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LOCALIZATIONS OF THE HEARTS OF COTORSION PAIRS

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Abstract. In this article, we study localizations of hearts of cotorsion pairs $(\mathcal{U}, \mathcal{V})$ where $\mathcal{U}$ is rigid on an extriangulated category $\mathcal{B}$. The hearts of such cotorsion pairs are equivalent to the functor categories over the stable category of $\mathcal{U}$ $(\mathrm{mod}\,\underline{\mathcal{U}})$. Inspired by Marsh and Palu (*Nagoya Math. J.* **225**(2017), 64–99), we consider the mutation (in the sense of Iyama and Yoshino, *Invent. Math.* **172**(1) (2008), 117–168) of $\mathcal{U}$ that induces a cotorsion pair $(\mathcal{U}', \mathcal{V}')$. Generally speaking, the hearts of $(\mathcal{U}, \mathcal{V})$ and $(\mathcal{U}', \mathcal{V}')$ are not equivalent to each other, but we will give a generalized pseudo-Morita equivalence between certain localizations of their hearts.

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1. Introduction. When we say localization in this article, we mean Gabriel–Zisman localization which is introduced in [7]. A well-known example of such localization is the bounded derived category of a module category; it is a localization of homotopy category of complexes. An example on triangulated categories is given in [4]. They proved that the category of finite-dimensional modules over the endomorphism algebra of a rigid object in a Hom-finite triangulated category is equivalent to the localization of the category with respect to a certain class of morphisms. More localizations are discussed in [5] and also in [13].

Exact categories are used widely in representative theory, and according to the well-known result by Happel, the stable category of a Frobenius category (a special case of exact category) is triangulated [8]. One may ask if we can investigate the similar kinds of localizations on exact categories. Since extriangulated category [17] generalizes both triangulated and exact categories, it is reasonable to study the subject on this more general structure.

In this article, let k be a field, $(\mathcal{B}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category defined in [17] (see Section 2 of [17] for details). Any subcategory discussed in this article will be full, additive, and closed under direct sums and isomorphisms. Let $\mathcal{P}$ (resp. $\mathcal{I}$) be the subcategory of projectives (resp. injectives). For a subcategory $\mathcal{C}$, let $\mathcal{C}^{\perp_1} = \{B \in \mathcal{B} \mid \mathbb{E}(\mathcal{C}, B) = 0\}$ and ${}^{\perp_1}\mathcal{C} = \{B \in \mathcal{B} \mid \mathbb{E}(B, \mathcal{C}) = 0\}$.

We first recall the definition of cotorsion pair and its heart.

Definition 1.1. Let $\mathcal{U}$ and $\mathcal{V}$ be subcategories of $\mathcal{B}$ which are closed under direct summands. We call $(\mathcal{U}, \mathcal{V})$ a *cotorsion pair* if it satisfies the following conditions:

(a) $\mathbb{E}(\mathcal{U}, \mathcal{V}) = 0$.
(b) For any object $B \in \mathcal{B}$, there exist two conflations:

$$V_B \rightarrowtail U_B \twoheadrightarrow B, \quad B \rightarrowtail V^B \twoheadrightarrow U^B$$

satisfying $U_B, U^B \in \mathcal{U}$, and $V_B, V^B \in \mathcal{V}$.

We denote the subcategory of all the objects B such that U_B, $V^B \in \mathcal{U} \cap \mathcal{V}$ in the conflations in (b) by $\mathcal{H}$. We call the ideal quotient $\mathcal{H}/(\mathcal{U} \cap \mathcal{V})$ the heart of $(\mathcal{U}, \mathcal{V})$. It is an abelian category by [12, Theorem 3.2].

DEFINITION 1.2. Let $\mathcal{B}'$, $\mathcal{B}''$ be two subcategories of $\mathcal{B}$, let

$$\mathrm{Cone}(\mathcal{B}', \mathcal{B}'') = \{X \in \mathcal{B} \mid X \text{ admits a conflation } B' \rightarrowtail B'' \twoheadrightarrow X, B' \in \mathcal{B}', B'' \in \mathcal{B}''\},$$

$$\mathrm{CoCone}(\mathcal{B}', \mathcal{B}'') = \{X \in \mathcal{B} \mid X \text{ admits a conflation } X \rightarrowtail B' \twoheadrightarrow B'', B' \in \mathcal{B}', B'' \in \mathcal{B}''\}.$$

When $\mathcal{B}$ has enough projectives and injectives, a rigid subcategory $\mathcal{C}$ which is contravariantly finite and contains $\mathcal{P}$ induces a cotorsion pair $(\mathcal{C}, \mathcal{C}^{\perp_1})$ (see Lemma 2.7 for details); the functor category (see [2]) $\mathrm{mod}(\mathcal{C}/\mathcal{P})$ is equivalent to the heart of $(\mathcal{C}, \mathcal{C}^{\perp_1})$. Let $\mathcal{D} \subset \mathcal{C}$, when we consider the mutation of $\mathcal{C}$: $\mathcal{C}' = \mathrm{CoCone}(\mathcal{D}, \mathcal{C}) \cap \mathcal{D}^{\perp_1}$ (see [9, Definition 2.5]) that induces a cotorsion pair $(\mathcal{C}', \mathcal{C}'^{\perp_1})$ where $\mathcal{C}'$ is also rigid, we can investigate the relation of two functor categories $\mathrm{mod}(\mathcal{C}/\mathcal{P})$ and $\mathrm{mod}(\mathcal{C}'/\mathcal{P})$ by studying the hearts. We know if $\mathcal{C}^{\perp_1} = \mathcal{C}'^{\perp_1}$, the hearts are equivalent [12, Proposition 3.12]. But here it is obviously not the case, since $\mathcal{C}^{\perp_1} = \mathcal{C}'^{\perp_1}$ implies $\mathcal{C} = \mathcal{C}'$.

Although these hearts (hence the functor categories) are not equivalent in general, we can consider the localizations of the hearts (for a quick understanding of localization, one can see [4, Section 3]), since on this level, we can find an equivalence. In this article (compare with [13]), we choose the language of hearts of cotorsion pairs since it simplifies some proofs; it also makes arguments simpler when we deal with general subcategories compared with the ones obtained from objects.

In this article, we will prove the following theorem. Please note that in [16], a similar result has been proved for triangulated category.

THEOREM 1.3. *Let $(\mathcal{B}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category with enough projectives and enough injectives. Let $\mathcal{C}, \mathcal{C}', \mathcal{D}$ be rigid subcategories such that $\mathcal{D} \subseteq \mathcal{C} \cap \mathcal{C}'$. Assume we have three pairs of cotorsion pairs $((\mathcal{C}, \mathcal{C}^{\perp_1}), (\mathcal{C}^{\perp_1}, \mathcal{M}))$, $((\mathcal{D}, \mathcal{D}^{\perp_1}), (\mathcal{D}^{\perp_1}, \mathcal{N}))$, $((\mathcal{C}', \mathcal{C}'^{\perp_1}), (\mathcal{C}'^{\perp_1}, \mathcal{M}'))$ such that* $\mathrm{CoCone}(\mathcal{D}, \mathcal{C}) = \mathrm{CoCone}(\mathcal{C}', \mathcal{D})$ *and* $\mathrm{Cone}(\mathcal{N}, \mathcal{M}) = \mathrm{Cone}(\mathcal{M}', \mathcal{N})$. *Let*

(a) *$\mathcal{H}/\mathcal{C}$ be the heart of $(\mathcal{C}, \mathcal{C}^{\perp_1})$. Denote $(\mathcal{H} \cap \mathcal{D}^{\perp_1})/\mathcal{C}$ by $\mathcal{A}$. Let $\mathcal{S}_{\mathcal{A}}$ be the class of epimorphisms in $\mathcal{H}/\mathcal{C}$ whose kernel belongs to $\mathcal{A}$.*

(b) *$\mathcal{H}'/\mathcal{M}'$ be the heart of $(\mathcal{C}'^{\perp_1}, \mathcal{M}')$. Denote $(\mathcal{H}' \cap \mathcal{D}^{\perp_1})/\mathcal{M}'$ by $\mathcal{A}'$. Let $\mathcal{S}_{\mathcal{A}'}$ be the class of monomorphisms in $\mathcal{H}'/\mathcal{M}'$ whose cokernel belongs to $\mathcal{A}'$.*

Then we have the following equivalences:

$$(\mathcal{H}/\mathcal{C})_{\mathcal{S}_{\mathcal{A}}} \simeq \mathrm{CoCone}(\mathcal{D}, \mathcal{C})/\mathcal{C}' \simeq \mathrm{Cone}(\mathcal{N}, \mathcal{M})/\mathcal{M} \simeq (\mathcal{H}'/\mathcal{M}')_{\mathcal{S}_{\mathcal{A}'}}$$

where $(\mathcal{H}/\mathcal{C})_{\mathcal{S}_{\mathcal{A}}}$ is the localization of $\mathcal{H}/\mathcal{C}$ at $\mathcal{S}_{\mathcal{A}}$ and $(\mathcal{H}'/\mathcal{M}')_{\mathcal{S}_{\mathcal{A}'}}$ is the localization of $\mathcal{H}'/\mathcal{M}'$ at $\mathcal{S}_{\mathcal{A}'}$.

Note that condition $\mathrm{CoCone}(\mathcal{D}, \mathcal{C}) = \mathrm{CoCone}(\mathcal{C}', \mathcal{D})$ implies $\mathcal{C}' = \mathrm{CoCone}(\mathcal{D}, \mathcal{C}) \cap \mathcal{D}^{\perp_1}$. This result allows us to study the similar mutations as in [13] on exact categories, where usually we do not have Serre functors. It generalizes the results by Marsh and Palu (see [13, Theorems 2.9, 3.2]) for the following reason:

REMARK 1.4. Let $\mathcal{B}$ be a Krull–Schmidt, k-linear, Hom-finite triangulated category with suspension functor Σ. Let C be a rigid object and D is a direct summand of C, we have $C = D \oplus R$. Let $\mathcal{C} = \mathrm{add}\, C$ and $\mathcal{D} = \mathrm{add}\, D$, we have the following triangle $R^* \to D_0 \xrightarrow{f} R \to \Sigma R^*$, where f is a minimal right $\mathcal{D}$-approximation. Let $C' = D \oplus R^*$ and $\mathcal{C}' = \mathrm{add}\, C'$,

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we assume $\mathcal{C}'$ is rigid. By [13, Lemma 2.7], we have $\mathrm{CoCone}(\mathcal{D}, \mathcal{C}) = \mathrm{CoCone}(\mathcal{C}', \mathcal{D})$. Under the assumptions for $\mathcal{B}$, we have cotorsoin pairs $(\mathcal{C}, \mathcal{C}^{\perp_1})$, $(\mathcal{D}, \mathcal{D}^{\perp_1})$, and $(\mathcal{C}', \mathcal{C}'^{\perp_1})$. The module category of the endomorphism algebra $\mathrm{End}_{\mathcal{B}}(C)^{\mathrm{op}}$ (resp. $\mathrm{End}_{\mathcal{B}}(C')^{\mathrm{op}}$) is equivalent to the heart of $(\mathcal{C}, \mathcal{C}^{\perp_1})$ (resp. $(\mathcal{C}'^{\perp_1}, \mathcal{M}')$) (see [12, Proposition 4.15]). Moreover, if $\mathcal{B}$ has a Serre functor S, then we also have cotorsion pairs $(\mathcal{C}^{\perp_1}, \Sigma^{-2}S\mathcal{C})$, $(\mathcal{D}^{\perp_1}, \Sigma^{-2}S\mathcal{D})$, and $(\mathcal{C}'^{\perp_1}, \Sigma^{-2}S\mathcal{C}')$. One can check that $\mathrm{Cone}(\Sigma^{-2}S\mathcal{D}, \Sigma^{-2}S\mathcal{C}) = \mathrm{Cone}(\Sigma^{-2}S\mathcal{C}', \Sigma^{-2}S\mathcal{D})$.

