A METRICAL THEOREM IN DIOPHANTINE APPROXIMATION

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Introduction. In this paper we prove a sharpening and generalization of the following Theorem of Khintchine **(4)**:

Let $\psi_1(q), \ldots, \psi_n(q)$ be n non-negative functions of the positive integer q and assume

$$\psi(q) = \prod_{i=1}^{n} \psi_i(q)$$

is monotonically decreasing. Then the set of inequalities

$$0 \leqq q\theta_i - p_i < \psi_i(q) \qquad (i = 1, \ldots, n) \tag{1}$$

has an infinity of integer solutions $q > 0$ and $p_1, \ldots, p_n$ for almost all or no sets of numbers $\theta_1, \ldots, \theta_n$, according as $\sum \psi(q)$ diverges or converges.

Actually, Khintchine proved the Theorem with $|q\theta_i - p_i| < \psi_i(q)$ instead of (1). The first author who used the one-sided inequalities (1) was Cassels **(1)**.

Surprisingly, the following sharpening of the Theorem seems to have escaped attention.

THEOREM 1. *Make the same assumptions as in Khintchine's Theorem. Let $\epsilon > 0$ be arbitrary. Write $N(h; \theta_1, \ldots, \theta_n)$ for the number of solutions of (1) with $1 \leqq q \leqq h$ and put*

$$\Psi(h) = \sum_{q=1}^{h} \psi(q) \tag{2}$$

$$\Omega(h) = \sum_{q=1}^{h} \psi(q) q^{-1}. \tag{3}$$

Then

$$N(h; \theta_1, \ldots, \theta_n) = \Psi(h) + O(\Psi^{\frac{1}{2}}(h)\Omega^{\frac{1}{2}}(h) \log^{2+\epsilon} \Psi(h)) \tag{4}$$

for almost all sets $\theta_1, \ldots, \theta_n$.

Note. In this paper, log α stands as an abbreviatson for

$$\begin{cases} \text{logarithm } \alpha, \text{ if } \alpha \geqq e \\ 1, \text{ if } \alpha < e. \end{cases}$$

Only $\log(1+1(1/q-1))$ in (10) means logarithm, in spite of $1+(1/q-1) < e$.

Next, we generalize Khintchine's Theorem to linear forms. We use the following notation. Throughout this paper, lower case italics denote rational

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integers. By $Q, R, \ldots,$ we denote lattice points $Q(q_1, \ldots, q_m)$ in R_m. Θ denotes points $(\theta_1, \ldots, \theta_m)$ in R_m. ρQ, where ρ is real, is the point with co-ordinates $\rho q_1, \ldots, \rho q_m$, and $Q\Theta$ is the scalar product $q_1\theta_1 + \ldots + q_m\theta_m$. We write $d(Q)$ for the number of common divisors of $q_1, \ldots, q_m$. Finally, we put $Q \leqq h$ if $q = \max(q_1, \ldots, q_m) \leqq h$, and similarly $h < Q$.

THEOREM 2. *Let $\epsilon > 0$ be arbitrary. Let $\psi_1(Q), \ldots, \psi_n(Q)$ be n bounded non-negative functions. We introduce*

$$\psi(Q) = \prod_{i=1}^{n} \psi_i(Q)$$

$$\Psi(h) = \sum_{Q \leqq h} \psi(Q)$$

$$\chi(h) = \sum_{Q \leqq h} \psi(Q) d(Q)$$

and write $N(h; \Theta_1, \ldots, \Theta_n)$ for the number of simultaneous solutions $Q \leqq h$, $p_1, \ldots, p_n$ of the system

$$0 \leqq Q\Theta_i - p_i < \psi_i(Q) \qquad (i = 1, \ldots, n). \tag{5}$$

Then for almost all n-tuples $\Theta_1, \ldots, \Theta_n$

$$N(h; \Theta_1, \ldots, \Theta_n) = \Psi(h) + O(\chi^{\frac{1}{2}}(h) \log^{3/2+\epsilon} \chi(h)). \tag{6}$$

Note. We need not assume $\psi(Q)$ to be monotonic in any co-ordinate.

