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COMPLETE AND ORTHOGONALLY COMPLETE RINGS

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This article continues the study of Abian's order on commutative semiprime rings (for such a ring R, the relation "$a \leqq b$ if and only if $ab = a^2$" makes R into a partially ordered multiplicative semigroup). The aim, here, is to extend as far as possible the theorem of Brainerd and Lambek which says that the completion of a Boolean ring is its complete ring of quotients. Only certain subsets of a ring may have upper bounds (in any extension ring) and these are called *boundable* (the notion is due to Haines). A ring will be called *complete* if every boundable subset has a supremum. If $R \subset S$ are (commutative semiprime) rings then S will be called a *completion* of R if S is complete and every element of S is the supremum of a subset of R. It is shown by example that not all rings have completions but completions exist if the ring has sufficiently many idempotents. Such rings will be called *i-dense* and they include regular and Baer rings (in fact all *pp*-rings and more). A technique, due to Banaschewski, yields a construction which gives the completion in the case of i-dense rings: this completion is a ring of quotients with respect to a certain torsion theory and, in the case of regular rings, this completion is the complete ring of quotients. The completion, in the i-dense case, has a weak form of self-injectivety and we get the theorem that *an i-dense ring is complete if and only if it is weakly self-injective.*

1. In [**1**], Abian initiated the study of a partial order relation for commutative semiprime rings defined by $a \leqq b$ if $ab = a^2$; and, although this order relation is known in the study of semigroups [**9**, p. 40] and is well known in the special case of Boolean rings, it will here be called *Abian's order*. In [**2**] it is remarked that for a ring A, commutative or not, the relation $\leqq$ defined above is an order relation if, and only if, A is reduced (i.e., 0 is its only nilpotent) and in this case $\leqq$ makes A an ordered semigroup. All order properties below refer to Abian's order.

The purpose of [**1**] (and, in the non-commutative case, [**8**]) was to characterize, in terms of Abian's order, those reduced rings which are products of fields. One of the conditions characterizing products of fields is "orthogonal completeness". A subset X of a reduced ring A is called *orthogonal* if for $a, b \in X$, $a \neq b$, $ab = 0$. A is *orthogonally complete* if every orthogonal set in A has a supremum. Orthogonally complete rings as well as orthogonal completions

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were studied in [**7**] and this article is a continuation of that one; the following definitions and results from it are quoted for convenience.

In what follows *all rings are assumed to be reduced with* 1 and, although some of what follows can be extended to the case of reduced rings A such that the complete ring of quotients, $Q(R)$, is strongly regular, we *assume that all rings mentioned are also commutative.* If $R \subset S$ are rings, S is an *orthogonal extension* of R if every element of S is the supremum of an orthogonal set in R, and S is an *orthogonal completion* of R if it is an orthogonally complete orthogonal extension. A regular ring R is orthogonally complete if, and only if, it is self-injective. Also a regular ring always has an orthogonal completion, namely $Q(R)$, the complete ring of quotients; and a Baer ring R has an orthogonal completion which may be smaller than $Q(R)$. For any R, an orthogonal extension must lie in $Q(R)$ and for any set X in R, $\sup_R X = \sup_{Q(R)} X$, if both exist. Finally not every ring has an orthogonal completion but every ring with ascending chain condition on annihilator ideals is orthogonally complete.

2. In [**11**], Haines introduces a generalization of orthogonality which he calls "quasiorthogonality"; we prefer the term "boundable". A subset X of a ring R is *boundable* if for all $a, b \in X$, $ab(a - b) = 0$. Note that if R is Boolean, every subset is boundable. The purpose of this section is to relate the notions of orthogonality and boundability and to show that certain orthogonally complete rings are *complete* in the sense that every boundable set has a supremum.

We say that an extension $R \subset S$ is an *order extension* if every element of S is the supremum of some set in R and S is a *completion* if it is a complete order extension. Clearly any order extension of R must lie in $Q(R)$.

