$$\left(\frac{d}{dx}\right)^{m}(x^{p}y^{n})$$

$$=m!\,\Sigma\left((n-r)!\,\frac{p(p-1)\ldots(p-r+1)}{r!}x^{p-r}\,\frac{(y_{r_1})^{\rho_1}(y_{r_2})^{\rho_2}\ldots\ldots}{(r_1!)^{\rho_1}(r_2!)^{\rho_2}\ldots\rho_1!\,\rho_2!\ldots\ldots}\right);$$

where

$$r \not< 0 \not> m,\ r_1 \not< 0 \not> m, \ldots\ldots;$$
$$\rho_1 \not< 1 \not> n,\ \rho_2 \not< 1 \not> n, \ldots\ldots;$$
$$r+\rho_1 r_1+\rho_2 r_2+\ldots\ldots=m;$$
$$\rho_1+\rho_2+\ldots\ldots=n.$$

Example

$$\left(\frac{d}{dx}\right)^{9}(y^3)=9!\,3!\left[\frac{y_9y^2}{9!2!}+\frac{y_8y_1y}{8!}+\frac{y_7y_1^{\,2}}{7!2!}+\frac{y_7y_2y}{7!2!}+\frac{y_6y_3y}{6!3!}+\frac{y_6y_2y_1}{6!2!}+\frac{y_5y_4y}{5!4!}\right.$$
$$\left.+\frac{y_5y_3y_1}{5!3!}+\frac{y_5y_2^{\,2}}{5!(2!)^3}+\frac{y_4^{\,2}y_1}{(4!)^2 2!}+\frac{y_4y_3y_2}{4!3!2!}+\frac{y_3^{\,3}}{(3!)^4}\right]$$
$$=3y_9y^2+54y_8y_1y+216y_7y_1^{\,2}+216y_7y_2y+504y_6y_3y_1+1512y_6y_2y_1$$
$$+756y_5y_4y+3024y_5y_3y_1+2268y_5y_2^{\,2}+1890y_4^{\,2}y_1+7560y_4y_3y_2+1680y_3^{\,3}.$$

On a method for obtaining the differential equation to an Algebraical Curve.

By Professor CHRYSTAL.

1. Consider the conic represented by the general equation

$$a_0+b_0x+b_1y+c_0x^2+c_1xy+c_2y^2=0. \quad \ldots \quad \ldots \quad (1)$$

Differentiating three times with respect to a we get

$$b_1(y)_3+c_2(y^2)_3+c_1(xy)_3=0 \ldots \quad \ldots \quad \ldots \quad (2)$$

where $(y)_3$ stands for $\left(\frac{d}{dx}\right)^3(y)$.

Again, from (2) by successive differentiation we derive

$$b_1(y)_4+c_2(y^2)_4+c_1(xy)_4=0 \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

$$b_1(y)_5+c_2(y^2)_5+c_1(xy)_5=0 \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

From (2) (3) (4), eliminating the remaining constants, we have

$$\begin{vmatrix} (y)_3 & (y^2)_3 & (xy)_3 \\ (y)_4 & (y^2)_4 & (xy)_4 \\ (y)_5 & (y^2)_5 & (xy)_5 \end{vmatrix}=0 \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

which is one form of the differential equation to the conic (1).

Since $$(xy)_3 = x(y)_3 + 3(y)_2,$$
$$(xy)_4 = x(y)_4 + 4(y)_3,$$
$$(xy)_5 = x(y)_5 + 5(y)_4,$$
we have rejecting redundant columns, and dropping brackets where they are no longer necessary

$$\begin{vmatrix} y_3 & 3y_2 & (y^2)_3 \\ y_4 & 4y_3 & (y^2)_4 \\ y_5 & 5y_4 & (y^2)_5 \end{vmatrix} = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

From which it already appears that the differential equation to the conic (1) does not contain the independent variable explicitly, and that it contains the highest differential coefficient y_5 in the first power only.

