will be considered ordered according to the usual ordering of $\mathbb{R}$.

Lemma 4. *Let β be an infinite countable ordinal, $\varepsilon > 0$ and $k \geqq 2$. Suppose that $\{I_\alpha : \alpha < \beta k\}$ is a discrete set of open intervals contained in $(a + \varepsilon, b - \varepsilon)$ such that $I_\alpha < I_\gamma$ for $\alpha < \gamma < \beta k$. Then there are points $a = x_0 < x_1 < \cdots < x_k = b$ and a homeomorphism $\phi : [x_0, x_{k-1}] \to [x_1, x_k]$ such that $I_{\beta i + \alpha} \subset [x_i, x_{i+1}]$ $(i = 0, \ldots, k-1, \alpha < \beta)$, ϕ maps $[x_{i-1}, x_i]$ onto $[x_i, x_{i+1}]$ and $I_{\beta(i-1)+\alpha}$ onto $I_{\beta i + \alpha}$ for each $i = 1, 2, \ldots, k-1$ and each $\alpha < \beta$.*

Proof. For every $\alpha < \beta k$ let $I_\alpha = (u_\alpha, v_\alpha)$ and define $w_i = \lim_{\alpha \to \beta i} v_\alpha$ for each $i = 1, 2, \ldots, k$. As $\{I_\alpha : \alpha < \beta k\}$ is discrete, $w_i < u_{\beta i}$ for $i \leqq k-1$ and $w_k \leqq b - \varepsilon$. Let $x_0 = a$, $x_i = (w_i + u_{\beta i})/2$ $(i = 1, 2, \ldots, k-1)$ and $x_k = b$. Then define $\phi(x_{k-1}) = b$; $\phi(x_i) = x_{i+1}$, $\phi(u_{\beta i + \alpha}) = u_{\beta(i+1)+\alpha}$ and $\phi(v_{\beta i + \alpha}) = v_{\beta(i+1)+\alpha}$ for $i = 0, 1, \ldots, k-2$ and $\alpha < \beta$. As ϕ is strictly increasing on its domain and $\{I_\alpha : \alpha < \beta k\}$ is discrete, ϕ can be extended to a strictly increasing continuous function defined on the closure of its domain. We further extend ϕ to the entire interval $[x_0, x_{k-1}]$ by defining the extension to be linear on each component of the complement of this closure. This completes the proof of Lemma 4.

Let $E \subseteq [0, 1] \times [0, 1]$ and $x, y \in [0, 1]$. We denote the vertical and horizontal sections of E by $E_x = \{y : (x, y) \in E\}$ and $E^y = \{x : (x, y) \in E\}$. Now let $\{J_\gamma : \gamma \in \Gamma\}$ be a discrete family of open intervals in $[a, b]$ and let $K = [a, b] \setminus \bigcup_{\gamma \in \Gamma} J_\gamma$. Then each portion of K has positive Lebesgue measure. Let $\{I_\gamma : \gamma \in \Gamma\}$ be a family of subintervals of $[0, 1]$ with rational endpoints, and define $G = \bigcup_{\gamma \in \Gamma} (J_\gamma \times I_\gamma)$. We define a map $T : [0, 1] \to \mathscr{C}[a, b]$ as follows ($\lambda \equiv$ Lebesgue measure).

$$T(y)(x) = \frac{\lambda([a, x] \setminus G^y)}{\lambda([a, b] \setminus G^y)}, \quad y \in [0, 1] \quad \text{and} \quad x \in [a, b]. \tag{5.1}$$

Lemma 5. *The map T defined above has the following properties.*

1. *$T(y)$ is increasing and continuous for every $y\in[0,1]$.*
2. *$T(y)(a)=0$ and $T(y)(b)=1$ for every $y\in[0,1]$.*
3. *$C(T(y))=G^y$ for every $y\in[0,1]$.*
4. *T, as a map from $[0,1]$ into $\mathscr{C}[a,b]$, is continuous at each irrational $y\in[0,1]$.*

Proof. Statements 1 and 2 are obvious and 3 follows from the fact that every portion of $[a,b]\setminus\bigcup_{\gamma\in\Gamma}J_\gamma$ has positive Lebesgue measure. To prove 4, first note that Γ is countable. Let $y_0\in[0,1]$ be irrational, and let $\{\gamma_1,\gamma_2,\ldots,\gamma_n\}\subset\Gamma$ be an arbitrary finite set of indices. As the endpoints of the I_γ are rational, there is a $\delta>0$ such that if $y\in(y_0-\delta,y_0+\delta)$ then $y\in I_{\gamma_j}$ if and only if $y_0\in I_{\gamma_j}$ for $j=1,2,\ldots,n$. The continuity of T at y_0 easily follows from this observation.