This theorem can be interpreted as a generalization of the well-known fact that the points $(Q\Theta_1, \ldots, Q\Theta_n)$ are uniformly distributed mod 1 for almost all $\Theta_1, \ldots, \Theta_n$. (See, for instance, (**3**, chapter IV).) Indeed, putting $\psi_i(Q) = \alpha_i$, $\alpha = \prod \alpha_i$, we have $\Psi(h) = \alpha h^m$ and

$$\chi(h) = \alpha \sum_{Q \leqq h} d(Q) = \begin{cases} O(h \log h), & \text{if } m = 1 \\ O(h^m), & \text{if } m > 1. \end{cases}$$

An interesting special case of Theorem 1 is when $\psi(Q) = \psi(q)$, where $q = \max(q_1, \ldots, q_m)$. Then

$$\chi(h) = O\Bigg(\sum_{d \leqq h} \sum_{\substack{q_1 \leqq h \\ d | q_1}} \sum_{\substack{q_2 \leqq q_1 \\ d | q_2}} \cdots \sum_{\substack{q_m \leqq q_1 \\ d | q_m}} \psi(q_1) \Bigg)$$

$$= O\Bigg(\sum_{d \leqq h} \sum_{\substack{q_1 \leqq h \\ d | q_1}} \psi(q_1) \left(\frac{q_1}{d}\right)^{m-1} \Bigg).$$

Thus we have

$$\chi(h) = O(\Psi(h))$$

if $m \geqq 3$, or if $m = 2$ and $q\psi(q)$ is monotonically decreasing, because in the latter case

$$\sum_{\substack{q_1 \leqq h \\ d | q_1}} \psi(q_1) q_1 \leqq d^{-1} \Psi(h).$$

For example, if $\psi_i(Q) = \psi_i(q) = q^{-m/n}$, $\psi(Q) = q^{-m}$, $\Psi(h) = m \log h + O(1)$, then for almost all $\Theta_1, \ldots, \Theta_m$

$$N(h; \Theta_1, \ldots, \Theta_m) = m \log h + O\,(\log^{\frac{1}{2}} h \operatorname{loglog}^{\alpha+\epsilon} h),$$

where we may take $\alpha = 2$ for $m = 1$, according to Theorem 1, and $\alpha = 3/2$ for $m > 1$, according to Theorem 2.

For the proof we have to modify the standard proof of Khintchine's Theorem and use some ideas of **(2)**. The new idea in Theorem 1 is to use fractions p/q with g.c.d.$(p, q) \leqq k$ where k is specified later, instead of p/q with g.c.d.$(p, q) = 1$, as employed in **(1; 3; 4)**. Theorems 1 and 2 should be compared with similar results I proved recently in the geometry of numbers **(5)**.

We give a detailed proof of Theorem 1 only. For convergent sums $\sum\psi(q)$ Theorem 1 follows from Khintchine's Theorem. Hence in §§ 1 to 4, which deal with Theorem 1, we assume without explicit mention that $\psi(q)$ is a non-negative, monotonically decreasing function with divergent sum $\sum\psi(q)$. $\Psi(h)$ and $\Omega(h)$ are defined by (2) and (3). The author is much indebted to the referee who discovered a mistake in the original draft and made valuable suggestions.

1. On certain intervals. Let $\omega(h)$, $h \geqq 1$, be a monotonically increasing integral-valued function which tends to infinity. We write $\omega(0) = 0$ and define S' to be the set consisting of 0 and of all integers $h > 0$ such that $\omega(h - 1) < \omega(h)$. We define S'' to be the set of integers $h \geqq 0$ having $\omega(h) < \omega(h + 1)$. Finally, S is the set of values of $\omega(h)$, $h \geqq 0$.

