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COMPLEMENTARY VARIATIONAL PRINCIPLES FOR A CLASS OF NONLINEAR BOUNDARY VALUE PROBLEMS

N. ANDERSON and A. M. ARTHURS

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Abstract

Complementary variational principles are presented for a class of nonlinear boundary value problems $S^* S\phi = g(\phi)$ in which g is not necessarily monotone. The results are illustrated by two examples, accurate variational solutions being obtained in both cases.

1. Introduction

Complementary variational principles are known [1] for boundary value problems described by equations of the form

$$T^*T\phi = f(\phi) \text{ in } V, \quad \phi = 0 \text{ on } \partial V. \tag{1}$$

Here V is some region of E^n with boundary ∂V, T and T^* form an adjoint pair of linear operators, and $f(\phi)$ is a monotonic *decreasing* function of ϕ. We assume the existence of a solution ϕ of (1) and view it as an element in the real Hilbert space H_ϕ with inner product $\langle\, ,\rangle$. The operator T acts on elements in H_ϕ and sends them to a second real Hilbert space H_u with inner product $(\, ,)$. The adjoint T^* of T is defined by

$$(v, T\psi) = \langle T^*v, \psi\rangle + (v, \sigma\psi) \tag{2}$$

for all v in H_u and all ψ in H_ϕ, where σ is a linear operator acting on functions on the boundary of V.

In this paper we investigate the possibility of extending these results to include boundary value problems in which $f(\phi)$ is not necessarily monotone decreasing.

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For instance, it may be monotone increasing, like e^{ϕ}, or it may not be monotone at all, like $\sin\phi$. To consider problems such as these, we therefore look at a class of boundary value problems

$$S^*S\phi = g(\phi) \text{ in } V, \quad \phi = 0 \text{ on } \partial V, \tag{3}$$

where S and S^* form an adjoint pair of operators and where $g(\phi)$ is not necessarily monotonic decreasing. Our aim here is to rewrite (3) in a form corresponding to (1), with $f(\phi)$ a monotonic decreasing function. This can be done for a certain class of problems.

First we shall suppose that S^*S is a strictly positive operator, that is there exists a positive number λ such that

$$\langle\psi, S^*S\psi\rangle = (S\psi, S\psi) \geqslant \lambda\langle\psi,\psi\rangle \tag{4}$$

for all non-zero ψ in H_ϕ. Then for some positive number p we can write

$$S^*S = T^*T + p, \quad p > 0, \tag{5}$$

for some positive self-adjoint operator T^*T, and equation (3) becomes

$$T^*T\phi = f(\phi), \tag{6}$$

with

$$T^*T = S^*S - p = L \quad \text{say}, \tag{7}$$

and

$$f(\phi) = g(\phi) - p\phi. \tag{8}$$

Now since $S^*S - p$ is positive, we must have

$$p \leqslant \lambda_0, \tag{9}$$

where λ_0 is the lowest (positive) eigenvalue of the eigenproblem

$$S^*S\theta = \lambda\theta \text{ in } V, \quad \theta = 0 \text{ on } \partial V. \tag{10}$$

In addition we want $f(\phi)$ to be monotone decreasing and this means that

$$\frac{g(\phi_i) - g(\phi_j)}{\phi_i - \phi_j} \leqslant p \tag{11}$$

for all ϕ_i and ϕ_j in H_ϕ. If $g(\phi)$ is differentiable this becomes

$$g'(\psi) \leqslant p, \quad \text{for all } \psi \text{ in } H_\phi. \tag{12}$$

Combining (9) and (12) we therefore find that p must satisfy the conditions

$$g'(\psi) \leqslant p \leqslant \lambda_0 \quad \text{for all } \psi \text{ in } H_\phi. \tag{13}$$

We shall assume that such a number p can be found.

To derive complementary variational principles associated with (3), rewritten as (6), we wish to use a canonical approach. We therefore require, at least in theory, the operators T and T^*, and if we assume that we can write

$$T = S + q, \tag{14}$$

then

$$T^* = S^* + q, \tag{15}$$

where q is some function as yet unknown. Then (7) requires that

$$S^*(q\psi) + qS\psi + q^2\psi + p\psi = 0 \quad \text{for all } \psi \text{ in } H_\phi. \tag{16}$$

This is of the form

$$A(q)\psi = 0,$$

and so we require a q which satisfies

$$A(q) = 0 \tag{17}$$

at all points of the space V. As we shall see later, beyond its existence, knowledge of q is not needed in practice for the class of problems under consideration.

2. Complementary principles

In Section 1 the boundary value problem (3) has been rewritten in the form

$$T^*T\phi = f(\phi) \text{ in } V, \quad \phi = 0 \text{ on } \partial V. \tag{18}$$

We now derive the associated complementary variational principles.