Lemma 6. *If $B\subset[0,1]$ is Borel, then there is a set $M\subseteq[0,1]\times[0,1]$ consisting of a countable union of vertical line segments with rational endpoints and a countable ordinal β such that:*

1. *If $y\in B$, then M^y is well ordered with ordertype less than β;*
2. *if $y\notin B$, then M^y is not well ordered, but every decreasing sequence in M^y converges to the same real number.*

Proof. For $i\in\mathbb{N}$ and $\sigma\in\mathbb{N}^{\mathbb{N}}$ we denote the restriction of σ to its first i coordinates by $\sigma|i$. The desired set M is a Lusin sieve for $\mathbb{R}\setminus B$ and the special characteristics of M are derived from the fact that B is Borel. Specifically, there is a set of closed intervals with rational endpoints, $\{I_\tau:\tau\in\mathbb{N}^i,\ i=1,2,\ldots\}$ satisfying the following conditions:

(i) $\mathbb{R}\setminus B=\bigcup_{\sigma\in\mathbb{N}^{\mathbb{N}}}\bigcap_{i=1}^{\infty}I_{\sigma|i}$.

(ii) If $n>m$ and $\sigma\in\mathbb{N}^{\mathbb{N}}$, then $I_{\sigma|n}\subset I_{\sigma|m}$.

(iii) For every $y\in\mathbb{R}\setminus B$ there is a unique $\sigma\in\mathbb{N}^{\mathbb{N}}$ such that $\{y\}=\bigcap_{i=1}^{\infty}I_{\sigma|i}$.

We form a Lusin sieve for $\mathbb{R}\setminus B$ by assigning to each finite sequence of natural numbers $\tau=(n_1,n_2,\ldots,n_i)$ the binary fraction $x_\tau=1-2^{-n_1}-\cdots-2^{-n_1-\cdots-n_i}$ and the closed interval I_τ. The set M is defined as $M=\cup(\{x_\tau\}\times I_\tau)$ where the union is taken over all finite sequences of natural numbers. It follows directly from the definition of the sieve that M^y is well ordered if and only if $y\in B$. The fact that there is a countable ordinal β bounding the ordinals of the sections M^y, $y\in B$ is the substance of Corollary 5a of Section 39, VIII in [1]. Finally, suppose $y\in\mathbb{R}\setminus B$. Then there is a unique sequence $\sigma=(n_1,n_2,\ldots)$ such that $y\in\bigcap_{i=1}^{\infty}I_{\sigma|i}$. We prove that every decreasing sequence in M^y converges to the point $1-x$ where $x=2^{-n_1}+2^{-n_1-n_2}+\ldots$.

Suppose $\{x_{\tau_i}\}$ is an increasing sequence of binary fractions such that $1-x_{\tau_i}\in M^y$ for every i. For each j we denote the jth coordinate of τ_i by $\tau_i(j)$. It is easily verified that for each fixed j, the sequence $\{\tau_i(j):i=1,2,\ldots\}$ is eventually decreasing and hence, is eventually stationary at a natural number which we denote by $\tau(j)$. If $\tau=(\tau(1),\tau(2),\ldots)$

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then $y \in \bigcap_{i=1}^{\infty} I_{\tau|i}$ and as σ is unique, $\sigma = \tau$. Hence, $\{x_{\tau_i}: i = 1, 2, \ldots\}$ converges to $x = 2^{-n_1} + 2^{-n_1-n_2} + \ldots$.

