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# ASYMPTOTIC EXPECTED NUMBER OF PASSAGES OF A RANDOM WALK THROUGH AN INTERVAL

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**Abstract**

In this note we find a new result concerning the asymptotic expected number of passages of a finite or infinite interval $(x, x+h]$ as $x \to \infty$ for a random walk with increments having a positive expected value. If the increments are distributed like $X$ then the limit for $0 < h < \infty$ turns out to have the form $\mathbb{E}\min(|X|, h)/\mathbb{E}X$, which unexpectedly is independent of $h$ for the special case where $|X| \le b < \infty$ almost surely and $h > b$. When $h = \infty$, the limit is $\mathbb{E}\max(X, 0)/\mathbb{E}X$. For the case of a simple random walk, a more pedestrian derivation of the limit is given.

*Keywords:* Random walk; passage; generalized renewal theorem; two-sided renewal theorem

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## 1. The result

In this note we prove an asymptotic formula for the expected number of passages of a random walk with positive drift through $(x, x+h]$ for $0 < h \le \infty$ as $x \to \infty$. In general, a *passage* of a stochastic sequence $(Y_n)_{n\ge 0}$ *through* a subset $A$ of its state space is defined to consist of an entry to, followed by a sojourn in, and then an exit from $A$. It is given by a sequence of epochs $n+1, \dots, n+i$ $(i \ge 1)$ such that $Y_n \notin A$, $Y_{n+1} \in A, \dots, Y_{n+i} \in A$, $Y_{n+i+1} \notin A$. It is natural to call $i$ the *length* of the passage.

Now, let $S_n = X_1 + \cdots + X_n$ $(S_0 = 0)$ be a real-valued random walk with independent and identically distributed (i.i.d.) increments $X_i$ distributed like $X$ with $\mathbb{E}|X| < \infty$ and having expected value $\mu = \mathbb{E}X > 0$. We fix a constant $0 < h \le \infty$ and denote by $N^x$, $x \in \mathbb{R}$, the number of passages of $S_n$ through the interval $(x, x+h]$ ($(x, \infty)$ if $h = \infty$). The classical two-sided renewal theorem (see, e.g. [2, p. 218] and [3, p. 172]) states that, when the distribution of $X$ is nonarithmetic, the expected number of visits of the interval $(x, x+h]$, denoted by $R((x, x+h])$, where

$$R(A) = \mathbb{E}\sum_{n=0}^{\infty} \mathbf{1}_{\{S_n \in A\}}, \tag{1}$$

converges to $h/\mu$ as $x \to \infty$ and to 0 as $x \to -\infty$ (with a slight adjustment in the case when the underlying distribution is arithmetic). The following two results can be viewed as a neat little supplement to this important theorem.

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**Theorem 1.** *Let* $0 < h < \infty$.

(a) *If* $X$ *has a nonarithmetic distribution,*

$$\lim_{x\to\infty} \mathbb{E}N^x = \frac{\mathbb{E}\min[|X|, h]}{\mu}. \tag{2}$$

(b) *If* $X$ *has an arithmetic distribution then (2) holds for every* $h > 0$ *which is divisible by the span.*

Although it would have been nice if, for the case $h = \infty$, we could simply replace $\min[|X|, h]$ or $\min[|X|, k\alpha]$ by $|X|$, this turns out to be false. Instead, the following holds, where throughout we use the notation $a^+ = \max(a, 0)$ and $a^- = \max(-a, 0)$.

**Theorem 2.** *Let* $h = \infty$. *Then (nonarithmetic or arithmetic),*

$$\lim_{x\to\infty} \mathbb{E}N^x = \frac{\mathbb{E}X^+}{\mu}. \tag{3}$$

In Theorem 2 we count in $N^x$ also the terminal entrance to and subsequent infinite sojourn in $(x, \infty)$. If we want to exclude this 'passage', the limit in (3) becomes $\mathbb{E}X^+/\mu - 1 = \mathbb{E}X^-/\mu$. In Section 2 we consider a few special cases; the proofs are carried out in Section 3.

## 2. Some special cases

### 2.1. Simple random walk with $0 < h < \infty$

We first consider the simple random walk with $\mathbb{P}(X = 1) = p$ and $\mathbb{P}(X = -1) = q = 1-p$, where $p > q$. Fix $x, h \geq 1$ (integers). Note that the expected number of passages through $\{x, \dots, x + h - 1\}$ when starting at 0 is the same for every $x > 0$ since the random walk is skip-free and converges to $\infty$ almost surely. Therefore, we set $x = 1$. Let $a_h$ and $b_h$ be the expected numbers of passages through $E = \{1, \dots, h\}$ when starting from 0 and $h + 1$, respectively. Then $\mathbb{E}N^x = a_h$ and we now give a direct proof that

$$\mathbb{E}N^x = a_h = \frac{\mathbb{E}\min[|X|, h]}{\mathbb{E}X} = \frac{\mathbb{E}|X|}{\mathbb{E}X} = \frac{1}{p-q} \quad \text{for all } h \geq 1$$

(note that $|X| \equiv 1$). It is remarkable that $\mathbb{E}N^x$ does not depend on $h$.