In Section 2, we introduce necessary background knowledge of cotorsion pairs and prove some lemmas which will be used later. In Section 3, we study a localization of the heart of a cotorsion pair related to the mutation, which is given as $(\mathcal{H}/\mathcal{C})_{\mathcal{S}_{\mathcal{A}}} \simeq \mathrm{CoCone}(\mathcal{D}, \mathcal{C})/\mathcal{C}'$ in the main theorem. In fact, we show the result in a more general setting; we only need $\mathcal{C}' = \mathrm{CoCone}(\mathcal{D}, \mathcal{C}) \cap \mathcal{D}^{\perp_1}$ without assuming its rigidity. In Section 4, we prove our main theorem. In Section 5, we discuss some localizations of $\mathcal{B}$ related to our main theorem. In the last section, we give some examples of our theorem.

2. Preliminaries. Throughout this article, let $(\mathcal{B}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category defined in [17] (see Section 2 of [17] for details). Let $\mathcal{P}$ (resp. $\mathcal{I}$) be the subcategory of projectives (resp. injectives). Let $\mathcal{C} \supset \mathcal{D}$ be subcategories of $\mathcal{B}$.

For a subcategory $\mathcal{B}'$, we define $\Omega^0\mathcal{B}' = \mathcal{B}'$ and $\Omega^i\mathcal{B}'$ for $i > 0$ inductively by $\Omega^i\mathcal{B}' = \mathrm{CoCone}(\mathcal{P}, \Omega^{i-1}\mathcal{B}')$. We call $\Omega^i\mathcal{B}'$ the *i-th syzygy* of $\mathcal{B}'$; by this definition, we have $\mathcal{P} \subseteq \Omega^i\mathcal{B}'$, $i > 0$. Dually we can define the *i-th cosyzygy* $\Sigma^i\mathcal{B}'$.

From Lemma 2.9 and also in the rest of the sections, we will always assume $\mathcal{B}$ has enough projectives and enough injectives.

We first recall the following proposition ([12, Proposition 1.20]), which (also the dual of it) will be used many times in the article.

Proposition 2.1. *Let $A \overset{x}{\rightarrowtail} B \overset{y}{\twoheadrightarrow} C \overset{\delta}{\dashrightarrow}$ be any $\mathbb{E}$-triangle, let $f : A \to D$ be any morphism, and let $D \overset{d}{\rightarrowtail} E \overset{e}{\twoheadrightarrow} C \overset{f_*\delta}{\dashrightarrow}$ be any $\mathbb{E}$-triangle realizing $f_*\delta$. Then there is a morphism g which gives a morphism of $\mathbb{E}$-triangles:*

$$\begin{array}{ccccccc}
A & \overset{x}{\rightarrowtail} & B & \overset{y}{\twoheadrightarrow} & C & \overset{\delta}{\dashrightarrow} & \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \| & & \\
D & \underset{d}{\rightarrowtail} & E & \underset{e}{\twoheadrightarrow} & C & \underset{f_*\delta}{\dashrightarrow} &
\end{array}$$

and moreover, the sequence $A \overset{\binom{f}{x}}{\rightarrowtail} D \oplus B \overset{(d\ -g)}{\twoheadrightarrow} E \overset{e^*\delta}{\dashrightarrow}$ becomes an $\mathbb{E}$-triangle.

Although most of the time we will deal with cotorsion pairs, it is still necessary to introduce the following more general concept used in the proof of our main theorem.

Definition 2.2. A pair of cotorsion pairs $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ on $\mathcal{B}$ is called a *twin cotorsion pair* if $\mathcal{S} \subseteq \mathcal{U}$.

Remark that any cotorsion pair $(\mathcal{U}, \mathcal{V})$ gives a twin cotorsion pair $((\mathcal{U}, \mathcal{V}), (\mathcal{U}, \mathcal{V}))$. Thus a cotorsion pair can be regarded as a special case of a twin cotorsion pair, satisfying $\mathcal{S} = \mathcal{U}$ and $\mathcal{T} = \mathcal{V}$.

Remark 2.3. For any cotorsion pair $(\mathcal{U}, \mathcal{V})$ on $\mathcal{B}$, the following holds:

(a) An object $B \in \mathcal{U}$ if and only if $\mathbb{E}(B, \mathcal{V}) = 0$.
(b) An object $B \in \mathcal{V}$ if and only if $\mathbb{E}(\mathcal{U}, B) = 0$.

(c) Subcategories $\mathcal{U}$ and $\mathcal{V}$ are closed under extension.
(d) $\mathcal{P} \subseteq \mathcal{U}$ and $\mathcal{I} \subseteq \mathcal{V}$.

DEFINITION 2.4. For any twin cotorsion pair $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$, put $\mathcal{W} = \mathcal{T} \cap \mathcal{U}$.

(a) Subcategory $\mathcal{B}^+$ is defined to be the full subcategory of $\mathcal{B}$, consisting of objects B which admit conflations $V_B \rightarrowtail W_B \twoheadrightarrow B$ where $W_B \in \mathcal{W}$ and $V_B \in \mathcal{V}$.
(b) Subcategory $\mathcal{B}^-$ is defined to be the full subcategory of $\mathcal{B}$, consisting of objects B which admit conflations $B \rightarrowtail W^B \twoheadrightarrow S^B$ where $W^B \in \mathcal{W}$ and $S^B \in \mathcal{S}$.

DEFINITION 2.5. Let $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be a twin cotorsion pair on $\mathcal{B}$, and write the ideal quotient of $\mathcal{B}$ by $\mathcal{W}$ as $\overline{\mathcal{B}} = \mathcal{B}/\mathcal{W}$. For any morphism $f \in \mathrm{Hom}_{\mathcal{B}}(X, Y)$, we denote its image in $\mathrm{Hom}_{\overline{\mathcal{B}}}(X, Y)$ by $\overline{f}$. For any full additive subcategory $\mathcal{B}_1$ of $\mathcal{B}$ containing $\mathcal{W}$, similarly we put $\overline{\mathcal{B}_1} = \mathcal{B}_1/\mathcal{W}$. This is a full subcategory of $\overline{\mathcal{B}}$ consisting of the same objects as $\mathcal{B}_1$.

Let $\mathcal{H} = \mathcal{B}^+ \cap \mathcal{B}^-$. Since $\mathcal{H} \supseteq \mathcal{W}$, we obtain a subcategory $\overline{\mathcal{H}} \subseteq \overline{\mathcal{B}}$, which we call the *heart* of the twin cotorsion pair [**15**, **16**]. It is semi-abelian by [**12**, Theorem 2.32]. In particular, the heart of the twin cotorsion pair $((\mathcal{U}, \mathcal{V}), (\mathcal{U}, \mathcal{V}))$ is the heart of $(\mathcal{U}, \mathcal{V})$ [**3**, **6**].

Let $\mathcal{B}_1 * \mathcal{B}_2 = \{B \in \mathcal{B} \mid B$ admits a conflation $B_1 \rightarrowtail B \twoheadrightarrow B_2, B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\}$, for a cotorsion pair $(\mathcal{U}, \mathcal{V})$ and its heart, according to [**12**, Theorems 3.2, 3.5, Corollary 3.8], we have the following theorem.

THEOREM 2.6. *Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair, then its hearts $\overline{\mathcal{H}}$ is abelian. Moreover, there exists an additive functor $H : \mathcal{B} \to \overline{\mathcal{H}}$ such that*

- *$H|_{\mathcal{H}} = \pi|_{\mathcal{H}}$, where $\pi : \mathcal{B} \to \overline{\mathcal{B}}$ is the quotient functor;*
- *for any object $X \in \mathcal{B}$, $H(X) = 0$ if and only if $X \in \mathrm{add}(\mathcal{U} * \mathcal{V})$; and*
- *for any conflation $A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C$ in $\mathcal{B}$, the sequence $H(A) \xrightarrow{H(f)} H(B) \xrightarrow{H(g)} H(C)$ is exact in $\overline{\mathcal{H}}$.*

We call H the *cohomological functor* associated with $(\mathcal{U}, \mathcal{V})$ [**1**, **12**].

A subcategory $\mathcal{B}'$ is called *contravariantly finite* if any object in $\mathcal{B}$ admits a right $\mathcal{B}'$-approximation. Moreover, it is called *fully* contravariantly finite if any object in $\mathcal{B}$ admits a right $\mathcal{B}'$-approximation which is also a deflation. Dually we can define (*faithfully*) *covariantly finite* subcategory.

LEMMA 2.7. *If $\mathcal{C}$ is rigid, closed under direct summands, fully contravariantly finite, and $\mathcal{B}$ has enough injectives, then $(\mathcal{C}, \mathcal{C}^{\perp_1})$ is a cotorsion pair.*

Proof. Since $\mathcal{B}$ has enough injectives, any object $A \in \mathcal{B}$ admits a conflation $A \rightarrowtail I \twoheadrightarrow B$ where I is injective. Since $\mathcal{C}$ is fully contravariantly finite, object B admits a conflation $B_1 \rightarrowtail C_0 \overset{f_0}{\twoheadrightarrow} B$ where f_0 is a right $\mathcal{C}$-approximation. The rigidity of $\mathcal{C}$ implies $B_1 \in \mathcal{C}^{\perp_1}$. We have the following commutative diagram:

$$\begin{array}{ccccc} & & A & = & A \\ & & \downarrow & & \downarrow \\ B_1 & \rightarrowtail & X & \twoheadrightarrow & I \\ \| & & \downarrow & & \downarrow \\ B_1 & \rightarrowtail & C_0 & \overset{f_0}{\twoheadrightarrow} & B \end{array}$$

where $X \in \mathcal{C}^{\perp_1}$. Hence, by definition, the pair $(\mathcal{C}, \mathcal{C}^{\perp_1})$ is a cotorsion pair. □

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The following lemma will be used later.