We write (18) in canonical form

$$\Omega_1\colon \quad T\phi = u = W_u, \quad \phi = 0 \text{ on } \partial V, \tag{19}$$

$$\Omega_2\colon \quad T^*u = f(\phi) = W_\phi \quad \text{in } V, \tag{20}$$

where subscripts on the Hamiltonian W denote abstract derivatives. A suitable W is given by

$$W(u, \phi) = \tfrac{1}{2}(u, u) + F(\phi), \tag{21}$$

where

$$F(\phi) = \int^\phi \langle f(\psi), d\psi \rangle. \tag{22}$$

Equations (19) and (20) are the Euler–Hamilton equations associated with the action functional

$$I(u,\phi) = (u, T\phi) - W(u,\phi) - (u, \sigma\phi) \tag{23a}$$

$$= \langle T^*u, \phi\rangle - W(u,\phi). \tag{23b}$$

Using (21) we see that

$$I(u,\phi) = (u, T\phi) - \tfrac{1}{2}(u,u) - F(\phi) - (u,\sigma\phi) \tag{24a}$$

$$= \langle T^*u, \phi\rangle - \tfrac{1}{2}(u,u) - F(\phi). \tag{24b}$$

The action I is stationary at the solution (u,ϕ) of equations (19) and (20).

Now we define a pair of dual functionals as follows:

$$\begin{aligned} J(\phi_1) &= I(u_1,\phi_1) \quad \text{via (24a), with } (u_1,\phi_1) \text{ in } \Omega_1 \\ &= \tfrac{1}{2}(T\phi_1, T\phi_1) - F(\phi_1) \\ &= \tfrac{1}{2}\langle \phi_1, L\phi_1\rangle - F(\phi_1), \quad \text{with } \phi_1 = 0 \text{ on } \partial V, \end{aligned} \tag{25}$$

and

$$\begin{aligned} K(u_2) &= I(u_2,\phi_2) \quad \text{via (24b), with } (u_2,\phi_2) \text{ in } \Omega_2 \\ &= \langle T^* u_2, f^{-1}(T^* u_2)\rangle - \tfrac{1}{2}(u_2,u_2) - F[f^{-1}(T^* u_2)]. \end{aligned} \tag{26}$$

If we take

$$u_2 = T\psi_2, \quad \psi_2 = 0 \quad \text{on } \partial V, \tag{27}$$

we have

$$K(T\psi_2) = \langle L\psi_2, f^{-1}(L\psi_2)\rangle - \tfrac{1}{2}\langle \psi_2, L\psi_2\rangle - F[f^{-1}(L\psi_2)]. \tag{28}$$

Here

$$L = T^*T = S^*S - p, \tag{29}$$

and

$$f(\psi) = g(\psi) - p\psi. \tag{30}$$

Since

$$f'(\psi) \leqslant 0 \quad \text{for all } \psi, \tag{31}$$

we have the complementary extremum principles [*cf.* 2]

$$K(T\psi_2) \leqslant K(T\phi) = J(\phi) \leqslant J(\phi_1), \tag{32}$$

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equality holding when ϕ_1 and ψ_2 are equal to the solution ϕ. With J in (25) and K in (28), we note that the function q of (14) is not required explicitly.

There is an alternative form for $J(\phi_1)$ which we give here. By (25)

$$\begin{aligned} J(\phi_1) &= \tfrac{1}{2}\langle \phi_1, L\phi_1\rangle - F(\phi_1) \\ &= \tfrac{1}{2}\langle \phi_1, (S^* S - p)\,\phi_1\rangle - \int^{\phi_1}\langle g(\psi) - p\psi, d\psi\rangle \\ &= \tfrac{1}{2}\langle \phi_1, S^* S\phi_1\rangle - G(\phi_1) \\ &= \tfrac{1}{2}(S\phi_1, S\phi_1) - G(\phi_1), \end{aligned} \tag{33}$$

where

$$G(\phi_1) = \int^{\phi_1}\langle g(\psi), d\psi\rangle. \tag{34}$$

Thus we have a formula (33) for $J(\phi_1)$ in terms of the original operators and functions of equation (3). In this form the number p drops out and the minimum principle for J holds provided that

$$g'(\psi) \leqslant \lambda_0 \quad \text{for all } \psi, \tag{35}$$

which of course is consistent with equation (13).