We may simplify (7) still further, for

$$\begin{vmatrix} y_3 & 3y_2 & (y^2)_3 \\ y_4 & 4y_3 & (y^2)_4 \\ y_5 & 5y_4 & (y^2)_5 \end{vmatrix} = \begin{vmatrix} y_3 & 3y_2 & 2yy_3 + 6y_1y_2 \\ y_4 & 4y_3 & 2yy_4 + 8y_1y_3 + 6y_2^2 \\ y_5 & 5y_4 & 2yy_5 + 10y_1y_4 + 20y_2y_3 \end{vmatrix} = 2y_2 \begin{vmatrix} y_3 & 3y_2 & 0 \\ y_4 & 4y_3 & 3y_2 \\ y_5 & 5y_4 & 10y_3 \end{vmatrix} = 0.$$

Whence we get immediately

$$9y_2^2y_5 - 45y_2y_3y_4 + 40y_3^3 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

The result and the process by which it has been obtained may be compared with Halphen's method (*Jordan Cours d'Analyse de l'Ecole Polytechnique*, t. i., § 53).

2. The method above applied to the general equation of the second degree, and some of the results are of a general character. This will perhaps be best seen by considering the general cubic, whose equation, for present convenience, I write as follows—

$$a_0 + a_1x + a_2x^2 + a_3x^3 + b_1y + b_2y^2 + b_3y^3 + c_1xy + c_2x^2y + d_1xy^2 = 0 \quad \ldots \quad (9)$$

From (9) we derive at once

$$b_1(y)_4 + b_2(y^2)_4 + b_3(y^3)_4 + c_1(xy)_4 + c_2(x^2y)_4 + d_1(xy^2)_4 = 0 \quad \ldots \quad (10)$$

From (10) by five successive differentiations we obtain five more equations, and using these along with (10), we eliminate the six constants, and obtain

$$\begin{vmatrix} (y)_4 & (y^2)_4 & (y^3)_4 & (xy)_4 & (x^2y)_4 & (xy^2)_4 \\ (y)_5 & (y^2)_5 & (y^3)_5 & (xy)_5 & (x^2y)_5 & (xy^2)_5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (y)_9 & (y^2)_9 & (y^3)_9 & (xy)_9 & (x^2y)_9 & (xy^2)_9 \end{vmatrix} = 0 \quad \ldots \quad \ldots \quad (11)$$

Now, observing that, if $(n,1)$, $(n,2)$, $(n,3)$, &c., denote the binomial coefficients of the n^{th} power, we have

$$(xy)_4 = x(y)_4 + (4,1).1.(y)_3, \quad (x^2y)_4 = x^2(y)_4 + (4,1).2.x(y)_3 + (4,2).2.1.(y)_2, \quad (xy^2)_4 = x(y^2)_4 + (4,1).1.(y^2)_3,$$
$$(xy)_5 = x(y)_5 + (5,1).1.(y)_4, \quad (x^2y)_5 = x^2(y)_5 + (5,1).2.x(y)_4 + (5,2).2.1.(y)_3, \quad (xy^2)_5 = x(y^2)_5 + (5,1).1.(y^2)_4,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(xy)_9 = x(y)_9 + (9,1).1.(y)_8, \quad (x^2y)_9 = x^2(y)_9 + (9,1).2.x(y)_8 + (9,2).2.1.(y)_7, \quad (xy^2)_9 = x(y^2)_9 + (9,1).1.(y^2)_8.$$

Hence rejecting what are obviously redundant columns, the determinantal equation (11) reduces to

$$\begin{vmatrix} (y)_4 & (y^2)_4 & (y^3)_4 & (4,1)(y)_3 & (4,2)(y)_2 & (4,1)(y^2)_3 \\ (y)_5 & (y^2)_5 & (y^3)_5 & (5,1)(y)_4 & (5,2)(y)_3 & (5,1)(y^2)_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (y)_9 & (y^2)_9 & (y^3)_9 & (9,1)(y)_8 & (9,2)(y)_7 & (9,1)(y^2)_8 \end{vmatrix} = 0 \quad \ldots \quad (12)$$

Equation (12) may be further simplified, but it is interesting to remark that here again we have *a differential equation, of the* 9^{th} *order linear in the highest differential coefficient and not explicitly containing the independant variable,* which must be satisfied by the ordinate of every cubic curve.