Next, we define for fixed $t > 0$ intervals of order t to be the half-open intervals

$$(u2^t + v_1,\ (u + 1)2^t + v_2],$$

where u, v_1, v_2 are non-negative integers such that $v_1 < 2^t$ and v_1, v_2 are the smallest non-negative integers satisfying $u2^t + v_1 \in S$, $(u + 1)2^t + v_2 \in S$. (It is possible, of course, that for given u, t there exists no such v_1.) The intervals of order t cover the positive axis exactly once.

LEMMA 1. *Every interval* $(0, x]$, $x \in S$, *can be expressed as union of intervals* $\bigcup I_i$ *of the type described above, where no two of the intervals* I_i *are of the same order.*

Proof. Write x in the binary scale,

$$x = \sum_{i=0}^{w} t_i 2^i$$

where t_i equals 0 or 1, but $t_w = 1$. There exists an interval $(0, j_1]$ of order w with $j_1 \leqq x$. If $j_1 = x$, then we are through. If not, and if

$$j_1 = \sum_{i=0}^{w} t_i^{(2)} 2^i,$$

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then $t_w^{(2)} = t_w = 1$ and there exists a largest integer w_2 having

$$t_{w_2}^{(2)} < t_{w_2}.$$

Hence there exists an interval $(j_1, j_2]$ of order w_2, $j_2 \leqq x$. If $j_2 = x$, then $(0, x] = (0, j_1] \cup (j_1, j_2]$. Otherwise, if

$$j_2 = \sum_{i=0}^{w} t_i^{(3)} 2^i, \quad t_w^{(3)} = t_w = 1, \ldots, t_{w_2}^{(3)} = t_{w_2} = 1,$$

then there exists a largest w_3, $w_3 < w_2$, having

$$t_{w_3}^{(3)} < t_{w_3}.$$

We proceed as before. Since $j_1 < j_2 < \ldots$, we finally arrive at $j_f = x$ and $(0, x] = (0, j_1] \cup \ldots \cup (j_{f-1}, j_f]$. The orders of the intervals are $w > w_2 > \ldots > w_f > 0$.

2. Sums involving a function $\phi(k, q)$. Let k, q be positive and write $\phi(k, q)$ for the number of integers x, $0 \leqq x < q$, so that g.c.d.$(x, q) \leqq k$.

LEMMA 2.

$$\sum_{q=1}^{v} \phi(k, q) q^{-1} = v + O(vk^{-1} + \log v \log k).$$

Note. Here and throughout the paper, the inequality indicated by the O-symbol holds for *all* values of *all* variables involved.

Proof. Clearly,

$$\phi(k, q) = \sum_{\substack{w \mid q \\ w \leqq k}} \phi\left(\frac{q}{w}\right),$$

where $\phi(x)$ is the Euler ϕ-function. Using the well-known relation

$$\phi(x) = x \sum_{y \mid x} \mu(y) y^{-1},$$

we obtain

$$\begin{aligned} &\sum_{q=1}^{v} \phi(k, q) q^{-1} \\ &\quad = \sum_{q=1}^{v} q^{-1} \sum_{\substack{w \mid q \\ w \leqq k}} q w^{-1} \sum_{y \mid q w^{-1}} \mu(y) y^{-1} \\ &\quad = \sum_{w=1}^{\min(k, v)} w^{-1} \sum_{y=1}^{[(v/w)]} \mu(y) y^{-1} \sum_{q=1}^{[(v/yw)]} 1, \end{aligned}$$

where $[\alpha]$ is the integral part of α. Thus

$$\sum_{q=1}^{v} \phi(k, q)q^{-1}$$
$$= v \sum_{w=1}^{\min(k,v)} w^{-2} \sum_{y=1}^{[(v/w)]} \mu(y)y^{-2} + O(\log v \log k)$$
$$= v \sum_{w=1}^{\min(k,v)} w^{-2}\xi(2)^{-1} + O\left(\sum_{w=1}^{\min(k,v)} w^{-1}\right) + O(\log v \log k)$$
$$= v + O(vk^{-1} + \log v \log k).$$