Lemma 7. *For every Borel set $B \subset \mathbb{N}^{\mathbb{N}}$ and every $k \geqq 2$ there is a continuous function $F: \mathbb{N}^{\mathbb{N}} \to \mathcal{N}$ such that $F(y) \in \mathscr{C}^k$ if and only if $y \in B$.*

Proof. For convenience, we identify the space $\mathbb{N}^{\mathbb{N}}$ with the irrational numbers in $[0, 1]$ (see [1], Section 3). We fix three numbers p, q, and r such that $a < p < q < r < b$. Let K denote a nowhere dense perfect subset of $[a, p]$. As the set of bounded intervals contiguous to K has order type η (dense, unbordered, countable) there is a 1-1 order preserving mapping, H, from the binary fractions onto this set of intervals. If x_τ is a binary fraction, we let J_τ denote the open interval concentric with $H(x_\tau)$ but of half the length. Next we apply Lemma 6 for the Borel set B and obtain the set

$$M = \bigcup_{\tau} (\{x_\tau\} \times I_\tau)$$

and the countable ordinal β satisfying 1. and 2. of Lemma 6. We define

$$G_1 = \bigcup (J_\tau \times I_\tau)$$

where the union is taken over all finite sequences of natural numbers. Let $\{L_\alpha: \alpha < \beta\omega k\}$ be a discrete set of open subintervals of $[q, r]$ of order type $\beta\omega k$. Then we define

$$G_2 = \bigcup_{\alpha > \beta\omega k} (L_\alpha \times [0, 1])$$

and

$$G = G_1 \cup G_2.$$

Now we define the map T by (5.1). We define $F: [0, 1] \to \mathscr{C}[0, 1]$ by $F(y)(x) = (c - b)T(y)(x) + b$ for $y \in [0, 1]$ and $x \in [a, b]$. We then set $F(y)(0) = 0$, $F(y)(1) = 1$ and complete the definition by insisting $F(y)$ be linear on each of the intervals $[0, a]$ and $[b, 1]$. It is evident from this definition that $F(y) \in \mathcal{N}$ for every $y \in [0, 1]$ and it follows directly from Lemma 5 that F is continuous at each irrational y. Finally, for each y, $C(F(y)) = G^y = G_1^y \cup G_2^y$. The set of components for G_2^y is precisely $\{L_\alpha: \alpha < \beta\omega k\}$ and has order type $\beta\omega k$. The nature of the components of G_1^y depends on whether $y \in B$ or not and we consider these cases separately.

Case 1. $y \notin B$.

It follows from Lemma 6 that the components of G_1^y contain a decreasing sequence of intervals converging, say, to x^*, and that every decreasing sequence of components converges to x^*. If $F(y) \in C^k$, then there are points $a = x_0 < x_1 < \cdots < x_k = b$ satisfying the conditions of Lemma 3. As $a \leqq x^* < b$ there is a unique $n \in \{0, 1, \ldots, k-1\}$ such that $x^* \in [x_n, x_{n+1})$. But then the components of $C(F(y) | [x_n, x_{n+1}))$ are not well ordered and for each $i \neq n$ the components of $C(F(y) | [x_i, x_{i+1}))$ are well ordered. Such a situation bodes ill for the homeomorphism guaranteed by Lemma 3. This contradiction entails that if $y \notin B$ then $F(y) \notin C^k$.

Case 2. $y \in B$.

In this case, Lemma 6 yields that the components of G_1^y are well ordered and of order type $\alpha < \beta$. As $\alpha + \beta\omega k = \beta\omega k$, the order type of $G_1^y \cup G_2^y$ is $\beta\omega k$. Further, as each set of components G_1^y and G_2^y is discrete and the two sets are separated by the interval (p, q), the entire collection of components of G^y is discrete. But then, Lemma 4 establishes the existence of the requisite points $a = x_0 < x_1 < \cdots < x_k = b$ and the increasing homeomorphism $\phi: [x_0, x_{k-1}] \to [x_1, x_k]$ which guarantee, via Lemma 3, that $F(y) \in \mathscr{C}^k$. This completes the proof of Lemma 7.

As we saw in the introduction to this section Lemma 7 can now be used to prove the following theorem.

Theorem 8. *Each of the sets* $\mathscr{C}^k$ *and* $\mathscr{I} \cap \mathscr{C}^k$ *is analytic and non-Borel in* $\mathscr{C}[0, 1]$ *for* $k = 2, 3, \ldots$.