3. Example 1

To illustrate these ideas we first consider the nonlinear two-point boundary value problem described by the equations

$$\frac{d^2\phi}{dx^2} = -e^{\phi}, \quad 0 < x < 1, \tag{36}$$

and

$$\phi(0) = \phi(1) = 0. \tag{37}$$

It is known [5] that there is a non-negative solution ϕ such that

$$0 \leqslant \phi < 0.142. \tag{38}$$

This is an example of our class of problems in (3) with

$$S = \frac{d}{dx}, \quad S^* = -\frac{d}{dx}, \tag{39}$$

$$g(\phi) = e^{\phi}, \tag{40}$$

and

$$\langle \phi, \psi \rangle = \int_0^1 \phi\psi \, dx, \quad (u, v) = \int_0^1 uv \, dx. \tag{41}$$

Here

$$g'(\psi) = e^{\psi} \geqslant 0, \tag{42}$$

and so $g(\phi)$ is monotone *increasing*. To reformulate the problem as in Section 1 we need to find a positive number p such that

$$g'(\psi) \leqslant p \leqslant \lambda_0 \quad \text{for all } \psi. \tag{43}$$

If we restrict all admissible functions ψ to the range

$$0 \leqslant \psi < 0.142, \tag{44}$$

this means that

$$\exp(0.142) \leqslant p \leqslant \pi^2, \tag{45}$$

which provides a choice of possible p values.

For this example we find that the function q of (16) and (17) must satisfy

$$-q' + q^2 + p = 0 \tag{46}$$

and this has the general solution

$$q = \sqrt{(p)} \tan\{\sqrt{(p)}(x+c)\}, \tag{47}$$

where c is a constant. With p in the range specified by (45), and $0 < x < 1$, the existence of q over the whole range is assured by the choice $c = -\frac{1}{2}$. We can therefore use the results of Section 2 with

$$L = -\frac{d^2}{dx^2} - p, \tag{48}$$

and

$$f(\psi) = e^{\psi} - p\psi. \tag{49}$$

By (25) and (28), the dual functionals J and K are

$$J(\phi_1) = \int_0^1 \{\tfrac{1}{2}\phi_1 L\phi_1 - e^{\phi_1} + \tfrac{1}{2}p\phi_1^2\} \, dx$$

$$= \int_0^1 \{\tfrac{1}{2}(\phi_1')^2 - e^{\phi_1}\} \, dx, \quad \phi_1(0) = \phi_1(1) = 0, \tag{50}$$

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and

$$K(T\psi_2) = \int_0^1 \{(L\psi_2) f^{-1}(L\psi_2) - \tfrac{1}{2}\psi_2 L\psi_2 - \exp[f^{-1}(L\psi_2)]\}\, dx,$$
$$\psi_2(0) = \psi_2(1) = 0. \tag{51}$$

Since

$$f'(\psi) = e^{\psi} - p < 0$$

for all functions ψ satisfying (44), we see that (31) is satisfied, and the global complementary principles

$$K(T\psi_2) \leqslant K(T\phi) = J(\phi) \leqslant J(\phi_1) \tag{52}$$

hold. The minimum principle for J was given previously by Arthurs and Winthrop [4], but the maximum principle for K appears to be new.

We can use (52) to obtain an approximation to the exact function ϕ. Taking $p = \exp(0.142)$ we have performed calculations with the trial functions

$$\left.\begin{aligned} \phi_1 &= \sum_{n=1}^{3} a_n (x - x^2)^n, \\ \psi_2 &= \sum_{n=1}^{3} b_n (x - x^2)^n, \end{aligned}\right\} \tag{53}$$

where the parameters a_n and b_n were determined by optimizing J and K.

The results are

$$\begin{aligned} a_1 &= 0.54920013, & b_1 &= 0.55020013, \\ a_2 &= 0.05310009, & b_2 &= 0.05300009, \\ a_3 &= -0.00498991, & b_3 &= -0.00301991, \\ J &= -1.0465168, & K &= -1.0465168. \end{aligned} \tag{54}$$

Since $J - K$ provides a measure of the mean square error in the function ϕ_1 [see 3], we conclude that ϕ_1 is a very good approximate solution of the problem in (36) and (37). The estimate (slightly corrected) given in [4] shows that

$$|\phi_1 - \phi| < 1.7 \times 10^{-4}.$$

4. Example 2

Our second example concerns the nonlinear two-point boundary value problem

$$\frac{d^2 y}{dx^2} + \sin y(x) = 0, \quad 0 < x < 3, \tag{55}$$

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with

$$y(0) = 0, \quad y(3) = B > 0. \tag{56}$$

This has been studied numerically by Bailey *et* al. [5] and they find that iteration methods provide relatively slow convergence to the unique solution y.