1. PROPOSITION. *The boundable sets of a commutative semi-prime ring R are exactly those which have suprema in $Q(R)$.*

Proof. We shall defer the proof of the fact that every boundable set X of R has a supremum in $Q(R)$ until Theorem 5. Let X be a subset of R with an upper bound $q \in Q(R)$. Then for $a, b \in X$, $ab(a - b) = a^2b - ab^2 = abq - abq = 0$. (In fact a subset X of R is boundable if it has an upper bound in any extension ring and this justifies the name.)

2. *Definition.* A ring R is called *i-dense* (*idempotent dense*) if it is commutative semiprime and every idempotent of $Q(R)$ is the supremum of a subset (necessarily a set of idempotents) of R.

The class of i-dense rings includes all *p.p.* rings [**4**, Lemma 31] and, hence, all regular and all Baer rings. However there are i-dense rings which are not *p.p.* rings (for example $C(\mathbf{Q})$). Any subdirect product of domains which includes the direct sum is i-dense and, if R is i-dense, so is any ring between R and $Q(R)$.

3. LEMMA. *If R has an order extension S which is Baer then R is i-dense.*

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Proof. This follows since the Baer ring S has all the idempotents of $Q(R)$ [**13**, 1.6 Lemma].

4. PROPOSITION. *A commutative semiprime ring R is i-dense if, and only if, every non-zero annihilator ideal contains a non-zero idempotent.*

Proof. Suppose R is i-dense. Let $X \subseteq R$ and $\text{Ann}_R X \neq 0$. We have $\text{Ann}_{Q(R)}X = eQ(R)$ for some $e^2 = e \in Q(R)$. Then $e = \sup \{e_\alpha\}$ for some set of idempotents in R. But $Xe_\alpha = Xe_\alpha e = 0$.

Conversely, let $e^2 = e \in Q(R)$, $e \neq 0$. Let D be a large ideal of R so that $(1 - e)D \subseteq R$ and $eD \subseteq R$. Then $(1 - e)DeD = 0$ so $(1 - e)D$ is annihilated by some $0 \neq g = g^2 \in R$. Thus $g(1 - e) = 0$ and $ge = g$ and e bounds some non-zero idempotents in R. Let

$$E = \{e_\alpha \in R | e_\alpha = e_\alpha^2, e_\alpha \leqq e\}.$$

Put $f = \sup_{Q(R)} E$. Then, $f \leqq e$ and $e - f$ is an idempotent, $e - f \leqq e$ and $(e - f)f = 0$. If $e - f \neq 0$, let $0 \neq e' \in R$ be an idempotent such that $e' \leqq e - f$. We get $e' \leqq e - f \leqq e$ so $e' \in E$ which implies $e' \leqq f$. Now $e' = e'(e - f) = e'e - e'f = e' - e'f$ so $e'f = 0$. This is a contradiction, so $e = f$.

5. THEOREM. *If R is i-dense then R is orthogonally complete if, and only if, it is complete. Further, if R is i-dense, any order extension of R is an orthogonal extension.*

Proof. It is easy to check that the supremum of a set X in R, if it exists, is an upper bound s so that $\text{Ann}X = \text{Ann}\{s\}$. Now if R is complete it is orthogonally complete. Conversely, if R is orthogonally complete and i-dense, it is Baer. Suppose X is boundable. Define $q \in Q(R)$ by $q : XR \oplus I \rightarrow R$ where $I = \text{Ann}_R X$ and $qx = x^2$ for $x \in X$ and $qa = 0$ for $a \in I$. Now q is well-defined since X is boundable ($\sum_X x^2 r = 0$ implies for $y \in X$, $\sum x^2 yr = \sum xy^2 r = 0$ so that $(\sum xr)y = 0$ for all $y \in X$; hence $\sum xr \in XR \cap I = 0$). Since $\text{Ann}_{Q(R)}X = \text{Ann}_{Q(R)}\{q\}$, $q = \sup_{Q(R)} X$.