Remark 9. As we saw at the beginning of this section, Lemma 7 actually proves the slightly stronger result that $\mathscr{N} \cap \mathscr{C}^k$ is analytic and non-Borel in $\mathscr{C}[0, 1]$ for $k = 2, 3, \ldots$.

Although this completes the proof of the main result of Section 1, there are some additional facts which we will need in the subsequent sections.

Proposition 10. *If* $f \in \mathscr{N}_{abc}$, *then for each* $i = 2, 3, \ldots$, $f^i \in \mathscr{N}_{A_i b C_i}$ *where* $A_i = a^i / b^{i-1}$ *and* $C_i = 1 - (1-c)^i / (1-b)^{i-1}$.

Proof. For each $i = 1, 2, 3, \ldots$ the facts that f^i is increasing, $f^i(0) = 0$, and $f^i(1) = 1$ follow directly from the hypothesis that $f \in \mathscr{N}_{abc}$. Further, as f is linear and increasing on $[0, a]$ and $[b, 1]$ it follows that f^i is linear on $[0, A_i]$ and $[b, 1]$. An easy computation shows that $f^i(A_i) = b$ and $f^i(b) = C_i$.

Proposition 11. *Suppose* $B \subset \mathbb{N}^{\mathbb{N}}$ *is Borel,* F *is as in Lemma 7, and* $y \notin B$. *Then* $F^i(y) \in \mathscr{C}^j$ *if and only if* j *divides* i.

Proof. The sufficiency is obvious; for the necessity we again rely on the structure of the intervals of local constancy. As $y \notin B$, every decreasing sequence of components of $C(F(y))$ converges to the same real number. As $F(y) \in N_{abc}$, $F(y)(a) = b$, $F(y)(b) = c$, and $F(y)$ is linear on each of the intervals $[0, a]$ and $[b, 1]$. From these it follows that any decreasing sequence of components of $C(F^i(y))$ converges to one of exactly i points, one in each of the intervals $[a,b)$, $[A_2,a)$, and $[A_n,A_{n-1})$ $n=2,3,\ldots,i$. From Proposition 10 we know that $F^i(y)\in\mathcal{N}_{A_ibC_i}$. If $F^i(y)\in\mathcal{C}^j$ then these i points must be equally distributed among the j intervals guaranteed by Lemma 3. This completes the proof of Proposition 11.

2. Decreasing iterates of odd exponent

In this section we show that if $k\geqq 3$ is odd and $\mathcal{D}$ denotes the set of decreasing functions in $\mathcal{C}$, then the set $\mathcal{D}\cap\mathcal{C}^k$ is analytic and non-Borel. The fact that $\mathcal{D}\cap\mathcal{C}^k$ is analytic is obvious as $\mathcal{D}$ is closed in $\mathcal{C}[0,1]$. Our method is to prove that for every odd $k\geqq 3$ and every Borel set $B\subset\mathbb{N}^{\mathbb{N}}$ there exists a continuous map $W:\mathbb{N}^{\mathbb{N}}\to\mathcal{D}$ such that if $y\in\mathbb{N}^{\mathbb{N}}$ then $W(y)\in\mathcal{C}^k$ if and only if $y\in B$. As we saw in Section 1, the existence of such a map proves that $\mathcal{D}\cap\mathcal{C}^k$ is non-Borel. We shall define W as $\Phi\circ F$, where Φ maps a certain subclass of $\mathcal{C}$ (containing $\mathcal{N}$) into $\mathcal{D}$ and F is the map found in Lemma 7.