As $p > q$, we have

$$a_h = 1 + \pi_h a_h + (1 - \pi_h) b_h, \tag{4}$$

where $\pi_h$ is the probability that 0 is reached before $h + 1$ when starting from 1. Indeed, when starting from a state to the left of $E$, the random walk enters $E$ at 1 with probability 1 and thereafter the next passage comes from the left with probability $\pi_h$ or, with probability $1 - \pi_h$, state $h+1$ is reached before 0. On the other hand, when starting from $h+1$, the set $E$ (actually, the state $h$) is reached with probability $q/p$ and then the next attained state outside $E$ is 0 or $h + 1$. Therefore, we obtain

$$b_h = \frac{q}{p}(1 + \rho_h a_h + (1 - \rho_h) b_h), \tag{5}$$

where $\rho_h$ is the probability that 0 is reached before $h+1$ when starting from $h$. The probabilities $\pi_h$ and $\rho_h$ are of course well known from the standard gambler's ruin problem:

$$\pi_h = \frac{(q/p) - (q/p)^{h+1}}{1 - (q/p)^{h+1}}, \qquad \rho_h = \frac{(q/p)^h - (q/p)^{h+1}}{1 - (q/p)^{h+1}}. \tag{6}$$

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Equation (4) yields

$$a_h = \frac{1}{1-\pi_h} + b_h. \tag{7}$$

Setting $r = q/p$, we get, from (5)–(7),

$$b_h = \frac{r}{1-r}\left(1 + \frac{\rho_h}{1-\pi_h}\right).$$

Next check that $\rho_h/(1-\pi_h) = r^h$. A little calculation now shows that

$$\begin{aligned} a_h &= \frac{1}{1-\pi_h} + \frac{r}{1-r}\left(1 + \frac{\rho_h}{1-\pi_h}\right) \\ &= \frac{1-r^{h+1}}{1-r} + \frac{r}{1-r}(1+r^h) \\ &= \frac{1+r}{1-r} \\ &= \frac{1}{p-q}, \end{aligned}$$

as was to be proved. Moreover, for $k \geq 1$, the expected number of passages through $E$ starting from $h+k$ is equal to $[1-r]^{-1}[1+r^h]r^k$.

The case of random walks having increments $-1, 0, 1$ with probabilities $p_{-1}, p_0, p_1$, reduces to the case above with $p = p_1/(p_{-1}+p_1)$ because here the number of passages is the same as that of the random walk which is embedded at state change epochs.

### 2.2. Simple random walk with $h = \infty$

In the setting of Subsection 2.1, when $h = \infty$, we are interested in the asymptotic expected number of passages through $(x, \infty)$. Since $x$ is hit with probability 1, then, for every $x > 0$, it is the same as the expected number of passages through $\{1, 2, \ldots\}$, which we denote by $a_\infty$. We want to verify that

$$a_\infty = \frac{\mathbb{E}X^+}{\mathbb{E}X} = \frac{p}{p-q}.$$

Indeed, since the probability to ever reach 1 starting from 0 is 1 and the probability to ever reach 0 from 1 is $q/p$, we have

$$a_\infty = 1 + \frac{q}{p}a_\infty,$$

so

$$a_\infty = \frac{1}{1-q/p} = \frac{p}{p-q}.$$

Of course, the last paragraph of Subsection 2.1 applies to this case as well.

### 2.3. Random walks with inequality constraints

In general, if $|X| \leq b < \infty$ almost surely, we have, for $b \leq h < \infty$,

$$\mathbb{E}N^x \to \frac{\mathbb{E}|X|}{\mathbb{E}X} = \frac{1 + (\mathbb{E}X^-/\mathbb{E}X^+)}{1 - (\mathbb{E}X^-/\mathbb{E}X^+)},$$

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so the limit depends only on the ratio $\mathbb{E}X^-/\mathbb{E}X^+$. This is also the case when $h = \infty$ as the limit may be written as follows:

$$\frac{\mathbb{E}X^+}{\mathbb{E}X} = \frac{1}{1 - (\mathbb{E}X^-/\mathbb{E}X^+)}.$$

If $X$ takes only nonnegative values, there is at most one passage through $(x, x+h]$ and

$$\mathbb{P}(N^x = 1) \to \frac{\mathbb{E}\min(X, h)}{\mu} = \int_0^h \frac{\mathbb{P}(X > s)}{\mu}\,\mathrm{d}s = F_{\mathrm{eq}}(h), \tag{8}$$

where $F_{\mathrm{eq}}$ is the equilibrium distribution associated with $X$. In this case it is interesting to note that (8) is valid regardless of whether $h$ is finite or not.