Let Y be a maximal orthogonal subset of XR with the property that for all $y \in Y$, $qy = y^2$. Then, $YR + I$ is large. Indeed, if $r \neq 0$ and $r(YR + I) = 0$ then $rI = 0$ and, so, $rXR \neq 0$. Hence for some $x \in X$, $rx \neq 0$. There is an idempotent $e \in Q(R)$ so that $er = r$ and $e = rr'$ for some $r' \in Q(R)$. Since R is Baer, $e \in R$. We get that $e(YR + I) = 0$ and $ex \neq 0$. From this, $qex = x^2 e = (xe)^2$, contradicting the maximality of Y. Now q is also defined by $qy = y^2$ for all $y \in Y$ and $qa = 0$ for $a \in I$.

The remaining part now follows since it has just been shown that the supremum, in $Q(R)$, of a boundable set in R is also the supremum of an orthogonal set in R.

Note that this completes the proof of Proposition 1 since $Q(R)$ has been shown to be complete. In fact this shows that a regular ring R is complete if and only if it is orthogonally complete if and only if it is self-injective (cf. [**7**, 2. Theorem]).

For rings which are not i-dense the situation is more complicated. We know rather little in this case but the following examples will show that a complete ring is not necessarily i-dense and that there are rings which are orthogonally complete which have no completion.

6. *Example.* Let R be the subring of $\prod_{n\in N}\mathbf{Z}$ generated by $\prod_{n\in N} n\mathbf{Z}$ and 1. A typical element of R has the form $r + m$ where $r \in \prod n\mathbf{Z}$, m an integer. Hence R has only two idempotents while its complete ring of quotients $\prod\mathbf{Q}$ has infinitely many. If X is a boundable set in R, and $x \in X$ is written $x = x' + n_x$, $x' \in \prod n\mathbf{Z}$, n_x an integer; then either all the n_x are zero or for some $n \neq 0$, $n_x = n$ or $n_x = 0$ and for some $x \in X$, $n_x = n$. In the first case X has a supremum which is in $\prod n\mathbf{Z}$. In the second, let $x \in X$, $n_x = n$. Then for all but finitely many i, the i component of x is non-zero. From this it follows that the supremum can be constructed in the form $z + m$, $z \in \prod n\mathbf{Z}$.

7. *Example.* Let R be the subring of $\mathbf{Z}[x] \times \mathbf{Z}[x] \times \mathbf{Z}[x]$ generated by $(x, x, 0)$, $(0, x, x)$ and $(1, 1, 1)$. A typical element of R is of the form $(f + n, f + g + h + n, g + n)$ where $n \in \mathbf{Z}$, f, g, h are polynomials of zero constant term and if $h \neq 0$, deg $h \geqq 2$. Clearly R has the ascending chain condition on annihilators so R is orthogonally complete. Hence [**12**, p. 113], $Q(R)$ is its total ring of fractions $Q_{cl}(R)$, which is seen to be $\mathbf{Q}(x) \times \mathbf{Q}(x) \times \mathbf{Q}(x)$. R is not complete since $\{(x, x, 0), (0, x, x)\}$ is boundable while its supremum $(x, x, x) \in Q(R)$ is not in R. Further, R has no completion since such a completion would contain $(x, x, x) + (x, 0, -x) = (2x, x, 0)$. There are no non-zero elements of R below $(2x, x, 0)$.

The subject of Abian's order in rings which are not i-dense remains to be studied.

3. In [**3**], Banaschewski gives a construction of $Q(R)$, where R is a (commutative semiprime) ring, which resembles that of [**10**] for rings of continuous functions. This method is used below to construct the completion for i-dense rings.

Let X = Spec R be the set of prime ideals of R where an element of X is denoted either by x or P_x, depending on the context. Then, as usual, X is topologized by taking the sets $\{\text{coz } r | r \in R\}$ as a base for the open sets $(\text{coz } r = \{x | r \notin P_x\}, z(r) = X\backslash \text{coz } r)$. Important for us is the observation that the clopen (closed and open) sets of X are of the form coz e, e an idempotent, and conversely.