LEMMA 2.8. *Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair and $\overline{\mathcal{H}}$ be its heart. If we have a short exact sequence $0 \to A \xrightarrow{\overline{f}} B \xrightarrow{\overline{g}} C \to 0$ in $\overline{\mathcal{H}}$, then we have a conflation $A' \overset{f'}{\rightarrowtail} B' \overset{g'}{\twoheadrightarrow} C$ where $A', B', C \in \mathcal{H}$ such that its image by applying cohomological functor H is isomorphic to $0 \to A \xrightarrow{\overline{f}} B \xrightarrow{\overline{g}} C \to 0$.*

Proof. For morphism g, we have the following commutative diagram:

$$\begin{array}{ccccc} V_C & \rightarrowtail & K_g & \overset{k_g}{\twoheadrightarrow} & B \\ \| & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle g} \\ V_C & \rightarrowtail & W_C & \underset{w_C}{\twoheadrightarrow} & C \end{array}$$

where $V_C \in \mathcal{V}$ and $W_C \in \mathcal{W}$. Then we obtain a conflation

$$K_g \overset{\binom{k_g}{-a}}{\rightarrowtail} B \oplus W_C \overset{(g\ w_C)}{\twoheadrightarrow} C.$$

By [**12**, Lemma 3.1], we have $K_g \in \mathcal{B}^-$; by [**12**, Lemma 2.10], we have $K_g \in \mathcal{B}^+$. Hence, $K_g \in \mathcal{H}$ and we get a short exact sequence $0 \to K_g \xrightarrow{\overline{k_g}} B \xrightarrow{\overline{g}} C \to 0$ in $\overline{\mathcal{H}}$ by the dual of [**12**, Corollary 3.7, 3.8]. Hence, $K_g \simeq A$ in $\overline{\mathcal{H}}$. □

LEMMA 2.9. *If we have a cotorsion pair $(\mathcal{C}, \mathcal{C}^{\perp_1})$, then any object X admits a commutative diagram:*

$$\begin{array}{ccccc} Y_0 & = & Y_0 & & \\ {\scriptstyle g_0}\downarrow & & \downarrow{\scriptstyle y_0} & & \\ U_0 & \overset{u_0}{\rightarrowtail} & P_0 & \twoheadrightarrow & C_0 \\ {\scriptstyle f_0}\downarrow & & \downarrow & & \| \\ X & \rightarrowtail & T_0 & \twoheadrightarrow & C_0 \end{array}$$

where $T_0 \in \mathcal{C}^{\perp_1}$, $C_0 \in \mathcal{C}$, $P_0 \in \mathcal{P}$, and f_0 is a right $\Omega\mathcal{C}$-approximation.

Proof. If we have a cotorsion pair $(\mathcal{C}, \mathcal{C}^{\perp_1})$, then any object X admits a conflation $X \rightarrowtail T_0 \twoheadrightarrow C_0$ where $T_0 \in \mathcal{C}^{\perp_1}$, $C_0 \in \mathcal{C}$. Since $\mathcal{B}$ has enough projectives, object T_0 admits a conflation $Y_0 \overset{y_0}{\rightarrowtail} P_0 \twoheadrightarrow T_0$ where $P_0 \in \mathcal{P}$. Hence, by the axiom of extriangulated category, we get a commutative diagram:

$$\begin{array}{ccccc} Y_0 & = & Y_0 & & \\ {\scriptstyle g_0}\downarrow & & \downarrow{\scriptstyle y_0} & & \\ U_0 & \overset{u_0}{\rightarrowtail} & P_0 & \twoheadrightarrow & C_0 \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle p_0} & & \| \\ X & \underset{t}{\rightarrowtail} & T_0 & \twoheadrightarrow & C_0 \end{array}$$

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which gives rise to a conflation $U_0 \overset{\binom{u_0}{f_0}}{\rightarrowtail} P_0 \oplus X \overset{(-p_0\ t)}{\twoheadrightarrow} T_0$. Let $f: U \to X$ be any morphism such that U admits a conflation $U \overset{u}{\rightarrowtail} P \twoheadrightarrow C$ where $P \in \mathcal{P}$ and $C \in \mathcal{C}$. Then we get the following commutative diagram:

$$\begin{array}{ccccc} U & \overset{u}{\rightarrowtail} & P & \twoheadrightarrow & C \\ {\scriptstyle \binom{0}{f}}\downarrow & & \vdots\,{\scriptstyle p} & & \\ U_0 & \underset{\binom{u_0}{f_0}}{\rightarrowtail} & P_0 \oplus X & \underset{(-p_0\ t)}{\twoheadrightarrow} & T_0 \end{array}$$

since $\mathbb{E}(C, T_0) = 0$. Object P is projective, hence there is a morphism $P \xrightarrow{\binom{-a}{b}} P_0 \oplus X$ such that $(-p_0\ t)\binom{-a}{b} = p$ and a morphism $c: P \to U_0$ such that $f_0 c = b$. Since $(-p_0\ t)(\binom{0}{f} - \binom{-au}{bu}) = 0$, there exists a morphism $d: U \to U_0$ such that $f = f_0 d + bu = f_0(d + cu)$. Hence, f_0 is a right $\Omega\mathcal{C}$-approximation. $\square$

REMARK 2.10. In the lemma above, if $\mathbb{E}(T_0, B) = 0$ for an object B, then $\mathrm{Hom}_{\mathcal{B}}(g_0, B)$ is surjective.

LEMMA 2.11. *If we have a cotorsion pair $(\mathcal{C}, \mathcal{C}^{\perp_1})$ where $\mathcal{C}$ is rigid, let $g: A \twoheadrightarrow B$ be a deflation in $\mathcal{B}$ such that $\overline{g}$ is an epimorphism where $A, B \in \mathcal{H}$, then* $\mathrm{Hom}_{\mathcal{B}}(X, g)$ *is surjective in $\mathcal{B}$ whenever $X \in \Omega\mathcal{C}$.*

Proof. Let $g: A \twoheadrightarrow B$ be a deflation in $\mathcal{H}$ such that $\overline{g}$ is an epimorphism. By [12, Corollary 2.26] g admits the following commutative diagram:

$$\begin{array}{ccccc} A & \overset{a_0}{\rightarrowtail} & C^0 & \twoheadrightarrow & C^1 \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle c_0} & & \| \\ B & \underset{h}{\rightarrowtail} & C_g & \twoheadrightarrow & C^1 \end{array}$$

where $C^0, C^1, C_g \in \mathcal{C}$. By Proposition 2.1, we have a conflation $A \overset{\binom{a_0}{g}}{\rightarrowtail} C_0 \oplus B \overset{(-c_0\ h)}{\twoheadrightarrow} C_g$. Let $f: X \to B$ be any morphism such that X admits a conflation $X \overset{u}{\rightarrowtail} P \twoheadrightarrow C$ where $P \in \mathcal{P}$ and $C \in \mathcal{C}$, then we get the following commutative diagram:

$$\begin{array}{ccccc} X & \overset{u}{\rightarrowtail} & P & \twoheadrightarrow & C \\ {\scriptstyle \binom{0}{f}}\downarrow & & \vdots\,{\scriptstyle p} & & \\ A & \underset{\binom{a_0}{g}}{\rightarrowtail} & C_0 \oplus B & \underset{(-c_0\ h)}{\twoheadrightarrow} & C_g \end{array}$$

since $\mathcal{C}$ is rigid. Object P is projective, hence there is a morphism $P \xrightarrow{\binom{-a}{b}} C_0 \oplus B$ such that $(-c_0\ h)\binom{-a}{b} = p$ and a morphism $c: P \to A$ such that $gc = b$. Since $(-c_0\ h)(\binom{0}{f} - \binom{-au}{bu}) = 0$, there exists a morphism $d: X \to A$ such that $f = gd + bu = g(d + cu)$. Hence, $\mathrm{Hom}_{\mathcal{B}}(X, g)$ is surjective. $\square$

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3. Localization of hearts.

Definition 3.1. Subcategory $\mathcal{C}$ satisfies condition (RCP) if $\mathcal{P} \subset \mathcal{C}$, $\mathcal{C}$ is rigid, contravariantly finite, and closed under direct summands.

("RCP" means rigid cotorsoin pair)

By Lemma 2.7, if $\mathcal{C}$ satisfies condition RCP , $(\mathcal{C}, \mathcal{C}^{\perp_1})$ is a cotorsion pair.

From now on, let $\mathcal{D} \subset \mathcal{C}$ be subcategories satisfying RCP. Let $\mathcal{U} = \Omega\mathcal{C}$ and $\mathcal{V} = \Omega\mathcal{D}$.

Since $(\mathcal{C}, \mathcal{C}^{\perp_1})$ is a cotorsion pair, we have a subcategory $\mathcal{H}$ according to the definition of the heart such that the heart of $(\mathcal{C}, \mathcal{C}^{\perp_1})$ is $\mathcal{H}/\mathcal{C} =: \overline{\mathcal{H}}$. In this case, $\mathcal{H} = \mathrm{CoCone}(\mathcal{C}, \mathcal{C})$.

Let $\mathcal{H}_{\mathcal{D}} = \mathrm{CoCone}(\mathcal{D}, \mathcal{C})$, then $\mathcal{U} \subseteq \mathcal{H}_{\mathcal{D}} \subseteq \mathcal{H}$.

The following remark is useful.

Remark 3.2. For any subcategory $\mathcal{B}_1$, let $\underline{\mathcal{B}_1^{\perp}} = \{B \in \mathcal{B} \mid \mathrm{Hom}_{\mathcal{B}/\mathcal{P}}(\mathcal{B}_1, B) = 0\}$. We have $\mathcal{D}^{\perp_1} = \underline{\mathcal{V}^{\perp}}$, since in the following commutative diagram:

$$\begin{array}{ccccc} V & \overset{p}{\rightarrowtail} & P & \twoheadrightarrow & D \\ & \searrow^{v} & & & \\ & & B & & \end{array}$$

where $D \in \mathcal{D}$ and $P \in \mathcal{P}$, morphism v factors through an object in $\mathcal{P}$ (then it factors through p because $\mathbb{E}(\mathcal{D}.\mathcal{P}) = 0$) if and only if $\mathbb{E}(D, B) = 0$.