3. Proceeding exactly as in § 2, if y be determined in terms of x by a *complete* rational integral equation of the n^{th} degree in x and y, and if $p = \frac{1}{2}n(n+3)$, then y satisfies the differential equation

$$\begin{vmatrix} (y)_{n+1} & (y^2)_{n+1}\cdots & (y^n)_{n+1} & (n+1,1)(y)_n\cdots & (n+1,n-1)(y)_2 & (n+1,1)(y^2)_n\cdots & (n+1,n-2)(y^2)_3\cdots & (n+1,1)(y^{n-1})_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (y)_p & (y^2)_p\cdots & (y^n)_p & (p,1)(y)_{p-1}\cdots & (p,n-1)(y)_{p-n+2} & (p,1)(y^2)_{p-1}\cdots & (p,n-2)(y^2)_{p-n+1}\cdots & (p,1)(y^{n-1})_p \end{vmatrix} = 0 \ldots (13);$$

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an equation of the $\frac{1}{2}$n(n + 3)th *order, linear in the highest differential coefficient, and not explicitly containing the independent variable.*

4. Adhering still to the supposition that all the coefficients of the primitive equation are independent, it is interesting to notice that the independent variable will not appear in the differential equation even if terms be omitted, provided that in those retained the powers of x which multiply the respective powers of y all occur in order without intermediate omissions. This is obvious on looking at § (2) and observing the reason for the disappearance of the redundant columns.

For example, let us take the cubic

$$a_0 + a_1x + a_2x^2 + (b_0 + b_1x + b_2x^2)y = 0.$$

The resulting differential equation is

$$\begin{vmatrix} (y)_3 & (xy)_3 & (x^2y)_3 \\ (y)_4 & (xy)_4 & (x^2y)_4 \\ (y)_5 & (xy)_5 & (x^2y)_5 \end{vmatrix} = 0,$$

which reduces successively to

$$\begin{vmatrix} (y)_3 & 3(y)_2 & 3.2.x(y)_2 + \ \ 3.2.(y)_1 \\ (y)_4 & 4(y)_3 & 4.2.x(y)_3 + \ \ 6.2.(y)_2 \\ (y)_5 & 5(y)_4 & 5.2.x(y)_4 + 10.2.(y)_3 \end{vmatrix} = 0, \quad \begin{vmatrix} y_3 & 3y_2 & 3y_1 \\ y_4 & 4y_3 & 6y_2 \\ y_5 & 5y_4 & 10y_3 \end{vmatrix} = 0 \ldots (14)$$

It is interesting to observe that (14) is a differential equation whose complete primitive is a quadratic rational fraction of the most general kind.

A similar equation could of course be obtained for a rational fraction of the most general kind, whose numerator and denominator are of the m^{th} and n^{th} degrees.

5. If the condition in § 4 be not fulfilled the differential equation may contain x.

For example,

$$a_0 + a_2x^2 + b_1xy + c_0y^2 = 0$$

gives

$$\begin{vmatrix} (x^2)_1 & (xy)_1 & (y^2)_1 \\ (x^2)_2 & (xy)_2 & (y^2)_2 \\ (x^2)_3 & (xy)_3 & (y^2)_3 \end{vmatrix} = 0,$$

$$\begin{vmatrix} 2x & xy_1 + \ y & 2yy_1 \\ 2 & xy_2 + 2y_1 & 2yy_2 + 2y_1^2 \\ 0 & xy_3 + 3y_2 & 2yy_3 + 6y_1y_2 \end{vmatrix} = 0,$$

whence $\qquad (y^2 - 2xyy_1 + x^2y_1^2)y_3 + 3x(y - xy_1)y_2^2 = 0,$

or throwing out the factor $y - xy_1$

$$(y - xy_1)y_3 + 3xy_2^2 = 0. \ldots \qquad \ldots \qquad \ldots \quad (15)$$

6. When the constants involved in the primitive integral equation are not independent, but subject to particular relations, the above method must of course be modified.