Now R may be represented as a subring of $\prod_{x\in X} R/P_x$ in an obvious way and, hence, as a subring of $S = \prod_{x\in X} Q(R/P_x)$; $Q(R/P)_x$ is a field. The components of $q \in S$ are denoted by $q(x)$, $x \in X$. For $q \in S$, coz $q \equiv \{x | q(x) \neq 0\}$ and

$z(q) \equiv \{x|q(x) = 0\}$. For each $q \in S$, we define, as in [**3**],

$$\mathscr{F}(q) = \{x|\text{ for some neighbourhood } N \text{ of } x, \text{ there are } r, s \in R \text{ with } N \subseteq \operatorname{coz} s \text{ so that for all } y \in N, q(y) = r(y)/s(y)\}$$

and

$$\mathscr{R}(q) = \{x|\text{ for some neighbourhood } N \text{ of } x, \text{ there is an } r \in R \text{ so that for all } y \in N, q(y) = r(y)\}.$$

Both $\mathscr{F}(q)$ and $\mathscr{R}(q)$ are open sets of X. We define

$$\mathscr{X}(R) \equiv \{q \in S|\mathscr{F}(q) \text{ is dense}\}$$

and $\mathscr{Y}(R) \equiv \{q \in S|\mathscr{R}(q) \text{ is dense}\}$. Hence elements of $\mathscr{X}(R)$ are "locally like" fractions of elements of R while those of $\mathscr{Y}(R)$ are "locally like" elements of R. It is clear that $\mathscr{Y}(R) \subseteq \mathscr{X}(R) \subset S$ are subrings. Next let $\mathscr{I}(R) = \{q \in S|z(q)$ contains a dense open set$\}$. Then $\mathscr{I}(R)$ is an ideal in $\mathscr{X}(R)$ and in $\mathscr{Y}(R)$ and, as Banaschewski showed, $\mathscr{X}(R)/\mathscr{I}(R) \simeq Q(R)$. We denote $\mathscr{Y}(R)/\mathscr{I}(R)$ by $C(R)$, it is a subring of $Q(R)$, in fact $R \subseteq C(R) \subseteq Q(R)$. Banaschewski remarks in [**3**] that the same construction, with X replaced by a dense subset, also yields $Q(R)$. It can be shown similarly that replacing X be a dense subset yields a ring isomorphic to $C(R)$.

For regular rings, R, $Q(R) = C(R)$, as will be seen later; but, in general, $Q(R) \neq C(R)$. In fact, if R is a domain, $C(R) = R$. Hence $C(R)$ is not always regular but it is Baer.

8. LEMMA. *For any ring R, $C(R)$ is Baer.*

Proof. We must show that any idempotent of $Q(R)$ is in $C(R)$. Let $\bar{e} \in Q(R)$ be an idempotent represented by $e \in \mathscr{X}(R)$ and $1 - \bar{e}$ represented by $f \in \mathscr{X}(R)$. We have $e^2 - e, ef, f^2 - f \in \mathscr{I}(R)$. Let U_1, U_2, U_3 be dense open sets in $z(e^2 - e)$, $z(ef)$, $z(f^2 - f)$, respectively. Then put $U = \mathscr{F}(e) \cap \mathscr{F}(f) \cap U_1 \cap U_2 \cap U_3$; U is a dense open set. For $x \in U$, $e(x) = 0$ or $e(x) = 1$. If $e(x) = 1$ then for some neighbourhood N of x, $N \subseteq U$, $e|N = r/s|N$ where $r, s \in R$, $N \subseteq \operatorname{coz} s$. Now on $N \cap \operatorname{coz} r$, $e = r/s = 1$. If $e(x) = 0$ then $f(x) = 1$ and, similarly, $f = 1$ on a neighbourhood $N \subseteq U$ of x. Hence e is 0 on N. If follows that $e \in \mathscr{Y}(R)$.