Proposition 3.3. *Any object X admits a conflation $Z \rightarrowtail Y \overset{f}{\twoheadrightarrow} X$ where f is a right $\mathcal{H}_{\mathcal{D}}$-approximation and $Z \in \mathcal{D}^{\perp_1}$. Moreover, morphism $x' : X \to X'$ factors through $\mathcal{C}^{\perp_1}$ if $x'f$ factors through $\mathcal{C}^{\perp_1}$.*

Proof. By Lemma 2.9, any object X admits the following commutative diagram:

$$\begin{array}{ccccc} Y_0 & = & Y_0 & & \\ {\scriptstyle g_0}\downarrow & & \downarrow{\scriptstyle y_0} & & \\ U_0 & \overset{u_0}{\rightarrowtail} & P_0 & \twoheadrightarrow & C_0 \\ {\scriptstyle f_0}\downarrow & & \downarrow & & \| \\ X & \rightarrowtail & T_0 & \twoheadrightarrow & C_0 \end{array}$$

where $T_0 \in \mathcal{C}^{\perp_1}$, $C_0 \in \mathcal{C}$, $P_0 \in \mathcal{P}$, and f_0 is a right $\mathcal{U}$-approximation. Object Y_0 also admits a conflation $Y_1 \overset{g_1}{\rightarrowtail} V_1 \overset{f_1}{\twoheadrightarrow} Y_0$ where f_1 is a right $\mathcal{V}$-approximation. Object V_1 admits a conflation $V_1 \overset{h_1}{\rightarrowtail} P_1 \twoheadrightarrow D_1$ where $P_1 \in \mathcal{P}_1$ and $D_1 \in \mathcal{D}$. Thus we have the following commutative diagram:

$$\begin{array}{ccccccccc} Y_1 & \overset{g_1}{\rightarrowtail} & V_1 & \overset{f_1}{\twoheadrightarrow} & Y_0 & \overset{g_0}{\rightarrowtail} & U_0 & \overset{f_0}{\twoheadrightarrow} & X \\ \| & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u} & & \| \\ Y_1 & \rightarrowtail & P_1 & \overset{p}{\twoheadrightarrow} & Z & \overset{g}{\rightarrowtail} & Y & \overset{f}{\twoheadrightarrow} & X \\ & & \downarrow & & \downarrow{\scriptstyle z} & & \downarrow{\scriptstyle y} & & \\ & & D_1 & = & D_1 & = & D_1 & & \end{array}$$

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where $\mathrm{Hom}_{\mathcal{B}}(U,f)$ is surjective for any $U\in\mathcal{U}$ since f_0 is a right $\mathcal{U}$-approximation. We claim $Z\overset{g}{\rightarrowtail} Y\overset{f}{\twoheadrightarrow} X$ is the conflation we need.
Since we have the following commutative diagram of conflations:

$$\begin{array}{ccccc} U_0 & \overset{u}{\rightarrowtail} & Y & \overset{y}{\twoheadrightarrow} & D_1 \\ \| & & \downarrow{\scriptstyle p_0} & & \downarrow \\ U_0 & \underset{u_0}{\rightarrowtail} & P_0 & \longrightarrow & C_0 \end{array}$$

where $P_0\in\mathcal{P}$ and $C_0\in\mathcal{C}$, by Proposition 2.1, we get a conflation $Y\overset{\binom{y}{p_0}}{\rightarrowtail} D_1\oplus P_0\twoheadrightarrow C_0$ which implies $Y\in\mathcal{H}_{\mathcal{D}}$.
Now we check that $Z\in\mathcal{D}^{\perp_1}$. It is enough to show $Z\in\underline{\mathcal{V}}^{\perp}$ since $\mathcal{D}^{\perp_1}=\underline{\mathcal{V}}^{\perp}$. Let V be an object in $\mathcal{V}$ that admits a conflation $V\overset{b}{\rightarrowtail} P\twoheadrightarrow D$ where $P\in\mathcal{P}$ and $D\in\mathcal{D}$. Let $a:V\to Z$ be any morphism, since $\mathcal{D}$ is rigid, morphism za factors through b. We have the following diagram:

$$\begin{array}{ccccccc} & & V & \overset{b}{\rightarrowtail} & P & \twoheadrightarrow & D \\ & {\scriptstyle e}\swarrow & \downarrow{\scriptstyle a} & {\scriptstyle d}\swarrow & \downarrow{\scriptstyle c} & & \\ Y_0 & \underset{v}{\rightarrowtail} & Z & \underset{z}{\twoheadrightarrow} & D_1 & & \end{array}$$

where $za=cb$. Since P is projective, there exists a morphism $d:P\to Z$ such that $c=zd$. Hence, $z(a-db)=0$ and there exists a morphism $e:V\to Y_0$ such that $ve=a-db$. Since f_1 is a right $\mathcal{V}$-approximation, there exists a morphism $h:V\to V_1$ such that $f_1h=e$. We get $vf_1h=ph_1h=a-db$, then a factors through $\mathcal{P}$, which implies $Z\in\underline{\mathcal{V}}^{\perp}$.

Let $Y'\in\mathcal{H}_{\mathcal{D}}$, then it admits the following commutative diagram:

$$\begin{array}{ccccc} V_{Y'} & = & V_{Y'} & & \\ \downarrow{\scriptstyle v'} & & \downarrow & & \\ U_{Y'} & \rightarrowtail & P_{Y'} & \twoheadrightarrow & C_{Y'} \\ \downarrow & & \downarrow & & \| \\ Y' & \rightarrowtail & D_{Y'} & \twoheadrightarrow & C_{Y'} \end{array}$$

where $D_{Y'}\in\mathcal{D}$, $C_{Y'}\in\mathcal{C}$, and $P_{Y'}\in\mathcal{P}$. Let $x:Y'\to X$ be a morphism, then we have a commutative diagram of conflations:

$$\begin{array}{ccccc} V_{Y'} & \overset{v'}{\rightarrowtail} & U_{Y'} & \twoheadrightarrow & Y' \\ \downarrow{\scriptstyle z'} & & \downarrow & & \downarrow{\scriptstyle x} \\ Z & \overset{g}{\rightarrowtail} & Y & \overset{f}{\twoheadrightarrow} & X. \end{array}$$

Since $Z\in\mathcal{D}^{\perp_1}=\underline{\mathcal{V}}^{\perp}$, morphism z' factor through $\mathcal{P}$. By Remark 2.10, morphism z' factors through v', which implies x factors through f by [**17**, Corollary 3.5]. Hence, f is a right $\mathcal{H}_{\mathcal{D}}$-approximation.

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Now we show the “moreover” part. Since we have the following commutative diagram of conflations:

$$\begin{array}{ccccc} Y_0 & \overset{v}{\rightarrowtail} & Z & \overset{z}{\twoheadrightarrow} & D_1 \\ \| & & \downarrow{\scriptstyle p_0 g} & & \vdots \\ Y_0 & \underset{y_0}{\rightarrowtail} & P_0 & \twoheadrightarrow & T_0 \end{array}$$

we get a commutative diagram of conflations:

$$\begin{array}{ccccc} Z & \overset{g}{\rightarrowtail} & Y & \overset{f}{\twoheadrightarrow} & X \\ \| & & \downarrow{\scriptstyle \binom{y}{p_0}} & & \vdots\, {\scriptstyle t} \\ Z & \underset{\binom{z}{p_0 g}}{\rightarrowtail} & D_1 \oplus P_0 & \twoheadrightarrow & T_0 \end{array}$$

For convenience, we denote the above diagram by

$$\begin{array}{ccccc} Z & \overset{g}{\rightarrowtail} & Y & \overset{f}{\twoheadrightarrow} & X \\ \| & & \downarrow{\scriptstyle y'} & & \downarrow{\scriptstyle t} \\ Z & \underset{z'}{\rightarrowtail} & D_1' & \twoheadrightarrow & T_0. \end{array}$$

Now assume there is a morphism $x' : X \to X'$ such that $x'f$ admits a commutative diagram

$$\begin{array}{ccc} Y & \overset{f}{\longrightarrow} & X \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle x'} \\ K & \underset{k}{\longrightarrow} & X' \end{array}$$

where $K \in \mathcal{C}^{\perp_1}$, then there exists a morphism $n : D_1' \to K$ such that $l = ny'$ since $\mathbb{E}(C_0, K) = 0$. Hence, there is a morphism $t' : T_0 \to X'$ such that $x' = t't$, which implies x itself factors through $\mathcal{C}^{\perp_1}$. □

The following corollary is an immediate consequence of Proposition 3.3.

COROLLARY 3.4. *In the proposition above, if $X \in \mathcal{H}$, then $\overline{f}$ is an epimorphism in $\overline{\mathcal{H}}$.*

We also have the following useful corollary.

COROLLARY 3.5. *Let $A \in \mathcal{H}_{\mathcal{D}}$. A morphism $g : A \to B$ factors through $\mathcal{D}^{\perp_1}$ only if it factors through $\mathcal{D}^{\perp_1} \cap \mathcal{H}_{\mathcal{D}}$.*

Proof. If $f : A \to B$ in $\mathcal{H}_{\mathcal{D}}$ factors through an object $X \in \mathcal{D}^{\perp_1}$, by Proposition 3.3, there is a conflation $Z \rightarrowtail Y \overset{f}{\twoheadrightarrow} X$ where f is a right $\mathcal{H}_{\mathcal{D}}$-approximation and $Z \in \mathcal{D}^{\perp_1}$. Hence, we have the following commutative diagram:

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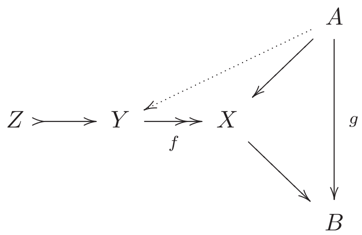
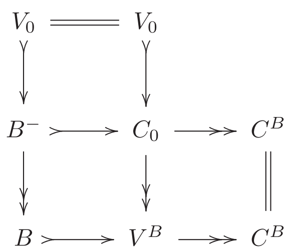


where $Y \in \mathcal{D}^{\perp_1}$ since $Z \in \mathcal{D}^{\perp_1}$. □

DEFINITION 3.6. Let $H : \mathcal{B} \to \overline{\mathcal{H}}$ be the cohomological functor as in Theorem 2.6. Denote $H(\mathcal{D}^{\perp_1})$ by $\mathcal{A}$. Let $\mathcal{S}_{\mathcal{A}}$ be the class of epimorphisms $\overline{f}$ whose kernel belong to $\mathcal{A}$.

LEMMA 3.7. *We have* $\mathcal{A} = (\mathcal{H} \cap \mathcal{D}^{\perp_1})/\mathcal{C}$.

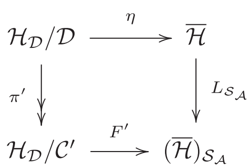
Proof. Since $H|_{\mathcal{H}} = \pi|_{\mathcal{H}}$, the image of $\mathcal{H} \cap \mathcal{D}^{\perp_1}$ lies in $\mathcal{A}$, hence $(\mathcal{H} \cap \mathcal{D}^{\perp_1})/\mathcal{C} \subseteq \mathcal{A}$. Let $B \in \mathcal{B}$, it admits a commutative diagram:



where V^B, $V_0 \in \mathcal{C}^{\perp_1}$, and $C_0, C^B \in \mathcal{C}$. Hence, $B^- \in \mathcal{B}^- = \mathcal{H}$, by definition in [12], we get $H(B) = B^-$. If $B \in \mathcal{D}^{\perp_1}$, we get $B^- \in \mathcal{D}^{\perp_1}$. Hence, $\mathcal{A} = H(\mathcal{D}^{\perp_1}) \subseteq (\mathcal{H} \cap \mathcal{D}^{\perp_1})/\mathcal{C}$. □

Now let $\mathcal{H}_{\mathcal{D}} \cap \mathcal{D}^{\perp_1} = \mathcal{C}'$, we call $\mathcal{C}'$ the *right* $\mathcal{D}$*-mutation* of $\mathcal{C}$. Note that we do not require $\mathcal{C}'$ to be rigid in this section.