For example, in the case of the circle $a+bx+cy+d(x^2+y^2)=0$, the differential equation is

$$\begin{vmatrix} y_2 & (x^2+y^2)_2 \\ y_3 & (x^2+y^2)_3 \end{vmatrix}=0 \text{; whence } \begin{vmatrix} y_2 & 2+(y^2)_2 \\ y_3 & (y^2)_3 \end{vmatrix}=0\text{;}$$

which gives $-2y_3+\begin{vmatrix} y_2 & 2yy_2+2y_1^2 \\ y_3 & 2yy_3+6y_1y_2 \end{vmatrix}=0$;

that is $y_3-y_1\begin{vmatrix} y_2 & y_1 \\ y_3 & 3y_2 \end{vmatrix}=0$;

that is $(1+y_2^2)y_3-3y_1y_2^2=0$ ... ... (16)

Again, taking the case of the parabola, $a+bx+cy+(dx+ey)^2=0$,

we derive $\begin{vmatrix} \{(ax+by)^2\}_2 & y_2 \\ \{(ax+by)^2\}_3 & y_3 \end{vmatrix}=0$,

whence $\begin{vmatrix} 2(ax+by)by_2+2(a+by_1)(a+by_1) & y_2 \\ 2(ax+by)by_3+6(a+by_1)by_2 & y_3 \end{vmatrix}=0$;

that is throwing out redundant rows and factors,

$$\begin{vmatrix} a+by_1 & y_2 \\ 3by_2 & y_3 \end{vmatrix}=0.$$

This last may be written

$$ay_3+b\begin{vmatrix} y_1 & y_2 \\ 3y_2 & y_3 \end{vmatrix}=0.$$

Hence the differential equation to the parabola is

$$\begin{vmatrix} y_3, & \begin{vmatrix} y_1 & y_2 \\ 3y_2 & y_3 \end{vmatrix} \\ y_4, & \begin{vmatrix} y_2 & y_3 \\ 3y_2 & y_3 \end{vmatrix}+\begin{vmatrix} y_1 & y_2 \\ 3y_3 & y_4 \end{vmatrix} \end{vmatrix}=0\text{;}$$

whence $y_3\begin{vmatrix} y_2 & y_3 \\ 2y_2 & 0 \end{vmatrix}+y_3\begin{vmatrix} y_1 & y_2 \\ 3y_3 & y_4 \end{vmatrix}-y_4\begin{vmatrix} y_1 & y_2 \\ 3y_2 & y_3 \end{vmatrix}=0$;

that is $3y_2^2y_4-5y_2y_3^2=0$,

or $3y_2y_4-5y_3^2=0$.... ... ... (17)

6. When owing to the omission of certain terms in the general equation, or to particular relations between the constants a differential equation of lower order than that corresponding to the general case is obtained, this differential equation must of course involve the truth of the more general one.

https://doi.org/10.1017/S0013091500037378 Published online by Cambridge University Press

For example, in the case of the parabola we have

$$3y_2y_4 - 5y_3^2 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

whence by differentiation $\quad 3y_2y_5 - 7y_3y_4 = 0.$

This may be written $\quad 9y_2^2y_5 - 21y_2y_3y_4 = 0$;

or $\quad 9y_2^2y_5 - 45y_2y_3y_4 + 24y_2y_3y_4 = 0$;

that is using (18)

$$9y_2^2y_5 - 45y_2y_3y_4 + 40y_3^3 = 0;$$

which is the general differential equation to a conic section.

Note on the Integration of $x^m(a+bx^n)^p dx$.

By Thomas Muir, LL.D.

The integration of differentials of the form $x^m(a+bx^n)^p dx$ seems to me to be susceptible of a more methodical mode of treatment than that commonly employed. In the ordinary way of presenting the matter there is little choice left to the student, when such an integration is required of him, between a haphazard, tentative process, and the consultation of a text-book, in which lists of "formulae of reduction" are given.

In beginning the subject with a learner, I should first state that the integration can be made dependent on any one of six different integrals, viz :—

(1) $\int x^{m-n}(a+bx^n)^p dx$,

(2) $\int x^{m+n}(a+bx^n)^p dx$,

(3) $\int x^m(a+bx^n)^{p-1} dx$,

(4) $\int x^m(a+bx^n)^{p+1} dx$,

(5) $\int x^{m-n}(a+bx^n)^{p+1} dx$,

(6) $\int x^{m+n}(a+bx^n)^{p-1} dx$;

that is to say, the integral can be expressed in terms of a like integral in which the index of the monomial factor is greater or less by n; in terms of a like integral in which the index of the binomial factor is greater or less by 1; in terms of a like integral in which the index of