It was seen in the previous section that every boundable set in R has a supremum in $Q(R)$. In fact the supremum is in $C(R)$.

9. THEOREM. *Every boundable set in a commutative semi-prime ring R has a supremum in $C(R)$. Further, $C(R)$ is complete.*

Proof. The first part will be done by exhibiting the supremum. Let $\{r_\alpha\}_{\alpha \in \Lambda}$ be a boundable set in R. Hence for all $x \in X$, all $r_\alpha(x)$ which are non-zero coincide. Define $q \in S$ by:

$$q(x) = \begin{cases} r_\alpha(x), & \text{if for some } \alpha, r_\alpha(x) \neq 0 \\ 0, & \text{if } r_\alpha(x) = 0 \text{ for all } \alpha \in \Lambda. \end{cases}$$

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Now $q \in \mathscr{Y}(R)$ since $\mathscr{R}(q)$ contains $\bigcup_\alpha \operatorname{coz} r_\alpha \cup \sim \operatorname{cl}(\bigcup_\alpha \operatorname{coz} r_\alpha)$.

Let $\bar{q} \in C(R)$ be the element represented by q. Clearly $\bar{q}$ is an upper bound for $\{r_\alpha\}$, since $(qr_\alpha - r_\alpha^2)(x) = 0$ for all $x \in X$. If $\bar{h}$ is another upper bound represented by $h \in \mathscr{Y}(R)$, consider $\{x \in X | (qh - q^2)(x) = 0\}$; this includes

$$V = \bigcup_\alpha [\mathscr{R}(h) \cap \operatorname{coz} r_\alpha \cap U_\alpha] \cup \left(\sim \operatorname{cl}\left(\bigcup_\alpha \operatorname{coz} r_\alpha\right)\right),$$

where U_α is the interior of $z(hr_\alpha - r_\alpha^2)$. Now V is dense open since $\mathscr{R}(h)$ and each U_α are dense open. Thus $\bar{q}\bar{h} = \bar{q}^2$.

For the second part we must show that each boundable set in $C(R)$ has a supremum there. Let $\{\bar{q}_\alpha\}_{\alpha \in \Lambda}$ be a boundable set in $C(R)$. This set has a supremum $\bar{q} \in Q(R)$ which will be shown to be in $C(R)$.

For each $t \in \mathscr{X}(R)$ let

$$\mathscr{K}(t) = \{x | \text{ on some neighbourhood } N \text{ of } x, t \text{ coincides with a non-zero fraction on } N\}.$$

Then, $\mathscr{K}(t) = \operatorname{coz} t \cap \mathscr{F}(t)$. Let U_α be the interior of $z(qq_\alpha - q^2)$, a dense open set, where q and q_α represent $\bar{q}$ and $\bar{q}_\alpha$, respectively. If $x \in \mathscr{F}(q) \cap \mathscr{K}(q_\alpha) \cap U_\alpha = V_\alpha$ then $q(x) = q_\alpha(x)$ and V_α is dense in $\mathscr{K}(q_\alpha)$. Hence, $L = \bigcup_\alpha V_\alpha \subseteq \mathscr{R}(q)$. Consider $Y = \sim \operatorname{cl}(\bigcup_\alpha \mathscr{K}(q_\alpha))$. If N is an open set on which all the q_α are zero then $N \subseteq Y$. Now define $p \in \mathscr{Y}(R)$ by $p(x) = q(x)$ for $x \in L$ and $p(x) = 0$, $x \notin L$. Then, $\mathscr{R}(p) \supseteq L \cup \sim \operatorname{cl}(L)$ so, indeed, $p \in \mathscr{Y}(R)$. Next, $\bar{p} \in C(R)$ is an upper bound of $\{\bar{q}_\alpha\}$. Indeed, $pq_\alpha - q_\alpha^2$ is zero on V_α and on $\mathscr{R}(q_\alpha) \backslash \mathscr{K}(q_\alpha)$. Hence $pq_\alpha - q_\alpha^2$ is zero on V_α, which is dense open. Also, $\bar{p} \leqq \bar{q}$ since $pq - p^2$ is zero on $L \cup \sim \operatorname{cl}(L)$. Hence $p - q \in \mathscr{I}(R)$. But $p \in \mathscr{Y}(R)$ so $\bar{q} \in C(R)$.