LEMMA 3.

$$\sum_{q=1}^{v} \psi(q)\phi(k, q)q^{-1} = \Psi(v) + O(\Psi(v)k^{-1} + \Omega(v) \log k).$$

Proof. Put $\Pi(k, 0) = 0$ and

$$\Pi(k, r) = \sum_{q=1}^{r} \phi(k, q)q^{-1}$$

for $r \geqq 1$. Lemma 2 yields

$$\Pi(k, r) = r + O(rk^{-1} + \log r \log k). \tag{7}$$

Using partial summation we obtain

$$\sum_{q=1}^{v} \psi(q)\phi(k, q)q^{-1}$$
$$= \sum_{q=1}^{v} \psi(q)(\Pi(k, q) - \Pi(k, q-1))$$
$$= \sum_{q=1}^{v-1} \Pi(k, q)(\psi(q) - \psi(q+1)) + \Pi(k, v)\psi(v) \tag{8}$$
$$= \sum_{q=1}^{v-1} q(\psi(q) - \psi(q+1)) + v\psi(v) + R(k, v)$$
$$= \Psi(v) + R(k, v),$$

where, according to (7),

$$R(k, v)$$
$$= O\left(\sum_{q=1}^{v-1} (qk^{-1} + \log q \log k)(\psi(q) - \psi(q+1)\right) \tag{9}$$
$$+ O(vk^{-1} + \log v \log k)\psi(v)$$
$$= O\left(\Psi(v)k^{-1} + \log k \sum_{q=2}^{v} \psi(q)(\log q - \log (q-1)) + \log k\psi(1)\right).$$

Now

$$\sum_{q=2}^{v} \psi(q)(\log q - \log (q-1))$$

$$(10) \qquad = O\Big(\sum_{q=2}^{v} \psi(q) \log \Big(1 + \frac{1}{q-1}\Big)\Big)$$

$$= O(\Omega(v)).$$

Lemma 3 is a consequence of (8), (9), and (10).

3. Bounds for certain integrals. We introduce the following functions and integrals.

$$\beta(q, \theta) = \begin{cases} 1, \text{ if } 0 \leqq \theta < \psi(q) \\ 0 \text{ otherwise,} \end{cases}$$

$$\gamma(q, \theta) = \sum_{p} \beta(q, q\theta - p),$$

$$\gamma(k, q, \theta) = \sum_{\substack{p \\ \text{g.c.d.}(p,q) \leqq k}} \beta(q, q\theta - p),$$

$$I(q) = \int_0^1 \gamma(q, \theta) d\theta,$$

$$I(k; q) = \int_0^1 \gamma(k, q, \theta) d\theta,$$

$$I(k; q, r) = \int_0^1 \gamma(k, q, \theta)\gamma(k, r, \theta) d\theta,$$

$$\Psi(u, v) = \sum_{u+1}^{v} \psi(q).$$

We observe

$$N(v, \theta) = \sum_{q=1}^{v} \gamma(q, \theta)$$

and put

$$N(k; u, v; \theta) = \sum_{q=u+1}^{v} \gamma(k, q, \theta).$$

LEMMA 4.

$$(11) \qquad I(q) = \psi(q); \qquad I(k; q) = \psi(q)\phi(k, q)q^{-1}$$

$$(12) \qquad \mathrm{I}(k; q, r) \leqq \psi(q)\psi(r) + A(k; q, r)\psi(q)q^{-1},$$

where $A(k; q, r)$ is the number of solutions p, s of

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(13) $$qs - rp = 0 \qquad 0 \leqq p < q$$

having

$$\text{g.c.d.}(p, q) \leqq k, \qquad \text{g.c.d.}(s, r) \leqq k.$$

Proof. $I(q) = \psi(q)$ is rather trivial, while the second half of (11) follows from

$$I(k, q) = \sum_{\substack{p \\ \text{g.c.d.}(p,q)\leqq k}} \int_0^1 \beta(q, \theta q - p)d\theta$$

$$= \phi(k, q)q^{-1} \int_{-\infty}^{\infty} \beta(q, \theta)d\theta.$$

As for $I(k; q, r)$, we have

$$I(k; q, r) = \sum_{\substack{p,\text{g.c.d.}(p,q)\leqq k \\ s,\text{g.c.d.}(s,r)\leqq k}} \int_0^1 \beta(q, \theta q - p)\beta(r, \theta r - s)d\theta.$$