We have a functor $\eta : \mathcal{H}_{\mathcal{D}}/\mathcal{D} \hookrightarrow \mathcal{H}/\mathcal{D} \twoheadrightarrow \overline{\mathcal{H}}$, and let F be composition of functor η and the localization functor $L_{\mathcal{S}_{\mathcal{A}}} : \overline{\mathcal{H}} \to (\overline{\mathcal{H}})_{\mathcal{S}_{\mathcal{A}}}$. By definition, we have $F(\mathcal{C}') = 0$ in $(\overline{\mathcal{H}})_{\mathcal{S}_{\mathcal{A}}}$. Hence, we have the following commutative diagram:



where π' is the quotient functor. For convenience, we still denote the morphisms in $\mathcal{H}_{\mathcal{D}}/\mathcal{D}$ by $\overline{f}$ (where f is the morphism in $\mathcal{H}_{\mathcal{D}}$) since f factors through $\mathcal{D}$ if and only if it factors through $\mathcal{C}$. We will show the following theorem, which is a generalization of the first part of [13, Theorem 3.2].

THEOREM 3.8. *The functor* $F' : \mathcal{H}_{\mathcal{D}}/\mathcal{C}' \to (\overline{\mathcal{H}})_{\mathcal{S}_{\mathcal{A}}}$ *is an equivalence.*

We shall prove it by several steps.

LEMMA 3.9. *Category* $\mathcal{A}$ *is closed under taking epimorphisms.*

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Proof. Let $\overline{f}: Y \to X$ be an epimorphism in $\overline{\mathcal{H}}$ such that $Y \in \mathcal{D}^{\perp_1}$; we show that $X \in \mathcal{D}^{\perp_1}$. By [12, Corollary 2.26], we have the following commutative diagram:

$$\begin{array}{ccccc} Y & \longrightarrow & C^0 & \longrightarrow & C^1 \\ {\scriptstyle f}\downarrow & & \downarrow & & \| \\ X & \longrightarrow & C_f & \longrightarrow & C^1 \end{array}$$

where C^0, C^1, $C_f \in \mathcal{C}$. Hence, we get a conflation $Y \rightarrowtail X \oplus C^0 \twoheadrightarrow C_f$ which implies $X \oplus C^0 \in \mathcal{D}^{\perp_1}$. Since $\mathcal{D}^{\perp_1}$ is closed under direct summands, we have $X \in \mathcal{D}^{\perp_1}$. □

By similar method, we can show that if $\overline{f}: Y \to X$ is a monomorphism in $\overline{\mathcal{H}}$ such that $X \in \mathcal{D}^{\perp_1}$, then $Y \in \mathcal{D}^{\perp_1}$.

PROPOSITION 3.10. *Functor F' is dense.*

Proof. It is enough to show F is dense.

By Proposition 3.3, any object $X \in \mathcal{H}$ admits a conflation $Z \overset{g}{\rightarrowtail} Y \overset{f}{\twoheadrightarrow} X$ where f is a right $\mathcal{H}_{\mathcal{D}}$-approximation and $Z \in \mathcal{D}^{\perp_1}$. By Corollary 3.4, morphism $\overline{f}$ is an epimorphism in $\overline{\mathcal{H}}$. By [12, Lemma 3.1], we get $Z \in \mathcal{H}$. Then we have an exact sequence $Z \xrightarrow{\overline{g}} Y \xrightarrow{\overline{f}} X \to 0$; there is an epimorphism from Z to the kernel of $\overline{f}$. By Lemma 3.9, the kernel of $\overline{f}$ is in $\mathcal{D}^{\perp_1}$. Hence, $Y \simeq X$ in $(\overline{\mathcal{H}})_{\mathcal{S}_{\mathcal{A}}}$. □

PROPOSITION 3.11. *Functor F' is full.*

Proof. It is enough to show F is full.

Consider a morphism $\alpha: X_1 \to Y_2$ in $(\overline{\mathcal{H}})_{\mathcal{S}_{\mathcal{A}}}$ having the form $X_1 \xrightarrow{\overline{x}} X_2 \xrightarrow{\overline{f}^{-1}} Y_2$ where $X_1, Y_2 \in \mathcal{H}_{\mathcal{D}}$; by definition, we have a short exact sequence $0 \to Z_2 \xrightarrow{\overline{g}} Y_2 \xrightarrow{\overline{f}} X_2 \to 0$ in $\overline{\mathcal{H}}$ where $Z_2 \in \mathcal{D}^{\perp_1}$. By Lemma 2.8, we have a conflation $Z_2' \overset{g'}{\rightarrowtail} Y_2' \overset{f'}{\twoheadrightarrow} X_2$ in $\mathcal{H}$ such that $\overline{f} = \overline{f'}$, $Y_2' = Y_2$ in $\overline{\mathcal{H}}$, and its image is isomorphic to the short exact sequence. Hence, $Z_2' \in \mathcal{D}^{\perp_1}$. By Lemma 2.11, morphism $\overline{f'}$ is an epimorphism that implies $\mathrm{Hom}_{\mathcal{B}}(U, f')$ is surjective for any $U \in \mathcal{U}$. Since X_1 admits a conflation $V_1 \overset{u}{\rightarrowtail} U_0 \twoheadrightarrow X_1$ where $U_0 \in \mathcal{U}$ and $V_1 \in \mathcal{V}$, we have the following commutative diagram:

$$\begin{array}{ccccc} V_1 & \overset{u}{\rightarrowtail} & U_0 & \twoheadrightarrow & X_1 \\ {\scriptstyle z}\downarrow & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle x} \\ Z_2' & \overset{g'}{\rightarrowtail} & Y_2' & \overset{f'}{\twoheadrightarrow} & X_2 \end{array}$$

where z factors through $\mathcal{P}$. Hence, by Remark 2.10, morphism z factors through u, then by [17, Corollary 3.5] there is a morphism $x': X_1 \to Y_2'$ such that $x = f'x'$. Hence, we have $\overline{x'} = \overline{f}^{-1}\overline{x}$ in $(\overline{\mathcal{H}})_{\mathcal{S}_{\mathcal{A}}}$, which shows F is full. □

PROPOSITION 3.12. *Functor F' is faithful.*

Proof. Since $(\mathcal{D}, \mathcal{D}^{\perp_1})$ is also a cotorsion pair, we denote its heart by $\mathcal{H}_0/\mathcal{D}$ and the associated cohomological functor by H_0. Since $\mathcal{C}^{\perp_1} \subseteq \mathcal{D}^{\perp_1}$ and $H_0(\mathcal{D}^{\perp_1}) = 0$ by [12, Corollary 3.8], we have the following commutative diagram:

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$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\quad H \quad} & \overline{\mathcal{H}}. \\ & {\scriptstyle H_0}\searrow \quad \swarrow{\scriptstyle K_0} & \\ & \mathcal{H}_0/\mathcal{D} & \end{array}$$

Now let $X, Y \in \mathcal{H}_{\mathcal{D}}$ and $Y \xrightarrow{\overline{f}} X$ be a morphism in $\mathcal{S}_{\mathcal{A}}$, then we have a conflation $Z' \overset{g'}{\rightarrowtail} Y' \overset{f'}{\twoheadrightarrow} X$ in $\mathcal{H}$ such that $\overline{f} = \overline{f}'$, $Y' = Y$ in $\overline{\mathcal{H}}$, and $Z' \in \mathcal{D}^{\perp_1} \cap \mathcal{H}$. Then $H_0(Z') = 0$, which means $H_0(f')$ is a monomorphism. Moreover, since $\overline{f}' = \overline{f}$ is an epimorphism, we have the following commutative diagram:

$$\begin{array}{ccccc} Y' & \xrightarrow{c} & C^0 & \longrightarrow & C^1 \\ {\scriptstyle f}\downarrow & & \downarrow & & \| \\ X & \longrightarrow & C_f & \longrightarrow & C^1 \end{array}$$

where $C^0, C^1, C_f \in \mathcal{C}$. Then we get a conflation $Y' \overset{\binom{f}{c}}{\rightarrowtail} X \oplus C^0 \twoheadrightarrow C_f$. By applying H_0 to this conflation, we get $H_0(f')$ is an epimorphism. Hence, $H_0(f') = K_0(\overline{f})$ is an isomorphism. By the universal property of $L_{\mathcal{S}_{\mathcal{A}}}$, there is a functor $J : (\overline{\mathcal{H}})_{\mathcal{S}_{\mathcal{A}}} \to \mathcal{H}_0/\mathcal{D}$ such that $JL_{\mathcal{S}_{\mathcal{A}}} = K_0$.

Now assume we have $X, Y \in \mathcal{H}_{\mathcal{D}}$ and $u, v \in \mathrm{Hom}_{\mathcal{B}}(X, Y)$ such that $F(\overline{u}) = F(\overline{v})$, then $H_0(u) = K_0(\overline{u}) = JL_{\mathcal{S}_{\mathcal{A}}}(\overline{u}) = JF(\overline{u}) = JF(\overline{v}) = JL_{\mathcal{S}_{\mathcal{A}}}(\overline{v}) = K_0(\overline{v}) = H_0(v)$. Morphism $K_0(\overline{a}) = 0$ if and only if a factors through $\mathcal{D}^{\perp_1}$ by [**12**, Proposition 2.22] then $u - v$ factors though $\mathcal{D}^{\perp_1}$. By Corollary 3.5, it factors through $\mathcal{C}'$. Hence, $\pi'(\overline{u}) = \pi'(\overline{v} + \overline{u} - \overline{v}) = \pi'(\overline{v})$, which shows F' is faithful. □

Since F' is fully faithful and dense, it is an equivalence. Now we finished the proof of Theorem 3.8.

Let $\mathcal{N} \subset \mathcal{M}'$ be rigid categories such that both $\mathcal{M}'$ and $\mathcal{N}$ are covariantly finite, closed under direct summands, and contain $\mathcal{I}$.

Since $({}^{\perp_1}\mathcal{M}', \mathcal{M}')$ is a cotorsion pair, denote $\mathrm{Cone}(\mathcal{M}', \mathcal{M}')$ by $\mathcal{H}'$, the heart of $({}^{\perp_1}\mathcal{M}', \mathcal{M}')$ is $\mathcal{H}'/\mathcal{M}' =: \overline{\mathcal{H}}'$.

Let $\mathcal{H}'_{\mathcal{N}} = \mathrm{Cone}(\mathcal{M}', \mathcal{N})$, $H' : \mathcal{B} \to \overline{\mathcal{H}}'$ be the associated cohomological functor. Denote $\mathcal{H}'_{\mathcal{N}} \cap {}^{\perp_1}\mathcal{N}$ by $\mathcal{M}$.

Denote $H'({}^{\perp_1}\mathcal{N})$ by $\mathcal{A}'$. Let $\mathcal{S}_{\mathcal{A}'}$ be the class of monomorphisms in $\overline{\mathcal{H}}'$ whose cokernels belong to $\mathcal{A}'$.