The question which arises naturally is: For which rings is $C(R)$ the (orthogonal) completion? The answer will be "i-dense rings".

10. LEMMA. *If R is i-dense then every non-empty open set of $X =$* Spec R *contains a non-empty open set of the form $A \cap V$, A clopen and V dense open.*

Proof. We must show that each set $\operatorname{coz} r$, $0 \neq r \in R$, contains a set of the indicated type. Let $\bar{f} \in Q(R)$ be an idempotent with $\bar{f}r = r$ and $\bar{f} = r\bar{r}$, for some $\bar{r}' \in Q(R)$. Let $f, r' \in \mathscr{X}(R)$ be representatives and so for some dense open set U, $f|U = rr'|U$. Hence for $x \in U$, $r(x) = 0$ if, and only if, $f(x) = 0$ and $r(x) \neq 0$ if, and only if $f(x) = 1$. Since R is i-dense, there is an idempotent $e \in R$, $e \neq 0$, with $e\bar{f} = e$. Hence $ef - e$ is zero on some dense open set U' and for $x \in U'$, $e(x) = 1$ implies $f(x) = 1$. Put $V = U \cap U'$. Then $\operatorname{coz} e \cap V \subseteq \operatorname{coz} r$.

Note that the family $\mathfrak{A}$ of sets of the form $A \cap V$, A clopen and V dense open, may not form a base for the topology, but for every open set U there is a disjoint family $\{A_\alpha\}$ from $\mathfrak{A}$ so that each $A_\alpha \subseteq U$ and $\bigcup A_\alpha$ is dense in U.

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11. THEOREM. *$C(R)$ is the completion of the commutative semiprime ring R if, and only if, R is i-dense; and, in this case, it is also the orthogonal completion.*

Proof. If the Baer ring $C(R)$ is the (orthogonal) completion of R then R is certainly i-dense.

Conversely, we shall use the family $\mathfrak{A}$ of open sets discussed in (10) to show that each element of $C(R)$ is the supremum of some orthogonal set in R.

Let $\bar{q} \in C(R)$ be represented by $q \in \mathscr{Y}(R)$. For each $x \in \mathscr{R}(q)$, there is an open set N, $x \in N$, so that for some $r \in R$, $q|N = r|N$. A maximal disjoint family, $\{U_\alpha\}_{\alpha\in\Lambda}$, of open subsets of $\mathscr{R}(q)$ such that $q|U_\alpha = r_\alpha|U_\alpha$ for some $r_\alpha \in R$, has union dense in $\mathscr{R}(q)$. For each U_α, in such a family, there is a disjoint family

$$\{\text{coz } e_{\alpha\beta} \cap V_{\alpha\beta}\}_{\beta\in\Lambda_\alpha},\ e_{\alpha\beta}{}^2 = e_{\alpha\beta} \in R,\ V_{\alpha\beta} \text{ dense open in } X,$$

such that its union is dense in U_α. Then $Y = \{r_\alpha e_{\alpha\beta}\}_{\beta\in\Lambda_\alpha,\alpha\in\Lambda}$ is orthogonal. Let $\bar{h} \in C(R)$ be the supremum of Y with representative $h \in \mathscr{Y}(R)$ as in the proof of (9). Then q and h coincide on $\bigcup\ (\text{coz } e_{\alpha\beta} \cap V_{\alpha\beta})$, and so $\bar{q} = \bar{h}$.

12. COROLLARY. *If R is regular then $C(R) = Q(R)$.*

Proof. This follows since R is i-dense with $Q(R)$ as orthogonal completion.