We split this sum into two parts,

$$I(k; q, r) = I_0(k; q, r) + I_1(k; q, r),$$

where I_0 consists of the terms with $qs - rp \neq 0$.

(14) $$I_0(k; q, r) \leqq \sum_{\substack{p,s, \\ qs-rp\neq 0}} \int_0^1 \beta(q, \theta q - p)\beta(r, \theta r - s)d\theta$$

$$= \sum_{\substack{p,s \\ qs-rp\neq 0}} \int_{-(p/q)}^{1-(p/q)} \beta(q, q\theta')\beta\left(r, r\theta' - \frac{qs - rp}{q}\right) d\theta'.$$

To find an estimate for this sum, write $q = q'd$, $r = r'd$, $qs - rp = hd$, where $d =$ g.c.d. (q, r). For given h, p is determined modulo q'. Hence

$$I_0(k; q, r)$$

$$\leqq d \sum_{h\neq 0} \int_{-\infty}^{\infty} \beta(q, q\theta')\beta\left(r, r\theta' - \frac{hd}{q}\right) d\theta'$$

$$\leqq d \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \beta(q, q\theta')\beta\left(r, r\theta' - \lambda dq^{-1}\right) d\theta' d\lambda$$

$$= \psi(q)\psi(r).$$

In changing from the summation over h to the continuous parameter λ we used the fact that the function

$$\int_{-\infty}^{\infty} \beta(q, q\theta')\beta(r, r\theta' - \lambda dq^{-1})\, d\theta'$$

is monotonically decreasing in λ when $\lambda \geqq 0$, and monotonically increasing when $\lambda \leqq 0$.

To prove Lemma 4 it remains to give an upper bound for $I_1(k; q, r)$. In analogy to (14), we find

$$I_1(k; q, r) = \sum_{\substack{p, \text{g.c.d.}(p,q) \leqq k \\ s, \text{g.c.d.}(s,r) \leqq k \\ qs - rp = 0}} \int_{-(p/q)}^{1-(p/q)} \beta(q, q\theta')\beta(r, r\theta')d\theta'$$

$$\leqq A(k; q, r)\psi(q)q^{-1}.$$

LEMMA 5.

$$\int_0^1 N(v, \theta)d\theta = \Psi(v)$$

$$\int_0^1 N(k; u, v; \theta)d\theta = \sum_{q=u+1}^{v} \psi(q)\phi(k, q)q^{-1}$$

$$\int_0^1 N^2(k; u, v; \theta)d\theta \leqq \Psi^2(u, v) + 2\sum_{q=u+1}^{v} \psi(q)d_k(q),$$

where $d_k(q)$ is the number of divisors of q not exceeding k.

Proof. The first two assertions follow from (11). As an immediate consequence of (12) we have

$$\int_0^1 N^2(k; u, v; \theta)d\theta \leqq \Psi^2(u, v) + 2\sum_{u<r\leqq q\leqq v} A(k; q, r)\psi(q)q^{-1}.$$

Now

$$\sum_{r=1}^{q} A(k; q, r)$$

is equal to the number of solutions r, p, s of

$$qs - rp = 0, \qquad 0 \leqq p < q, \qquad 1 \leqq r \leqq q$$
$$\text{g.c.d.}(p, q) \leqq k, \qquad \text{g.c.d.}(s, r) \leqq k.$$