We have a functor $\eta' : \mathcal{H}'_{\mathcal{N}}/\mathcal{N} \hookrightarrow \mathcal{H}'/\mathcal{N} \twoheadrightarrow \overline{\mathcal{H}}'$. Let G be composition of functor η and the localization functor $L_{\mathcal{S}_{\mathcal{A}'}} : \overline{\mathcal{H}}' \to (\overline{\mathcal{H}}')_{\mathcal{S}_{\mathcal{A}'}}$. Since $H'(\mathcal{M}) \subseteq \mathcal{A}'$, we have $G(\mathcal{M}) = 0$ in $(\overline{\mathcal{H}}')_{\mathcal{S}_{\mathcal{A}'}}$. Hence, we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{H}'_{\mathcal{N}}/\mathcal{N} & \xrightarrow{\eta'} & \overline{\mathcal{H}}' \\ {\scriptstyle \pi''}\downarrow & & \downarrow{\scriptstyle L_{\mathcal{S}_{\mathcal{A}'}}} \\ \mathcal{H}'_{\mathcal{N}}/\mathcal{M} & \xrightarrow{G'} & (\overline{\mathcal{H}}')_{\mathcal{S}_{\mathcal{A}'}} \end{array}$$

where π'' is the quotient functor.

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THEOREM 3.13. *Functor* $G' : \mathcal{H}'_{\mathcal{N}}/\mathcal{M} \to (\overline{\mathcal{H}}')_{\mathcal{S}_{\mathcal{A}'}}$ *is an equivalence.*

Proof. This is a dual of Theorem 3.8. □

4. Pseudo-Morita equivalences. In this section, we prove the main theorem of this paper. Please note that a similar result for triangulated category has been proved in [**16**].

THEOREM 4.1. *Let* $(\mathcal{B}, \mathbb{E}, \mathfrak{s})$ *be an extriangulated category with enough projectives and enough injectives. Assume we have three twin cotorsion pairs* $((\mathcal{C}, \mathcal{C}^{\perp_1}), (\mathcal{C}^{\perp_1}, \mathcal{M}))$, $((\mathcal{D}, \mathcal{D}^{\perp_1}), (\mathcal{D}^{\perp_1}, \mathcal{N}))$, *and* $((\mathcal{C}', \mathcal{C}'^{\perp_1}), (\mathcal{C}'^{\perp_1}, \mathcal{M}'))$ *such that* $\mathcal{D} \subseteq \mathcal{C} \cap \mathcal{C}'$, $\mathcal{H}_{\mathcal{D}} := \mathrm{CoCone}(\mathcal{D}, \mathcal{C}) = \mathrm{CoCone}(\mathcal{C}', \mathcal{D})$*, and* $\mathrm{Cone}(\mathcal{N}, \mathcal{M}) = \mathrm{Cone}(\mathcal{M}', \mathcal{N}) =: \mathcal{H}'_{\mathcal{N}}$*. Let*

(a) $\overline{\mathcal{H}}$ *be the heart of* $(\mathcal{C}, \mathcal{C}^{\perp_1})$*. Denote* $(\mathcal{H} \cap \mathcal{D}^{\perp_1})/\mathcal{C}$ *by* $\mathcal{A}$*. Let* $\mathcal{S}_{\mathcal{A}}$ *be the class of epimorphisms in* $\overline{\mathcal{H}}$ *whose kernel belongs to* $\mathcal{A}$*.*

(b) $\overline{\mathcal{H}}'$ *be the heart of* $(\mathcal{C}'^{\perp_1}, \mathcal{M}')$*. Denote* $(\mathcal{H}' \cap \mathcal{D}^{\perp_1})/\mathcal{M}'$ *by* $\mathcal{A}'$*. Let* $\mathcal{S}_{\mathcal{A}'}$ *be the class of monomorphisms in* $\overline{\mathcal{H}}'$ *whose cokernel belongs to* $\mathcal{A}'$*.*

Then we have the following equivalences:

$$(\overline{\mathcal{H}})_{\mathcal{S}_{\mathcal{A}}} \simeq \mathcal{H}_{\mathcal{D}}/\mathcal{C}' \simeq \mathcal{H}'_{\mathcal{N}}/\mathcal{M} \simeq (\overline{\mathcal{H}}')_{\mathcal{S}_{\mathcal{A}'}}.$$

Inspired by [**13**], we call the equivalence $(\overline{\mathcal{H}})_{\mathcal{S}_{\mathcal{A}}} \simeq (\overline{\mathcal{H}}')_{\mathcal{S}_{\mathcal{A}'}}$ *pseudo-Morita equivalence*.

REMARK 4.2. If $\mathcal{C}$, $\mathcal{C}'$, and $\mathcal{D}$ are subcategories satisfying condition RCP and $D \subseteq \mathcal{C} \cap \mathcal{C}'$, then $\mathrm{CoCone}(\mathcal{D}, \mathcal{C}) = \mathrm{CoCone}(\mathcal{C}', \mathcal{D})$ if and only if $\mathcal{C}' = \mathrm{CoCone}(\mathcal{D}, \mathcal{C}) \cap \mathcal{D}^{\perp_1}$.

REMARK 4.3. By [**12**, Propositions 3.12, 4.15], heart $\overline{\mathcal{H}}'$ is equivalent to the heart of $(\mathcal{C}', \mathcal{C}'^{\perp_1})$, which is equivalent to $\mathrm{mod}(\mathcal{C}'/\mathcal{P})$. We also have $\overline{\mathcal{H}} \simeq \mathrm{mod}(\mathcal{C}/\mathcal{P})$.

According to the previous results, we only need to show $\mathcal{H}_{\mathcal{D}}/\mathcal{C}' \simeq \mathcal{H}'_{\mathcal{N}}/\mathcal{M}$.

Since $((\mathcal{C}^{\perp_1}, \mathcal{M}), (\mathcal{D}^{\perp_1}, \mathcal{N}))$ is a twin cotorsion pair, by Definition 2.4, we have a subcategory $\mathcal{B}^+$ associated with this twin cotorsion pair. In fact, $\mathcal{B}^+ = \mathcal{H}'_{\mathcal{N}}$. By [**12**, Definition 2.21], the inclusion functor $i^+ : \mathcal{B}^+/\mathcal{M} = \mathcal{H}'_{\mathcal{N}}/\mathcal{M} \hookrightarrow \mathcal{B}/\mathcal{M}$ has a right adjoint functor σ^+ such that every object B admits the following commutative diagram:

$$\begin{array}{ccccc}
V_B & \rightarrowtail & U_B & \twoheadrightarrow & B \\
\| & & \downarrow & & \downarrow{\scriptstyle f} \\
V_B & \rightarrowtail & T & \twoheadrightarrow & B^+ \\
 & & \downarrow & & \downarrow \\
 & & S & = & S
\end{array}$$

where $U_B \in \mathcal{D}^{\perp_1}$, $V_B \in \mathcal{N}$, $T \in \mathcal{M}$, $S \in \mathcal{C}^{\perp_1}$, and $\sigma^+(B) = B^+$, and morphism f is a left $\mathcal{H}'_{\mathcal{N}}$-approximation. Let H'_0 be the cohomological functor associated with the heart of $(\mathcal{D}^{\perp_1}, \mathcal{N})$, then $H'_0(f)$ is an isomorphism. We call the conflation $B \overset{f}{\rightarrowtail} B^+ \twoheadrightarrow S$ a *reflection conflation* of B. For every object B, we fix a reflection conflation of it. Then

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for any morphism $x\colon B \to C$, we define $\sigma^+(\overline{x})$ as the unique image of the morphism which makes the following diagram commute (see [12, Definition 2.21]):

$$\begin{array}{ccc} B & \xrightarrow{\ x\ } & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B^+ & \underset{x^+}{\dashrightarrow} & C^+ \end{array}$$

By [12, Proposition 2.22], morphism $\sigma^+(\overline{f}) = 0$ if and only if f factors through $\mathcal{D}^{\perp_1}$. Since $\mathcal{C}' \subseteq \mathcal{D}^{\perp_1}$, we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\ \pi_{\mathcal{C}'}\ } & \mathcal{B}/\mathcal{C}' \\ {\scriptstyle \pi_{\mathcal{M}}}\downarrow & & \vdots\,{\scriptstyle \overline{\sigma}^+} \\ \mathcal{B}/\mathcal{M} & \underset{\sigma^+}{\longrightarrow} & \mathcal{H}'_{\mathcal{N}}/\mathcal{M} \end{array}$$

where $\pi_{\mathcal{C}'}$ and $\pi_{\mathcal{M}}$ are quotient functors. Hence, we have a functor $K : \mathcal{H}_{\mathcal{D}}/\mathcal{C}' \hookrightarrow \mathcal{B}/\mathcal{C}' \xrightarrow{\overline{\sigma}^+} \mathcal{H}'_{\mathcal{N}}/\mathcal{M}$.

On the other hand, since we have a twin cotorsion pair $((\mathcal{D}, \mathcal{D}^{\perp_1}), (\mathcal{C}', \mathcal{C}'^{\perp_1}))$, by the definition, we have a subcategory $\mathcal{B}^-$ associated with this twin cotorsion pair. By [12, Definition 2.21], the inclusion $i^- : \mathcal{B}^-/\mathcal{C}' = \mathcal{H}_{\mathcal{D}}/\mathcal{C}' \hookrightarrow \mathcal{B}/\mathcal{C}'$ has a left adjoint functor σ^- such that every object B admits a conflation $V \rightarrowtail B^- \overset{f'}{\twoheadrightarrow} B$ where $\sigma^-(B) = B^-$, $V \in \mathcal{C}'^{\perp_1}$, and f is a right $\mathcal{H}_{\mathcal{D}}$-approximation; we call this conflation a *coreflection conflation* of B. Then dually we have a functor $K' : \mathcal{H}'_{\mathcal{N}}/\mathcal{M} \to \mathcal{H}_{\mathcal{D}}/\mathcal{C}'$. We will prove that $K'K \simeq \mathrm{id}_{\mathcal{H}_{\mathcal{D}}/\mathcal{C}'}$, and dually we can get $KK' \simeq \mathrm{id}_{\mathcal{H}'_{\mathcal{N}}/\mathcal{M}}$, which shows $\mathcal{H}'_{\mathcal{N}}/\mathcal{M}$ and $\mathcal{H}_{\mathcal{D}}/\mathcal{C}'$ are equivalent.

PROPOSITION 4.4. *There is a natural isomorphism between functors $K'K$ and $\mathrm{id}_{\mathcal{H}_{\mathcal{D}}/\mathcal{C}'}$.*

Proof. Let $B \in \mathcal{H}_{\mathcal{D}}$, we have a reflection conflation $B \overset{f}{\rightarrowtail} B^+ \overset{g}{\twoheadrightarrow} S$, and let $V \rightarrowtail B' \overset{f'}{\twoheadrightarrow} B^+$ be a coreflection conflation of B^+. By the proof of Proposition 3.10, we have $L_{\mathcal{S}_{\mathcal{A}}} H(f') : B' \xrightarrow{\simeq} H(B^+)$ in $(\overline{\mathcal{H}})_{\mathcal{S}_{\mathcal{A}}}$. Note that $K'K(B) = B'$ in $\mathcal{H}_{\mathcal{D}}/\mathcal{C}'$.

