THE HEREDITARY PROPERTY IN THE LOWER RADICAL CONSTRUCTION

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All rings considered are associative. We show that if a homomorphically closed class $\mathbf{P}_1$ of rings is hereditary in the sense that every ideal of a ring in $\mathbf{P}_1$ is also in $\mathbf{P}_1$, then the lower Kurosh radical construction terminates at $\mathbf{P}_3$. This is an improvement on the result of Anderson, Divinsky, and Sulinski **(3)** showing that the lower radical construction terminates at $\mathbf{P}_2$ provided $\mathbf{P}_1$ is homomorphically closed, hereditary, and contains all zero rings. Examples are given to show that the third step is actually attained in some constructions.

1. Preliminaries. The class of zero rings will be denoted by $\mathbf{Z}$. Let $\mathbf{P}_1$ be a homomorphically closed class of rings. For each ordinal $\alpha > 1$, $\mathbf{P}_\alpha$ is defined as the class of all rings each of whose non-zero homomorphic images contains a non-zero ideal from $\mathbf{P}_\beta$ for some $\beta < \alpha$. The union of the classes $\mathbf{P}_\alpha$ is the lower radical $\mathbf{LP}_1$ of Kurosh, and is the smallest radical class containing $\mathbf{P}_1$. If A_0 denotes the zero ring on the infinite cyclic group and $\mathbf{P}_1$ consists of all rings which are homomorphic images of A_0, then it is known **(7)** that $\mathbf{LP}_1 = \mathbf{LZ} = \mathbf{Z}_2$, while Anderson, Divinsky, and Sulinski **(3)** have shown that $\mathbf{LP}_1 = \mathbf{P}_3$. For a prime p, let $\mathbf{G}_p$ denote the class of all rings whose additive group is p-primary, and let $\mathbf{A}_p$ denote the zero ring on the cyclic group of order p. The class $\mathbf{M}_p = \{A_p, (0)\}$ is hereditary and homomorphically closed.

LEMMA 1. *For each prime p, $\mathbf{G}_p$ is a hereditary radical class and* $\mathbf{LZ} \cap \mathbf{G}_p = \mathbf{LM}_p = (\mathbf{M}_p)_3$.

Proof. It is easily verified that $\mathbf{G}_p$ is hereditary and homomorphically closed. If K/I and I are in $\mathbf{G}_p$ for I an ideal of a ring K, then K is in $\mathbf{G}_p$. Also in a ring R the union of a well-ordered sequence of ideals in $\mathbf{G}_p$ is again an ideal in $\mathbf{G}_p$. Thus $\mathbf{G}_p$ is a radical class by **(1**, p. 105**)**. Evidently, $\mathbf{M}_p \subseteq \mathbf{LZ} \cap \mathbf{G}_p$ and since $\mathbf{LZ} \cap \mathbf{G}_p$ is a radical class, $\mathbf{LM}_p \subseteq \mathbf{LZ} \cap \mathbf{G}_p$. For the opposite inclusion we observe that if $K \neq (0)$ is a homomorphic image of a ring in $\mathbf{Z} \cap \mathbf{G}_p$, then, choosing $0 \neq a \in K$ of additive order p, we have that the additive cyclic group $\{a\}$ generated by a is isomorphic to A_p, and $\{a\}$ is an ideal of K. Thus $\mathbf{Z} \cap \mathbf{G}_p \subseteq (\mathbf{M}_p)_2$. Then

$$\mathbf{LZ} \cap \mathbf{G}_p = \mathbf{Z}_2 \cap \mathbf{G}_p \subseteq (\mathbf{M}_p)_3 \subseteq \mathbf{LM}_p.$$

Remark. The class $\mathbf{P}_1$ consisting of A_0 and its homomorphic images is homomorphically closed, hereditary, and $\mathbf{LP}_1 = \mathbf{P}_3 \neq \mathbf{P}_2$. We give another

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example of a class with these properties. Let K be the ring generated by the commuting and associative symbols $\{x_i: i \geqslant 1\}$, such that for a fixed prime p, $px_i = 0$ for all i, $x_1^2 = 0$, and $x_{i+1}^2 = x_i$ for all $i \geqslant 1$ (cf. **(6)**). It is clear that K is in $\mathbf{G}_p$. In addition, every non-zero homomorphic image of K contains a non-zero ideal in $\mathbf{Z}$ so that $K \in \mathbf{LZ} \cap \mathbf{G}_p = \mathbf{LM}_p$. Now K itself can contain no non-zero ideal in $\mathbf{M}_p$. For if $I \neq (0)$ is an ideal of K, let $0 \neq a \in I$. Then at most a finite number of the x_j's occur in a. Letting t denote the maximum of these indices, we obtain a set of p distinct non-zero elements ax_{t+1}, $ax_{t+1}x_{t+2}, \ldots, ax_{t+1} \ldots x_{t+p}$ and so K is in $(\mathbf{M}_p)_3 \neq (\mathbf{M}_p)_2$.