Let $\mathcal{M}$ denote the set of functions $f\in\mathcal{C}$ such that $f(1)=1$ and $f(x)<1$ for $x\in[0,1)$. For $f\in\mathcal{M}$ we define

$$\Phi(f)(x)=\begin{cases}1-\frac{1}{2}f(2x), & x\in[0,\frac{1}{2}]\\ \frac{1}{2}f(2-2x), & x\in(\frac{1}{2},1].\end{cases}$$

Lemma 12. *The map Φ defined above has the following properties:*

(i) *$\Phi(f)\in\mathcal{C}$ for every $f\in\mathcal{M}$.*

(ii) *$\Phi(f)\in\mathcal{D}$ for every $f\in\mathcal{I}\cap\mathcal{M}$.*

(iii) *$\Phi(f)\circ\Phi(g)=1-\Phi(f\circ g)$ for every $f,g\in\mathcal{M}$.*

(iv) *$\Phi(\mathcal{N}\cap\mathcal{C}^k)\subset\mathcal{C}^k$ for every odd k.*

(v) *If k is odd, then $f^2\in\mathcal{C}^k$ whenever $f\in\mathcal{M}$ and $\Phi(f)\in\mathcal{C}^k$.*

Proof. Property (i) follows from the fact that $f(1)=1$ for every $f\in\mathcal{M}$ and (ii) is obvious from the definition of Φ. An easy computation gives (iii). To prove (iv) let k be odd and $f\in\mathcal{N}\cap\mathcal{C}^k$. From Lemma 2 we deduce that $f=g^k$ where $g\in\mathcal{I}$, $g(1)=1$ and g is strictly increasing on $[b,1]$. These imply that $g\in\mathcal{M}$. It now follows easily from (iii) and the fact that k is odd that $\Phi(f)=\Phi(g^k)=(\Phi(g))^k\in\mathcal{C}^k$.

To prove (*v*) suppose that $f\in\mathcal{M}$ and $\Phi(f)=g^k$ where $g\in\mathcal{C}$. It follows from the definitions of $\mathcal{M}$ and Φ that $\Phi(f)$ has a unique fixed point at $x=\frac{1}{2}$ and that $\Phi(f)$ attains the value of $\frac{1}{2}$ only at $\frac{1}{2}$. Therefore g has the same two properties. Consequently, either $g(x)<\frac{1}{2}$ for every $x\in[0,\frac{1}{2})$ or $g(x)>\frac{1}{2}$ for every $x\in[0,\frac{1}{2})$. The former is impossible since $\Phi(f)=g^k$ and $\Phi(f)(x)>\frac{1}{2}$ on $[0,\frac{1}{2})$. Hence $g(x)>\frac{1}{2}$ on $[0,\frac{1}{2})$ and the same argument shows that $g(x)<\frac{1}{2}$ on $(\frac{1}{2},1]$. This, together with the definition of Φ, implies that there are functions $g_1,g_2\in\mathcal{M}$ such that $\Phi(g_1)\,|\,[0,\frac{1}{2}]=g\,|\,[0,\frac{1}{2}]$ and $\Phi(g_2)\,|\,[\frac{1}{2},1]=g\,|\,[\frac{1}{2},1]$. Then for $x\in[\frac{1}{2},1]$, $g(x)=\Phi(g_2)(x)\in[0,\frac{1}{2}]$ and hence, $g^2(x)=\Phi(g_1)\circ\Phi(g_2)(x)\in[\frac{1}{2},1]$. This implies that

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$$g^{2k}(x)=(\Phi(g_1)\circ\Phi(g_2))^k(x) \quad \text{for} \quad x\in[\tfrac{1}{2},1]. \tag{12.1}$$

By (iii), $1-\Phi(f^2)=(\Phi(f))^2=g^{2k}$. On the other hand, (iii) implies that whenever m is even and $f_1,f_2,\dots,f_m\in\mathcal{M}$, we have

$$\Phi(f_1)\circ\Phi(f_2)\circ\cdots\circ\Phi(f_m)=1-\Phi(f_1\circ f_2\circ\cdots\circ f_m).$$

Hence

$$(\Phi(g_1)\circ\Phi(g_2))^k=1-\Phi((g_1\circ g_2)^k).$$

By (12.1), we have

$$\Phi(f^2)\,|\,[\tfrac{1}{2},1]=\Phi((g_1\circ g_2)^k)\,|\,[\tfrac{1}{2},1]. \tag{12.2}$$

But, if $f_1,f_2\in\mathcal{M}$ and $\Phi(f_1)\,|\,[\tfrac{1}{2},1]=\Phi(f_2)\,|\,[\tfrac{1}{2},1]$ then $f_1=f_2$. Hence, it follows from (12.2) that $f^2=(g_1\circ g_2)^k\in\mathscr{C}^k$.