We have the following commutative diagram:

$$\begin{array}{ccccc} \Omega S & \overset{q_S}{\rightarrowtail} & P_S & \overset{p_S}{\twoheadrightarrow} & S \\ {\scriptstyle s}\downarrow & & \downarrow & & \| \\ B & \underset{f}{\rightarrowtail} & B^+ & \underset{g}{\twoheadrightarrow} & S \end{array}$$

which induces a conflation $\Omega S \overset{\binom{s}{q_S}}{\rightarrowtail} B \oplus P_S \overset{(f\ -p')}{\twoheadrightarrow} B^+$. Since $H'_0(f)$ is an isomorphism, we have $H'_0(s) = 0$, which implies s factors through $\mathcal{D}^{\perp_1}$. Object ΩS admits a conflation $\Omega S \rightarrowtail T \twoheadrightarrow D_1$ where $T \in \mathcal{D}^{\perp_1}$ and $D_1 \in \mathcal{D}$, hence s factors through T. We have the following commutative diagram in $\overline{\mathcal{H}}$ by applying H:

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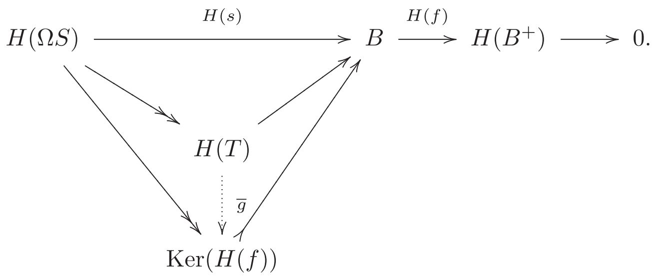

It implies $\overline{g}$ is an epimorphism. Hence, by Lemma 3.9, $\mathrm{Ker}(H(f)) \in \mathcal{A}$. This implies $L_{\mathcal{S}_\mathcal{A}} H(f) : B \overset{\simeq}{\rightarrow} H(B^+)$ in $(\overline{\mathcal{H}})_{\mathcal{S}_\mathcal{A}}$.

Let $x : B_0 \to B_1$ be a morphism in $\mathcal{H}_\mathcal{D}$, denote its image in $\mathcal{H}_\mathcal{D}/\mathcal{C}'$ by $\underline{x}$, then we have the following commutative diagram:

$$\begin{array}{ccccc} B_0 & \xrightarrow{f_0} & B_0^+ & \xleftarrow{f_0'} & B_0' \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle x^+} & & \downarrow{\scriptstyle y} \\ B_1 & \xrightarrow[f_1]{} & B_1{}^+ & \xleftarrow[f_1']{} & B_1' \end{array}$$

where the image of y in $\mathcal{H}_D/\mathcal{C}'$ is $K'K(\underline{x})$. Since $L_{\mathcal{S}_\mathcal{A}} H(f_i)$ and $L_{\mathcal{S}_\mathcal{A}} H(f_i')$ are invertible, $i \in \{0, 1\}$, by Proposition 3.11, we have isomorphisms $\underline{b_i} : B_i \to K'K(B_i)$ such that $F'(\underline{b_i}) = L_{\mathcal{S}_\mathcal{A}} H(f_i')^{-1} L_{\mathcal{S}_\mathcal{A}} H(f_i)$. Then we have the following commutative diagram in $\mathcal{H}_\mathcal{D}/\mathcal{C}'$:

$$\begin{array}{ccc} B_0 & \overset{\underline{b_0}}{\underset{\simeq}{\longrightarrow}} & K'K(B_0) \\ {\scriptstyle \underline{x}}\downarrow & & \downarrow{\scriptstyle K'K(\underline{x})} \\ B_1 & \overset{\simeq}{\underset{\underline{b_1}}{\longrightarrow}} & K'K(B_1) \end{array}$$

Hence, $K'K \simeq \mathrm{id}_{\mathcal{H}_\mathcal{D}/\mathcal{C}'}$. □

Dually we can show $KK' \simeq \mathrm{id}_{\mathcal{H}'_\mathcal{N}/\mathcal{M}}$. Now we finished the proof of Theorem 4.1.

5. More Localizations.

Let $\mathcal{D} \subset \mathcal{C}$ be subcategories satisfying (RCP), and let $\mathcal{H}_\mathcal{D} \cap \mathcal{D}^{\perp_1} = \mathcal{C}'$ (the same as in Section 3).

Let $f : Y \to X$ be a morphism in $\mathcal{B}$, then it admits the following commutative diagram ($\diamond$):

$$\begin{array}{ccccccccc} \Omega X & \rightarrowtail & Z_1 & \overset{g}{\twoheadrightarrow} & Y & \rightarrowtail & I^Y & \twoheadrightarrow & \Sigma Y \\ \Vert & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow & & \Vert \\ \Omega X & \rightarrowtail & P_X & \twoheadrightarrow & X & \overset{h}{\rightarrowtail} & Z_2 & \twoheadrightarrow & \Sigma Y \end{array}$$

where $P_X \in \mathcal{P}$ and $I^Y \in \mathcal{I}$. Let $\widetilde{\mathcal{R}_1}$ be the class of morphisms f such that there is a commutative diagram ($\diamond$) where g factors through $\mathcal{D}^{\perp_1}$ and h factors through $\mathcal{C}^{\perp_1}$. Let $\mathcal{R}_1$ be

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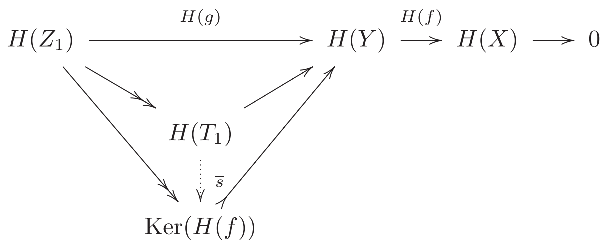
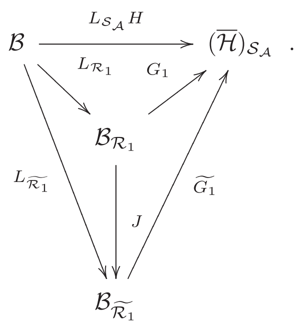
the class of morphisms f such that there is a commutative diagram ($\diamond$) where $Z_1 \in \mathcal{D}^{\perp_1}$ and h factors through $\mathcal{C}^{\perp_1}$. Then $\widetilde{\mathcal{R}_1} \supseteq \mathcal{R}_1$. Let $\mathcal{B}_{\mathcal{R}_1}$ (resp. $\mathcal{B}_{\widetilde{\mathcal{R}_1}}$) be the localization of $\mathcal{B}$ at $\mathcal{R}_1$ (resp. $\widetilde{\mathcal{R}_1}$) and $L_{\mathcal{R}_1}$ (resp. $L_{\widetilde{\mathcal{R}_1}}$) be the localization functor. If $f \in \widetilde{\mathcal{R}_1}$, object Z_1 admits a conflation $Z_1 \rightarrowtail T_1 \twoheadrightarrow D_1$ where $T_1 \in \mathcal{D}^{\perp_1}$ and $D_1 \in \mathcal{D}$, hence g factors through T_1. We have the following commutative diagram in $\overline{\mathcal{H}}$ by applying H:



which implies $\bar{s}$ is an epimorphism. Hence, by Lemma 3.9, $\mathrm{Ker}(H(f)) \in \mathcal{A}$. This implies $H(f) \in \mathcal{S}_{\mathcal{A}}$. Then we have the following commutative diagram:



We will show that G_1 is an equivalence. This implies $L_{\mathcal{R}_1}$ inverts all the morphism in $\widetilde{\mathcal{R}_1}$, then we have a unique functor $I : \mathcal{B}_{\widetilde{\mathcal{R}_1}} \to \mathcal{B}_{\mathcal{R}_1}$ such that $L_{\mathcal{R}_1} = L_{\widetilde{\mathcal{R}_1}} I$. Hence, $JI = \mathrm{id}$ and $IJ = \mathrm{id}$, which means $\widetilde{G_1}$ is also an equivalence.

REMARK 5.1. In the conflation $Z \rightarrowtail Y \xrightarrow{f} X$ in Proposition 3.3, morphism $f \in \mathcal{R}_1$.

The following theorem together with the arguments above generalizes [13, Theorem 3.19].

THEOREM 5.2. *Functor* $G_1 : \mathcal{B}_{\mathcal{R}_1} \to (\overline{\mathcal{H}})_{\mathcal{S}_{\mathcal{A}}}$ *is an equivalence.*

Proof. Since $H|_{\mathcal{H}} = \pi|_{\mathcal{H}}$, we get G_1 is dense. We show G_1 is full.
Let $\alpha : G_1(X_1) \to G_1(X_2)$ be a morphism. By Proposition 3.3, for $i \in \{1, 2\}$, X_i admits a conflation $Z_i \rightarrowtail Y_i \xrightarrow{f_i} X_i$ where $f_i \in \mathcal{R}_1$ and $Y_i \in \mathcal{H}_{\mathcal{D}}$. By the definition of localization, morphism f_i becomes invertible in $\mathcal{B}_{\mathcal{R}_1}$. We have isomorphisms $L_{\mathcal{S}_{\mathcal{A}}} H(f_i)$, $i \in \{1, 2\}$ and $L_{\mathcal{S}_{\mathcal{A}}} H(f_1) \alpha L_{\mathcal{S}_{\mathcal{A}}} H(f_2)^{-1} : Y_1 \to Y_2$. By Proposition 3.11, there exists a morphism $g : Y_1 \to Y_2$ such that $L_{\mathcal{S}_{\mathcal{A}}} H(g) = L_{\mathcal{S}_{\mathcal{A}}} H(f_1) \alpha L_{\mathcal{S}_{\mathcal{A}}} H(f_2)^{-1}$. Hence, $\alpha = L_{\mathcal{S}_{\mathcal{A}}} H(f_1)^{-1} L_{\mathcal{S}_{\mathcal{A}}} H(g) L_{\mathcal{S}_{\mathcal{A}}} H(f_2) = G_1(f_1^{-1} g f_2)$.

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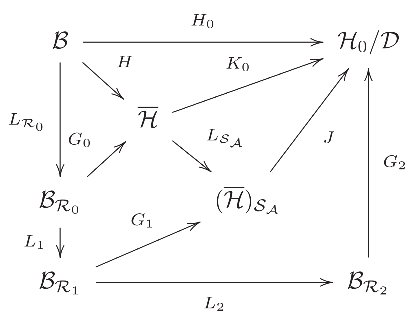
We show G_1 is faithful. It is enough to check $G_1L_{\mathcal{R}_1}(a_1) = G_1L_{\mathcal{R}_1}(a_2)$ implies $a_1 = a_2$ in $\mathcal{B}_{\mathcal{R}_1}$ where a_1, a_2 are morphisms from X_1 to X_2 in $\mathcal{B}$.