A subring I of a ring R is *accessible* if there is a sequence $I = I_0 \subseteq I_1 \subseteq \ldots \subseteq I_n = R$ of subrings of R with I_j an ideal of I_{j+1} for $j = 0, 1, \ldots, n-1$. In **(2)** it was shown that for a radical class $\mathbf{M}$ the semisimple class $\mathbf{SM} = \{K: K$ has no non-zero ideals in $\mathbf{M}\}$ is hereditary. This implies the following

LEMMA 2 **(4)**. *For any ring K, if I is an accessible subring of K, then K contains a non-zero ideal in $\mathbf{LM}_1$, where $\mathbf{M}_1$ consists of all homomorphic images of I.*

The next lemma has been shown to be valid in the class of all (not necessarily associative) rings.

LEMMA 3 **(5)**. *If $\mathbf{M}_1$ is hereditary and homomorphically closed, then $\mathbf{M}_\alpha$ is hereditary for each ordinal α and $\mathbf{LM}_1$ is a hereditary radical class.*

2. Main results. We are now in a position to prove the following

THEOREM. *Let $\mathbf{P}_1$ be a hereditary and homomorphically closed class of (associative) rings. Then the lower Kurosh radical $\mathbf{LP}_1 = \mathbf{P}_3$.*

Proof. We show that $\mathbf{P}_4 \subseteq \mathbf{P}_3$; i.e., that every non-zero homomorphic image of a ring in $\mathbf{P}_4$ contains a non-zero ideal in $\mathbf{P}_2$. Let $K \neq (0)$ be a homomorphic image of a ring in $\mathbf{P}_4$. Then there is a sequence of non-zero subrings of K, $I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4 = K$, such that I_j is an ideal of I_{j+1} and $I_j \in \mathbf{P}_j$ for $j = 1, 2, 3$. Letting B denote the ideal of K generated by I_2, we have $B^3 \subseteq I_2$. If $B^3 \neq (0)$, then B^3 is an ideal of K in $\mathbf{P}_2$, since $\mathbf{P}_2$ is hereditary by Lemma 3. Thus assume that $B^3 = (0)$. Now $I_1 \subseteq B$ and one of I_1^2 or I_1 is non-zero and is in $\mathbf{P}_1$. Letting S denote this non-zero subring of K, we have $S \in \mathbf{Z} \cap \mathbf{P}_1$. If S contains a non-zero element of infinite additive order, then A_0 is in $\mathbf{P}_1$ as are all homomorphic images of A_0. Then $\mathbf{Z} \subseteq \mathbf{P}_2$ and since $\mathbf{P}_2$ is hereditary, in this case $\mathbf{LP}_1 = \mathbf{P}_3$ by **(3**, Theorem 2**)**. If $A_0 \subseteq S$, then A_p is an ideal of S for some prime p so that $\mathbf{M}_p \subseteq \mathbf{P}_1$. By Lemma 1, $\mathbf{LM}_p \subseteq \mathbf{P}_3$, so by Lemma 2, K contains an ideal $J_3 \neq (0)$ in $\mathbf{LM}_p = (\mathbf{M}_p)_3$. Thus we have accessible subrings $A_p \subseteq J_2 \subseteq J_3 \subseteq K$ where J_2 is in $(\mathbf{M}_p)_2 = \mathbf{Z} \cap \mathbf{G}_p$. Then if N is the ideal of K generated by J_2, $N^3 \subseteq J_2$ so that $N^6 = (0)$. Thus $N^3 \in \mathbf{Z} \cap \mathbf{G}_p = (\mathbf{M}_p)_2 \subseteq \mathbf{P}_2$. If $N^3 \neq (0)$, we are done. If $N^3 = (0)$, let L be the non-zero one of N or N^2 with $L^2 = (0)$. Since $J_3 \in \mathbf{G}_p$, $L \in \mathbf{Z} \cap \mathbf{G}_p \subseteq (\mathbf{M}_p)_2$ as in the proof of Lemma 1. Hence $L \in \mathbf{P}_2$.

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