Lemma 13. *For every Borel set $B\subset\mathbb{N}^{\mathbb{N}}$ and odd $k\geqq 3$, there is a continuous map $W:\mathbb{N}^{\mathbb{N}}\to\mathscr{D}$ such that $W(y)\in\mathscr{C}^k$ if and only if $y\in B$.*

Proof. We put $W=\Phi\circ F$, where Φ is the mapping described above and F is the function defined for the Borel set B in Lemma 7. If $y\in B$ then, by Lemma 7, $F(y)\in\mathscr{N}\cap\mathscr{C}^k$ and hence $W(y)=\Phi(F(y))\in\mathscr{D}\cap\mathscr{C}^k$ by (ii) and (iv) of Lemma 12. On the other hand, if $y\in\mathbb{N}^{\mathbb{N}}$ and $W(y)=\Phi(F(y))\in\mathscr{C}^k$ then, by (v) of Lemma 12, $(F(y))^2\in\mathscr{C}^k$. But 2 does not divide k and hence Proposition 11 implies that $y\in B$.

As we saw at the beginning of this section, Lemma 13 establishes the following result.

Theorem 14. *If $k\geqq 3$ is odd, then $\mathscr{D}\cap\mathscr{C}^k$ is analytic and non-Borel in $\mathscr{C}[0,1]$.*

3. Decreasing iterates of even exponent

Our goal in this section is to prove the following characterization of the class $\mathscr{D}\cap\mathscr{C}^k$, k even.

Theorem 15. *For each even k, $\mathscr{D}\cap\mathscr{C}^k=\mathscr{D}\cap\mathscr{C}^2$. Moreover, if $f\in\mathscr{D}$ and k is even then $f\in\mathscr{C}^k$ if and only if $C(f)$ contains the range of f.*

From this we can immediately infer that for even k, $\mathscr{D}\cap\mathscr{C}^k$ is Borel and indeed, is F_σ.

Corollary 16. *For every even k, $\mathscr{D}\cap\mathscr{C}^k$ is an F_σ subset of $\mathscr{C}[0,1]$.*

Proof. It is easy to see that if $p<q<r<s$ then the set of functions $f\in\mathscr{D}$ such that f

is constant on $[p,s]\cap[0,1]$ and the range of f is contained in $[q,r]$ is closed in $\mathscr{C}[0,1]$. By Theorem 15, $\mathscr{D}\cap\mathscr{C}^k$ is the union of all such sets where $p,q,r,$ and s are rational.

We turn now to the proof of Theorem 15 which is accomplished via a series of results.

Theorem 17. *If $f\in\mathscr{C}$ and $C(f)$ contains the range of f, then $f\in\mathscr{C}^k$ for every $k=1,2,\ldots$.*

Proof. The range of f is a closed interval while $C(f)$ is relatively open in $[0,1]$. Let I denote the component of $C(f)$ containing the range of f and set $u\equiv f\,|\,I$. If $I=[0,1]$, f is constant on I and the conclusion follows as $f=f^k$. Therefore we may assume $I\neq[0,1]$, and we first assume $I=(a,b)$ where $0<a<b<1$. If $m=\min\{f(x):x\in[0,1]\}$ and $M=\max\{f(x):x\in[0,1]\}$ then the hypothesis implies $a<m\leqq u\leqq M<b$.

Let $k\geqq 2$ be fixed, and choose points x_i, $i=1,2,\ldots,k$ such that $0=x_1<a<x_2<\cdots<x_k<m$. We define the function g to be the increasing linear map from $[x_{i-1},x_i]$ onto $[x_i,x_{i+1}]$, $i=2,3,\ldots,k-1$. Then g^{k-2} maps $[0,x_2]$ onto $[x_{k-1},x_k]$. Let

$$c=g^{k-2}(a)\in(x_{k-1},x_k).$$

Define g to be linear and increasing on each of the intervals $[x_{k-1},c]$ and $[c,x_k]$, mapping them respectively onto $[x_k,m]$ and onto $[m,f(0)]$. Next define

$$g(x)=f\left(\frac{a}{m-x_k}(x-x_k)\right)\quad\text{if}\quad x\in[x_k,m],\quad\text{and}$$

$$g(x)=u\quad\text{if}\quad x\in[m,u].$$

At this point g has been defined on $[0,u]$ and it is easy to check that g is continuous here using the fact that $g(m)=f(a)=u$. The definition of g on $[u,1]$ is analogous but using M and b in place of m and a respectively.