Define a, b by

$$\frac{a}{b} = \frac{p}{q} = \frac{s}{r}, \qquad \text{g.c.d. } (a, b) = 1.$$

Then b/q and g.c.d.$(p, q) \leqq k$ implies $qb^{-1} \leqq k$. Thus the number of possible choices for b is $d_k(q)$. Furthermore, there are $\phi(b) \leqq b$ possibilities for a and qb^{-1} possibilities for r, once b is given. Hence

$$\sum_{r=1}^{q} A(k; q, r) \leqq q\, d_k(q)$$

and

$$\sum_{u<r\leqq q\leqq v} A(k; q, r)\psi(q)q^{-1} \leqq \sum_{q=u+1}^{v} \psi(q)\, d_k(q).$$

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4. Proof of Theorem 1 ($n = 1$). Write $\omega(h) = [\Psi(h)\Omega(h)]$ and define S, S', S'' as in § 1. Let L_s be the set of all pairs (u, v), $u \in S'$, $v \in S'$, so that $(\omega(u), \omega(v)]$ is an interval of any order t with respect to ω (see § 1), and $\omega(v) \leqq 2^s$. From now on, the numbers k, s are always connected by the relation

$$k = 2^s. \tag{15}$$

From here on, we make heavy use of the methods developed in **(2)**. Write $h^* = h^*(s)$ for the largest integer h^* having $\omega(h^*) \leqq 2^s$.

LEMMA 6.

$$0 \leqq \int_0^1 (N(h^*, \theta) - N(k; 0, h^*; \theta))d\theta = O(s\, 2^{s/2}) \tag{16}$$

$$\sum_{(u,v)\in L_s} \int_0^1 (N(k; u, v; \theta) - \Psi(u, v))^2 d\theta = O(s^2\, 2^s). \tag{17}$$

Proof. The first two equations of Lemma 5 give

$$\int_0^1 (N(h^*, \theta) - N(k; 0, h^*, \theta))d\theta$$

$$= \Psi(h^*) - \sum_{q=1}^{h^*} \psi(q)\phi(k, q)q^{-1}$$

$$= O(\Psi(h^*)k^{-1} + \Omega(h^*) \log k$$

according to Lemma 3. Since

$$\Omega(h^*) = O(2^{\frac{1}{2}s}),$$

(16) follows.

Using Lemma 5 again we see that a single integral in (17) does not exceed

$$2 \sum_{q=u+1}^{v} \psi(q)d_k(q) + 2\Psi(u, v)(\Psi(u, v) - \sum_{q=u+1}^{v} \psi(q)\phi(k, q)q^{-1}).$$

We first take the sum over those pairs $(u, v) \in L_s$ where $(\omega(u), \omega(v)]$ is an interval of fixed order t. Since intervals of order t cover the positive axis exactly once, we obtain the upper bound

$$2 \sum_{q=1}^{h^*} \psi(q)d_k(q) + 2\Psi(h^*)(\Psi(h^*) - \sum_{q=1}^{h^*} \psi(q)\phi(k, q)q^{-1}).$$

We observe

$$\sum_{q=1}^{h^*} \psi(q)d_k(q) \leqq 2^s \sum_{t=1}^{k} t^{-1} = O(2^s \log k)$$

and using Lemma 3 we find the upper bound

$$O(2^s \log k) + O(\Psi^2(h^*)k^{-1} + \Psi(h^*)\Omega(h^*) \log k) = O(s2^s).$$

Summing over t and observing $t \leqq s$ we obtain (17).

LEMMA 7. *There is a sequence of subsets $\sigma_1, \sigma_2, \ldots$ of the unit-interval with measures*

$$\mu_s = \int_{\sigma_s} d\theta = O(s^{-1-\epsilon})$$

such that

$$N(h, \theta) = \Psi(h) + O(2^{s/2} s^{2+\epsilon})$$

for any h with $\omega(h) \leqq 2^s$, $h \in S'$, and any θ in $0 \leqq \theta < 1$, but not in σ_s.