Since the boundary conditions in (56) are not homogeneous we shall make them so by setting

$$y(x) = \phi(x) + \tfrac{1}{3}Bx, \tag{57}$$

which gives the new problem

$$\frac{d^2\phi}{dx^2} + \sin(\phi + \tfrac{1}{3}Bx) = 0, \quad 0 < x < 3, \tag{58}$$

with

$$\phi(0) = \phi(3) = 0. \tag{59}$$

Equations (58) and (59) provide an example of our class of problems in (3) with

$$S = \frac{d}{dx}, \quad S^* = -\frac{d}{dx}, \tag{60}$$

$$g(\phi) = \sin(\phi + \tfrac{1}{3}Bx), \tag{61}$$

and

$$\langle \phi, \psi \rangle = \int_0^3 \phi\psi\, dx, \quad (u, v) = \int_0^3 uv\, dx. \tag{62}$$

Here we see that $g(\phi)$ is not monotone.

To reformulate the problem as in Section 1 we need to find a positive number p such that

$$g'(\psi) \leqslant p \leqslant \lambda_0 \quad \text{for all } \psi. \tag{63}$$

For this example, (63) is satisfied by choosing p in the range

$$1 \leqslant p \leqslant \tfrac{1}{9}\pi^2. \tag{64}$$

The function q in the decomposition (14) again satisfies (46) and is given over the whole range by (47) with c chosen to be $c = -\frac{3}{2}$. We can therefore use the results of Section 2 with

$$L = -\frac{d^2}{dx^2} - p \tag{65}$$

and

$$f(\psi) = \sin(\psi + \tfrac{1}{3}Bx) - p\psi. \tag{66}$$

By (25) and (28) the dual functionals J and K are

$$J(\phi_1) = \int_0^3 \{\tfrac{1}{2}\phi_1 L\phi_1 + \cos(\phi_1 + \tfrac{1}{3}Bx) + \tfrac{1}{2}p\phi_1^2\}\,dx$$

$$= \int_0^3 \{\tfrac{1}{2}(\phi_1')^2 + \cos(\phi_1 + \tfrac{1}{3}Bx)\}\,dx, \quad \phi_1(0) = \phi_1(3) = 0, \tag{67}$$

and

$$K(T\psi_2) = \int_0^3 \left\{(L\psi_2)f^{-1}(L\psi_2) - \tfrac{1}{2}\psi_2 L\psi_2 + \tfrac{1}{2}p[f^{-1}(L\psi_2)]^2 + \cos\left[f^{-1}(L\psi_2) + \frac{Bx}{3}\right]\right\} dx,$$

$$\psi_2(0) = \psi_2(3) = 0. \tag{68}$$

Since

$$f'(\psi) = \cos(\psi + \tfrac{1}{3}Bx) - p \leqslant 0 \tag{69}$$

for values of p in the range (64), we see that (31) is satisfied and the global complementary principles

$$K(T\psi_2) \leqslant K(T\phi) = J(\phi) \leqslant J(\phi_1) \tag{70}$$

hold. These principles appear to be new.

To obtain an approximate solution of (58) and (59), and hence of (55) and (56), we have performed calculations with the trial functions

$$\phi_1 = \sum_{n=1}^{7} a_n\{(\tfrac{1}{3}x)^{n+1} - \tfrac{1}{3}x\}, \tag{71}$$

and

$$\psi_2 = \sum_{n=1}^{3} b_n\{x(3-x)\}^n. \tag{72}$$

To avoid difficulties with f^{-1} in (68) we took

$$p = \tfrac{1}{9}\pi^2, \tag{73}$$

and for comparison with the results in [5] we chose

$$B = 2.7. \tag{74}$$

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The parameters a_n and b_n were found by optimizing J and K, and the results are

$$\begin{aligned} a_1 &= -2.445, & a_5 &= -0.423, & b_1 &= 0.320, \\ a_2 &= -2.937, & a_6 &= -0.078, & b_2 &= 0.0356, \\ a_3 &= 1.322, & a_7 &= -0.145, & b_3 &= -0.0042, \\ a_4 &= 1.285, & J &= -0.2546, & K &= -0.2822. \end{aligned} \tag{75}$$

By (57), our variational solution of the original problem in (55) and (56) is

$$y_1 = \phi_1 + \tfrac{1}{3}Bx. \tag{76}$$

The variational bounds in (75) indicate that y_1 is quite a good approximate solution and to check this we have also obtained a numerical solution. Table 1 provides a comparison between these two solutions which are seen to be in very close agreement.

TABLE 1

Comparison of variational and numerical solutions in example 2

x	y_1	y (numerical)
0	0	0
0.5	0.938	0.938
0.75	1.335	1.343†
1.0	1.678	1.688
1.5	2.193	2.197†
2.0	2.502	2.505
2.25	2.595	2.599†
2.5	2.657	2.661
3.0	2.700	2.700

† These values agree with the numerical values given in [5].

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Department of Mathematics
University of York
York, YO1 5DD
England