We prove that $f=g^k$. Since g^{k-1} maps $[0,a]$ linearly onto $[x_k,m]$, we have, for $x\in[0,a]$,

$$g^{k-1}(x)=\frac{m-x_k}{a}x+x_k.$$

Therefore, by the definition of g in $[x_k,m]$ we deduce that $g^k(x)=f(x)$ for $x\in[0,a]$. Further, since g^{k-1} maps $[a,x_2]$ into $[m,f(0)]\subset[m,M]$ and $g([m,M])=\{u\}$, we have $g^k(x)=u=f(x)$ whenever $x\in[a,x_2]$. Since $g([x_{k-1},x_k])=[x_k,f(0)]$ and

$$g([x_k,f(0)])=g([x_k,m])\cup g([m,f(0)])\subset[m,M]\cup g([m,M])=[m,M]\cup\{u\}=[m,M],$$

we have $g^3([x_{k-1},x_k])=\{u\}$. Therefore, if $3\leqq i\leqq k$ then

$$g^k([x_{i-1}, x_i]) = g^i([x_{k-1}, x_k]) = g^{i-3}(\{u\}) = \{u\}.$$

Since $f(x) = u$ for $x \in [x_2, x_k]$, this proves that $f(x) = g^k(x)$ on $[x_2, x_k]$. If $x \in [x_k, u]$, then $g(x) \in [m, M]$ so that $g^2(x) = g^k(x) = u = f(x)$. The same argument applies if $x \in [u, 1]$ and, as such, the proof that $f \in \mathscr{C}^k$ is complete.

Next consider the case $I = (a, 1]$. Then for each k, the function g is defined as above on the interval $[0, u]$ but is defined to be the constant u on $[u, 1]$. The proof that $f(x) = g^k(x)$ for $x \in [0, u]$ is exactly as that given above while the fact that $g^k(x) = f(x)$ for $x \in [u, 1]$ now follows from the fact that $f(x) = g(x) = u$ on $[u, 1]$.

The case $I = [0, b)$ is analogous and this completes the proof of Theorem 17.

Lemma 18. *Let $f \in \mathscr{D} \cap \mathscr{C}^2$ and let u denote the (unique) fixed point of f. Then $f(x) = u$ holds whenever $x \in [f(1), f(0)]$.*

Proof. Let $f^{-1}(\{u\}) = [\alpha, \beta]$; we must prove that $\alpha \leqq f(1)$ and $\beta \geqq f(0)$. Suppose this is not true and assume, for example, that $f(1) < \alpha$. Let $f = g^2$, $g \in \mathscr{C}$. Since u is the sole fixed point of f, u is also the only fixed point of g. As $f = g^2$ is decreasing, this implies that $g(x) > x$ for $x < u$ and $g(x) < x$ for $x > u$. Set $g(1) = w$.

As a first case, suppose $w > \beta$. Then $g(w) = f(1) \leqq u$ and $g(1) = w > \beta \geqq u$. Hence there is a $y \in [w, 1]$ such that $g(y) = u$. Then $f(y) = g^2(y) = g(u) = u$ which is impossible as $y \geqq w > \beta$.

Next, suppose $w < \alpha$. Then $g(g(w)) = f(w) > u$ and $g(w) = f(1) \leqq u$. Hence, there is a $y \in [w, g(w)]$ with $g(y) = u$. Again, $f(y) = g(u) = u$ which is impossible since $y \leqq g(w) = f(1) < \alpha$.

Therefore, we may suppose $w \in [\alpha, \beta]$ and hence that $f(w) = u$. Now, $f^2(1) = g^4(1) = g^3(w) = g(f(w)) = g(u) = u$ which again is impossible as $f(1) < \alpha$. This final contradiction completes the proof of Lemma 18.

We now turn to the proof of Theorem 15.