Proof. We define σ_s to be the set of all θ in $0 \leqq \theta < 1$, for which not both of the following two inequalities hold:

$$0 \leqq N(h^*, \theta) - N(k; 0, h^*; \theta) \leqq s^{2+\epsilon} 2^{\frac{1}{2}s} \tag{18}$$

$$\sum_{(u,v)\in L_s} (N(k; u, v; \theta) - \Psi(u, v))^2 \leqq s^{3+\epsilon} 2^s. \tag{19}$$

As a consequence of Lemma 6,

$$\mu_s = O(s^{-1-\epsilon}).$$

If $h \leqq h^*$, $h \in S'$, then the interval $(0, \omega(h)]$ is the union of at most s intervals $(\omega(u), \omega(v)]$, where $(u, v) \in L_s$.

$$N(k; 0, h; \theta) - \Psi(h) = \sum (N(k; u, v; \theta) - \Psi(u, v)),$$

where the sum is over at most s pairs $(u, v) \in L_s$. This fact, together with (19) and Cauchy's inequality yields for $0 \leqq \theta < 1$, $\theta \notin \sigma_s$,

$$(N(k; 0, h; \theta) - \Psi(h))^2 \leqq s^{4+\epsilon} 2^s.$$

The last equation together with (18) gives Lemma 7.

Proof of Theorem 1 ($n = 1$). Since $\sum s^{-1-\epsilon}$ is convergent, there exists for almost all θ, $0 \leqq \theta < 1$, an $s_0 = s_0(\theta)$ such that $\theta \notin \sigma_s$ for $s \geqq s_0$. Assume θ has such an $s_0(\theta)$ and assume h to be so large that $\omega(h) \geqq 2^{s_0}$. Choose s so that $2^{s-1} \leqq \omega(h) < 2^s$.

Suppose $h \in S'$. Then we have with Lemma 7

$$\begin{aligned} N(h, \theta) &= \Psi(h) + O(2^{\frac{1}{2}s} s^{2+\epsilon}) \\ &= \Psi(h) + O(\Psi^{\frac{1}{2}}(h)\Omega^{\frac{1}{2}}(h) \log^{2+\epsilon}\Psi(h)). \end{aligned}$$

Hence Theorem 1 holds for $h \in S'$. By the same argument we can prove the Theorem for $h \in S''$.

To any h there exist h', h'' with $h' \in S'$, $h'' \in S''$ and

$$\omega(h') = \omega(h) = \omega(h'').$$
$$|\Psi(h)\Omega(h) - \Psi(h')\Omega(h')| \leqq 1.$$

Then

$$|\Psi(h) - \Psi(h')| \leqq \Omega(h)^{-1} \leqq \Omega(1)^{-1} = \psi(1)^{-1},$$

and similarly for $\Psi(h'')$. Since

$$N(h', \theta) \leqq N(h, \theta) \leqq N(h'', \theta),$$

the case $n = 1$ of Theorem 1 follows.

5. The case $n \geqq 2$. Using

$$\begin{aligned} v - \sum_{q=1}^{v} \phi^n(k, q)q^{-n} &= \sum_{q=1}^{v} (q^n - \phi^n(k, q))q^{-n} \\ &\leqq n \sum_{q=1}^{v} (q - \phi(k, q))q^{n-1}q^{-n} \\ &= n\left(v - \sum_{q=1}^{v} \phi(k, q)q^{-1}\right) \end{aligned}$$

we easily generalize Lemmas 2, 3 to

$$\sum_{q=1}^{v} \phi^n(k, q)q^{-n} = v + O(vk^{-1} + \log k \log v),$$

$$\sum_{q=1}^{v} \psi(q)\phi^n(k, q)q^{-n} = \Psi(v) + O(\Psi(v)k^{-1} + \Omega(v) \log k).$$