By the discussion above, we have $b_i : Y_1 \to Y_2$ such that $f_2b_i = a_if_1$, $i \in \{1, 2\}$. Hence,

$$\begin{aligned} L_{\mathcal{S}_\mathcal{A}}H(b_1) &= L_{\mathcal{S}_\mathcal{A}}H(f_2)^{-1}L_{\mathcal{S}_\mathcal{A}}H(a_1)L_{\mathcal{S}_\mathcal{A}}H(f_1) = L_{\mathcal{S}_\mathcal{A}}H(f_2)^{-1}L_{\mathcal{S}_\mathcal{A}}H(a_2)L_{\mathcal{S}_\mathcal{A}}H(f_1) \\ &= L_{\mathcal{S}_\mathcal{A}}H(b_2) \end{aligned}$$

This implies $b_1 - b_2$ factors through $\mathcal{C}' \subseteq \mathcal{D}^{\perp_1}$ by Proposition 3.12. Hence, we have $b_1 = b_2 + (b_1 - b_2) = b_2$ and $a_1 = f_1^{-1}b_1f_2 = f_1^{-1}b_2f_2 = a_2$ in $\mathcal{B}_{\mathcal{R}_1}$. □

Now let $\mathcal{R}_0$ be the class of morphisms f such that there is a commutative diagram ($\diamond$) where $Z_1 \in \mathcal{C}^{\perp_1}$ and h factors through $\mathcal{C}^{\perp_1}$, let $\mathcal{R}_2$ be the class of morphisms f such that there is a commutative diagram ($\diamond$) where $Z_1 \in \mathcal{D}^{\perp_1}$ and h factors through $\mathcal{D}^{\perp_1}$. Then $\mathcal{R}_0 \subseteq \mathcal{R}_1 \subseteq \mathcal{R}_2$. Let $\mathcal{B}_{\mathcal{R}_0}$ (resp. $\mathcal{B}_{\mathcal{R}_2}$) be the localization of $\mathcal{B}$ at $\mathcal{R}_0$ (resp. $\mathcal{R}_2$) and $L_{\mathcal{R}_0}$ (resp. $L_{\mathcal{R}_2}$) be the localization functor. Since $H(f)$ (resp. $H_0(f)$) is an isomorphism if $f \in \mathcal{R}_0$ (resp. $f \in \mathcal{R}_2$) , we have the following commutative diagram:



where $L_1L_{\mathcal{R}_0} = L_{\mathcal{R}_1}$ and $L_2L_1L_{\mathcal{R}_0} = L_{\mathcal{R}_2}$.

REMARK 5.3. By the similar method as Proposition 3.3, we can show the following statement:

Any object X admits a conflation $Z \rightarrowtail Y \xrightarrow{f} X$ where f is a right $\mathcal{H}$-approximation (resp. $\mathcal{H}_0$-approximation) and $Z \in \mathcal{C}^{\perp_1}$ (resp. $\mathcal{D}^{\perp_1}$). Moreover, morphism $x : X \to X'$ factors through $\mathcal{C}^{\perp_1}$ (resp. $\mathcal{D}^{\perp_1}$) if xf factors through $\mathcal{C}^{\perp_1}$ (resp. $\mathcal{D}^{\perp_1}$).

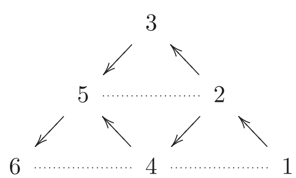
One can check morphism f in the statement belongs to $\mathcal{R}_0$ (resp. $\mathcal{R}_2$). Then by the similar method as Theorem 5.2, we can show that G_0 (resp. G_2) is an equivalence.

The detail of the proofs is left to the readers.

6. Example.

In the last section, we give an example of our result in module category.

EXAMPLE 6.1. *Let Λ be the k-algebra given by the quiver*



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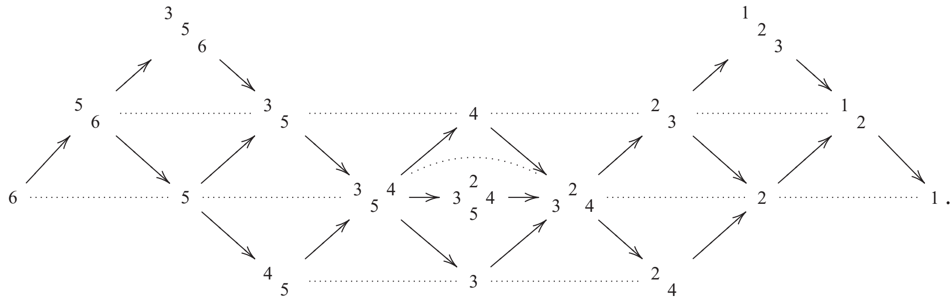
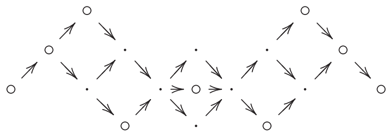
with mesh relations. The AR-quiver of $\mathcal{B} := \operatorname{mod} \Lambda$ *is given by*

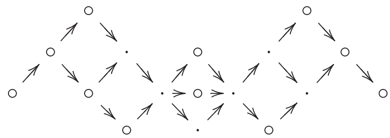


We denote by "∘" in the AR-quiver the indecomposable objects that belong to a subcategory and by "·" the indecomposable objects that do not belong to it.

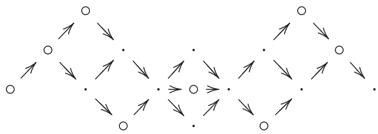
$\mathcal{C}$:



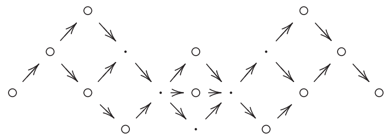
$\mathcal{C}^{\perp_1}$:



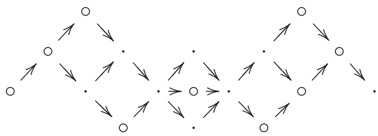
$\mathcal{D}$:



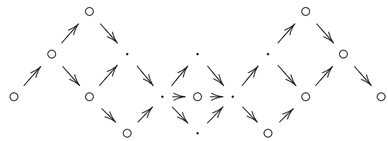
$\mathcal{D}^{\perp_1}$:



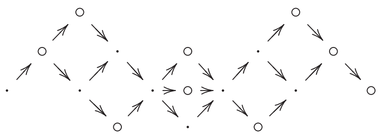
$\mathcal{C}'$:



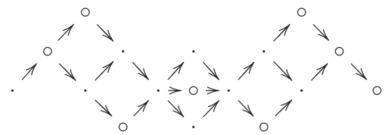
$\mathcal{C}'^{\perp_1}$:



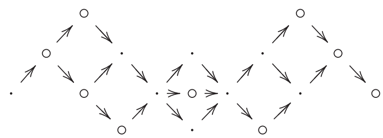
$\mathcal{M}$:



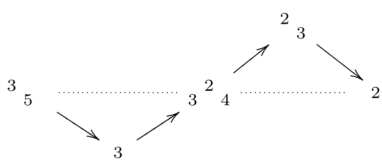
$\mathcal{N}$:



$\mathcal{M}'$:



The heart of $(\mathcal{C}, \mathcal{C}^{\perp_1})$ *is the following:*



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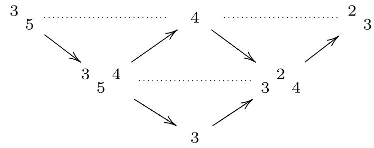
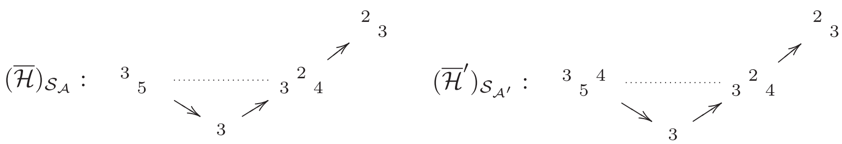
and the heart of $(\mathcal{C}'^{\perp_1}, \mathcal{M}')$ *is the following:*

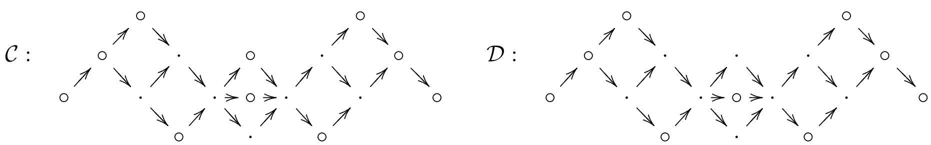


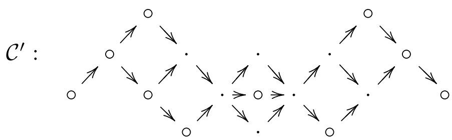
Obviously they are not equivalent, but if we take localization as in the last section, we get



When we consider the right $\mathcal{D}$-mutation of $\mathcal{C}$, we cannot always get a rigid subcategory $\mathcal{C}' = \mathrm{CoCone}(\mathcal{D}, \mathcal{C}) \cap \mathcal{D}^{\perp_1}$. See the following example in mod Λ of Example 6.1.

EXAMPLE 6.2.





But in the following cases, we can always get a rigid subcategory $\mathcal{C}'$.

PROPOSITION 6.1. *Let* $\mathcal{C}$, $\mathcal{D}$ *be subcategories satisfying condition RCP and* $\mathcal{D} \subset \mathcal{C}$. *Let* $\mathcal{C}' = \mathrm{CoCone}(\mathcal{D}, \mathcal{C}) \cap \mathcal{D}^{\perp_1}$. *In the following cases,* $\mathcal{C}'$ *is rigid:*

(a) $\mathcal{B}$ *is a Krull–Schmidt, k-linear, Hom-finite, 2-Calabi–Yau triangulated category, and* $\mathcal{D}$ *is functorially finite.*
(b) $\mathbb{E}^2(\mathcal{C}, \mathcal{D}) = 0$.

Proof. (a) By the assumptions, we have a cotorsion pair $(\mathcal{C}, \mathcal{C}^{\perp_1})$. By [**18**, Theorem 3.3], we get a cotorsion pair $(\mathcal{C}', \mathcal{C}'^{\perp_1})$ such that $\mathcal{C}' \cap \mathcal{C}'^{\perp_1} = \mathcal{C}'$, which implies $\mathcal{C}'$ is rigid.

(b) Let $C' \in \mathcal{C}'$, it admits a conflation $(\star)$ $C' \rightarrowtail D \xrightarrow{f} C$ where $D \in \mathcal{D}$ and $C \in \mathcal{C}$. By applying $\mathbb{E}(\mathcal{C}, -)$ to conflation $(\star)$, we get an exact sequence $0 = \mathbb{E}(\mathcal{C}, C) \to \mathbb{E}^2(\mathcal{C}, C') \to \mathbb{E}^2(\mathcal{C}, D) = 0$, hence $\mathbb{E}^2(\mathcal{C}, C') = 0$. Since C' is arbitrary, we get $\mathbb{E}^2(\mathcal{C}, \mathcal{C}') = 0$. Then we apply $\mathbb{E}(-, \mathcal{C}')$ to conflation $(\star)$; we get an exact sequence $0 = \mathbb{E}(D, \mathcal{C}') \to \mathbb{E}(C', \mathcal{C}') \to \mathbb{E}^2(C, \mathcal{C}') = 0$, hence $\mathbb{E}(C', \mathcal{C}') = 0$. Since C' is arbitrary, we get $\mathcal{C}'$ is rigid. $\square$

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