Proof of Theorem 15. Let $f \in \mathscr{D}$. If $C(f)$ contains the range of f then, by Theorem 17, $f \in D \cap C^k$ for every k. If $f \in \mathscr{D} \cap \mathscr{C}^k$ with k even then, obviously, $f \in \mathscr{D} \cap \mathscr{C}^2$ so that, by Lemma 18, f is constant on the interval $[f(1), f(0)]$. To complete the proof of Theorem 15 we must show that there is an $\varepsilon > 0$ such that f is constant on the interval $[f(1) - \varepsilon, f(0) + \varepsilon] \cap [0, 1]$. As in Lemma 18 we let u denote the only fixed point of f, let $[\alpha, \beta] = f^{-1}(\{u\})$, and let $g \in \mathscr{C}$ be such that $g^2 = f$. We must show that

(i) either $\beta = 1$ or $f(0) < \beta$ and

(ii) either $\alpha = 0$ or $f(1) > \alpha$.

Suppose, for example, that (i) is false, that is, $f(0) = \beta < 1$. We prove that this implies that g is not constant on $[u, f(0)]$.

First we show $f(0) > u$. Indeed, if $u = f(0) = \beta$, then $f(z) = u$ for every $z \in [0, u]$. Hence either $g \equiv u$ in $[0, u]$ or $g([0, u])$ contains a one sided neighbourhood of u on which $g \equiv u$. In each of these cases, $g \equiv u$ in a one sided neighbourhood of u. If this is a right

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neighbourhood then $f \equiv u$ in that neighbourhood which is impossible since $f(0)=u=\beta$. We conclude that g is not constant in $[u,1]$ so that $g([u,1])$ contains a one sided neighbourhood of u and $g<u$ in this neighbourhood. Thus, there is a $\delta>0$ such that $g(z)<u$ for $z\in(u,u+\delta)$ which again is a contradiction. Hence, $f(0)=\beta>u$.

Suppose g is constant on $[u,f(0)]$. Then $g\equiv u$ on $[u,f(0)]$. Further, since $f(0)=\beta$, $g^2(z)=f(z)<u$ for $z\in(f(0),1]$. But this entails that g is not constant on $(f(0),1]$ else this constant would be $g(f(0))=u$ which would imply that $g^2(z)=u$ on $(f(0),1]$. Thus, $g((f(0),1])$ contains a one sided neighbourhood of u and $g<u$ on this neighbourhood. Since $g\equiv u$ on $[u,f(0)]$ there is a $\delta>0$ such that $g(z)<u$ for $z\in(u-\delta,u)$. Then there is an $\eta>0$ such that $f(z)=g^2(z)<u$ for $z\in(u-\eta,u)$ which is not the case. Hence g cannot be constant on $[u,f(0)]$ as we claimed.

Now, set $v=g(0)$ so that $g(v)=f(0)$. We consider three cases.

Case 1. $v>f(0)$.

In this case $f(v)<u$, $g(f(0))=f(v)<u$, and $g(v)=f(0)\geqq u$. Hence, there is a $y\in(f(0),v]$ with $g(y)=u$. But then $f(y)=u$ which is impossible as $y>f(0)=\beta$.

Case 2. $f(1)\leqq v\leqq f(0)$.

Here we have $f(v)=u$ and $f(g(0))=g(f(0))=u$. As $g(v)=f(0)$, it follows that $g([v,f(0)])\supset[u,f(0)]$. But $f\equiv u$ on $[v,f(0)]$ and hence g is constant on $[u,f(0)]$ which, as we saw above, is impossible.

Case 3. $v<f(1)$.

In this case, $g(v)=f(0)\geqq u$ and $g(0)=v<u$ so that there is a $y\in[0,v]$ such that $g(y)=u$. Then $f(y)=u$ and hence, as $y\leqq v<f(1)$, $f(v)=u$. Therefore $f\equiv u$ on $[v,f(0)]$. But $g([v,f(0)])\supset[u,f(0)]$ and we again conclude that g is constant on $[u,f(0)]$. This final contradiction completes the proof.

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Department of Mathematics
St. Olaf College
Northfield, MN, U.S.A.

Department of Analysis
Eötvös Loránd University
Budapest, Hungary

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