In analogy to $\beta(q, \theta)$ of § 3 we define $\beta(q, \theta_1, \ldots, \theta_n)$ to be the characteristic function of the rectangle

$$0 \leqq \theta_i < \psi_i(q) \qquad (i = 1, \ldots, n)$$

and put

$$\gamma(q, \theta_1, \ldots, \theta_n) = \sum_{p_1, \ldots, p_n} \beta(q, q\theta_1 - p_1, \ldots, q\theta_n - p_n)$$

$$\gamma(k; q, \theta_1, \ldots, \theta_n) = \sum_{\substack{p_i, \text{g.c.d.}(p_i, q) \leqq k \\ i=1, \ldots, n}} \beta(q, q\theta_1 - p_1, \ldots, q\theta_n - p_n).$$

$I(q)$, $I(k, q)$, $I(k; q, r)$ are now n-dimensional integrals. To find an upper bound for

$$I(k; q, r) = \sum_{\substack{p_i, \text{g.c.d.}(p_i, q) \leqq k \\ s_i, \text{g.c.d.}(s_i, r) \leqq k \\ i=1, \ldots, n}} \int_0^1 \ldots \int_0^1 \beta(q, q\theta_1 - p_1, \ldots,) \, \beta(r, r\theta_1 - s_1, \ldots,) d\theta_1 \ldots d\theta_n,$$

we split this sum into $n + 1$ parts,

$$I(k; q, r) = I_0 + \ldots + I_n,$$

where I_j consists of the terms with exactly j indices $i_1, \ldots, i_j$ having $qs_i - rp_i = 0$. We find

$$I_0(k; q, r) \leqq \psi(q)\psi(r)$$

and

$$I_j(k; q, r) \leqq c^{(j)} A^j(k; q, r)\psi(q)q^{-j}$$
$$\leqq c^{(j)} A(k; q, r)\psi(q)q^{-1}.$$

There are no other modifications of any depth.

6. On the proof of Theorem 2. For simplicity assume $n = 1$. We put

$$\beta(Q, \theta) = \begin{cases} 1, \text{ if } 0 \leqq \theta < \psi(Q) \\ 0 \text{ otherwise} \end{cases}$$

and define $\gamma(Q, \theta)$, $I(Q)$ in an obvious way. Further

$$I(Q, R) = \int_0^1 \gamma(Q, \theta)\gamma(R, \theta)d\theta,$$

$$\Psi(u, v) = \sum_{u<Q\leqq v} \psi(Q).$$

We observe

$$N(v, \theta) = \sum_{Q\leqq v} \gamma(Q, \theta)$$

and put

$$N(u, v, \theta) = \sum_{u<Q\leqq v} \gamma(Q, \theta).$$

We do not need the parameter k now, which was essential in Theorem 1. Lemma 4 now reads

LEMMA 4*a*.

$$I(Q) = \psi(Q) \tag{20}$$

$$I(Q, R) = \psi(Q)\psi(R), \tag{21}$$

if Q, R are linearly independent (there exists no ρ having $Q = \rho R$).

$$I(Q, R) \leqq \psi(Q)\psi(R) + c\, A(q_1, r_1)\psi(Q)q_1^{-1}, \tag{22}$$

if Q, R are linearly dependent. Here q_1, r_1 are the first co-ordinates of Q, R and $A(q_1, r_1)$ is the number of solutions p, s of

$$q_1 s - r_1 p = 0 \qquad 0 \leqq p < q.$$

(20) *and* (21) *are proved like* (11), *while the proof of* (22) *is like the one given for* (12). *Lemma* 5 *becomes*

LEMMA 5*a*.

$$\int_0^1 N(u, v, \theta)d\theta = \sum_{u<Q\leqq v} \psi(Q) = \Psi(u, v)$$

$$\int_0^1 N^2(u, v, \theta)d\theta \leqq \Psi^2(u, v) + c \sum_{u<Q\leqq v} \psi(Q)d(Q).$$

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All the other changes in the proof are obvious, except perhaps the definition of $\omega(h)$, namely $\omega(h) = [\chi(h)]$.

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