The converse, however, is false since the subring R of $\prod_N \mathbf{Q}$, consisting of elements which are almost everywhere integers, is Baer, but not regular, while $C(R) = Q(R)$.

Just as in [**7**, 18. Theorem] it will be shown that if R is i-dense, $C(R)$ is the partial ring of quotients with respect to an idempotent topologizing family, $\mathscr{E}$, of ideals of R. This is done by making precise the isomorphism $Q(R) \to \mathscr{X}(R)/\mathscr{I}(R)$. Let $s \in Q(R)$ be represented by $\phi : D \to R$ where D is a large ideal of R. Let D' be a maximal orthogonal family from D, D' necessarily has zero annihilator. Define $q \in \mathscr{X}(R)$ by:

$$q(x) = \begin{cases} \dfrac{\phi(d)(x)}{d(x)}, & \text{if } x \in \text{coz } d \text{ for some } d \in D' \\ 0, & \text{otherwise.} \end{cases}$$

Then define $\Psi : Q(R) \to \mathscr{X}(R)/\mathscr{I}(R)$ by $\Psi(s) = \bar{q}$.

1. $\bar{q}$ is independent of the choice of D and D'. (This is easy to see).

2. Ψ is a ring isomorphism. The verification that Ψ is an injective ring homomorphism is straightforward. It must be shown to be surjective. Consider $\bar{q} \in \mathscr{X}(R)/\mathscr{I}(R)$ represented by $q \in \mathscr{X}(R)$. Let $\mathscr{N}$ be the set of open sets N of X so that for some $r, s \in R$, $N \subseteq \text{coz } s$, $q|N = r/s|N$. A maximal disjoint family $\mathscr{U}$ from $\mathscr{N}$ has union which is dense in X, since it is dense in $\bigcup_{\mathscr{N}} N = \mathscr{F}(q)$, which is dense in X. For each $U_\alpha \in \mathscr{U}$, let $q|U_\alpha = r_\alpha/s_\alpha|U_\alpha$ and choose a maximal orthogonal set $\{t_{\alpha\beta}\}$ of elements of R so that $\text{coz } t_{\alpha\beta} \subseteq U_\alpha$. Then,

$$q\,\Big|\,\text{coz } t_{\alpha\beta} = \frac{r_\alpha t_{\alpha\beta}}{s_\alpha t_{\alpha\beta}}\,\Big|\,\text{coz } t_{\alpha\beta}.$$

Note that $\bigcup_\beta$ coz $t_{\alpha\beta}$ is dense in U_α. Then, $T = \{s_\alpha t_{\alpha\beta}\}_{\alpha,\beta}$ is orthogonal and $\bigcup_{\alpha,\beta}$ coz $s_\alpha t_{\alpha\beta}$ is dense. Also, TR is a large ideal and T is a maximal orthogonal set in it. Define $\phi : TR \rightarrow R$ by $\phi(s_\alpha t_{\alpha\beta}) = r_\alpha t_{\alpha\beta}$. The corresponding element of $\mathscr{X}(R)$ defined by T and ϕ is q' where

$$q'(x) = \begin{cases} \dfrac{\phi(s_\alpha t_{\alpha\beta})(x)}{s_\alpha t_{\alpha\beta}(x)} = \dfrac{r_\alpha t_{\alpha\beta}(x)}{s_\alpha t_{\alpha\beta}(x)}, & \text{for } x \in \text{coz } t_\alpha s_{\alpha\beta} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly q and q' coincide on the dense open set $\bigcup_{\alpha,\beta}$ coz $s_\alpha t_{\alpha\beta}$ and, hence, $\bar{q} = \bar{q}'$.

13. THEOREM. *For each large ideal D of a commutative semi-prime ring R, let* Hom$'$D $= \{\phi : D \rightarrow R | \phi(d) = r_d d$ *for some* $r_d \in R\}$. *Then*

$$C(R) = \varinjlim_{D \text{ large}} \text{Hom}'D.$$

If R is i-dense, let $\mathscr{E} = \{D | D$ contains a set of idempotents E so that Ann $E = 0\}$. *Then $\mathscr{E}$ is an idempotent topologizing family and $C(R) = Q_{\mathscr{E}}(R)$.*

Proof. A general reference for rings of quotients is [**5**, Chapitre II, § 2, Exercices] or [**14**, Chapter 2]. Clearly $\varinjlim$ Hom$'D$ is a subring of $Q(R)$ since each Hom$'D$ is a subgroup of Hom(D, R) which is preserved by restrictions and compositions. We next imitate the constructions given above. In fact if $\phi \in$ Hom$'D$, the corresponding element $q \in \mathscr{X}(R)$ is in $\mathscr{Y}(R)$ (for $x \in$ coz d, $q(x) = \phi(d)(x)/d(x) = r_d(x)d(x)/d(x) = r_d(x)$). Similarly, if $q \in \mathscr{Y}(R)$, the homomorphism $\phi : TR \rightarrow R$ constructed above is in Hom$'TR$ since $\phi(s_\alpha t_{\alpha\beta}) = r_\alpha t_{\alpha\beta}$; but, here, s_α may be taken to be 1.

In [**7**], it is shown that $\mathscr{E}$ is a topologizing idempotent family. If $D \in \mathscr{E}$ with E the set of idempotents in D, then ER is also large; and, the set of ideals of the form ER, E a set of idempotents with Ann $E = 0$, is cofinal in $\mathscr{E}$. But, Hom$(ER, R) =$ Hom$'ER$. Hence, in general, $Q_{\mathscr{E}}(R) \subseteq C(R)$. If R is i-dense the first part of this proof will be refined.

Indeed, let $\{t_{\alpha\beta}\}$ be as in the first part of the proof. Then, in coz $t_{\alpha\beta}$ find a maximal disjoint family of sets from $\mathfrak{A}$ (as in (10)). That is, sets of the form coz $e_{\alpha\beta\gamma} \cap V_{\alpha\beta\gamma}$, where $e_{\alpha\beta\gamma}$ is an idempotent and $V_{\alpha\beta\gamma}$ is dense open. Then $\bigcup_\gamma$ (coz $e_{\alpha\beta\gamma} \cap V_{\alpha\beta\gamma}$) is dense in coz $t_{\alpha\beta}$. Now let $E = \{e_{\alpha\beta\gamma}\}_{\alpha,\beta,\gamma}$; E is orthogonal and has zero annihilator, since $\bigcup_{\alpha\beta\gamma}$ coz $e_{\alpha\beta\gamma}$ is dense. Define $\phi : ER \rightarrow R$ by: $\phi(e_{\alpha\beta\gamma}) = e_{\alpha\beta\gamma} r_\alpha t_{\alpha\beta}$. Then ϕ gives rise to an element equivalent to q.

The rings $C(R)$ have a kind of weakened injectivety.

14. *Definition.* A commutative ring R is *weakly self-injective* if for every ideal I of R and homomorphism $\phi : I \rightarrow R$, so that for all $a \in I$ there is $r_a \in R$ with $\phi(a) = ar_a$, ϕ lifts to an endomorphism of R.

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This allows an extension to i-dense rings of the theorems of Brainerd and Lambek, [**6**], for Boolean rings and those of [**7**] for regular rings.

15. THEOREM. *Let R be i-dense. Then R is complete if, and only if, R is weakly-self-injective. Also, $C(R)$ is the completion of R.*

Proof. This follows from (11) and the observation, based on (13), that R is weakly self-injective if, and only if, $R = C(R)$.

Products of domains are weakly self-injective and the following characterization is given without proof, since it is straightforward.

16. PROPOSITION. *A commutative semiprime ring R is isomorphic to a product of domains if, and only if, it is orthogonally complete, i-dense and its algebra of idempotents is atomic.*

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