If $|X| > h$ then $\mathbb{E}N^x \to h/\mu$. Every passage through $(x, x+h]$ corresponds to exactly one visit of this interval (since every entrance to $(x, x+h]$ is immediately followed by an exit). Therefore, $\mathbb{E}N^x = R((x, x+h])$ in this case and we are back to the classical two-sided renewal theorem.

Finally, consider the case when $X$ takes only values in $[-h, 0] \cup (h, \infty)$. Then it follows that

$$\mathbb{E}N^x \to \frac{\mathbb{E}X^- + h\mathbb{P}(X > h)}{\mathbb{E}X} = \frac{\mathbb{E}X^-}{\mathbb{E}X} + hf_{\mathrm{eq}}(h),$$

where $f_{\mathrm{eq}}$ denotes the equilibrium density of $X$.

## 3. Proofs

We only treat the nonarithmetic case. The proof of the arithmetic case follows along the same lines. The following lemma will prove useful.

**Lemma 1.** *Let $(X_n)_{n\geq 0}$ be a stationary and ergodic sequence, and let $A$ be a measurable subset of its state space, satisfying $\mathbb{P}(X_0 \in A^{\mathrm{c}},\ X_1 \in A) > 0$ (thus, $\mathbb{P}(X_0 \in A) = \mathbb{P}(X_1 \in A) > 0$). Let $V_1, V_2, \dots$ be the lengths of the successive passages through $A$. Then, as $n \to \infty$,*

$$n^{-1}\sum_{i=1}^{n} V_i \to (1 - \mathbb{P}(X_1 \in A \mid X_0 \in A))^{-1} \quad \textit{almost surely.}$$

*Proof.* Let $J_i = \mathbf{1}_{\{X_i \in A\}}$ and $K_i = \mathbf{1}_{\{X_i \in A^{\mathrm{c}},\, X_{i+1} \in A\}}$. Let $L_n$ be the last time of the $n$th passage. As $\mathbb{P}(X_0 \in A^{\mathrm{c}},\ X_1 \in A) > 0$, then $L_n \to \infty$ almost surely and, by the ergodic theorem for stationary sequences,

$$\begin{aligned}
\lim_{n\to\infty} n^{-1}\sum_{i=1}^{n} V_i &= \lim_{n\to\infty} \frac{\sum_{i=1}^{L_n} J_i}{\sum_{i=1}^{L_n} K_i} \\
&= \frac{\lim_{n\to\infty} L_n^{-1}\sum_{i=1}^{L_n} J_i}{\lim_{n\to\infty} L_n^{-1}\sum_{i=1}^{L_n} K_i} \\
&= \frac{\mathbb{P}(J_0 = 1)}{\mathbb{P}(K_0 = 1)} \\
&= \frac{\mathbb{P}(X_0 \in A)}{\mathbb{P}(X_0 \in A^{\mathrm{c}}, X_1 \in A)} \quad \text{almost surely,}
\end{aligned}$$

completing the proof.

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### 3.1. Proof of Theorem 1: $0 < h < \infty$

We introduce the auxiliary regenerative process $X_n^x$ that is identical to $S_n$ until the level $2x$ is exceeded (at which time the first cycle is completed), then restarts from 0 until $2x$ is exceeded again, etc. ($2x$ could be replaced by any $f(x)$ such that $f(x) - x \to \infty$ as $x \to \infty$). Let $\tilde{N}^x$ be the number of passages of $X_n^x$ through $(x, x+h]$ in the first cycle. Observe that, for $x > h$, a passage cannot be interrupted by an end of a cycle. We recall from (1) that $R(A)$ is the expected number of epochs at which $S_n$ is in $A$ (the renewal measure) and denote by $R_x(A)$ the expected number of epochs at which $X_n^x$ is in $A$ during the first cycle. $R(x+I)$ tends to the length of $I$ divided by $\mu$ as $x \to \infty$ and to 0 as $x \to -\infty$ for all bounded intervals $I$. Clearly, $R_x \le R$ and since $R(A)$ and $R_x(A)$ differ at most by the expected number of points of $S_n$ that return to $[\inf A, \sup A]$ after $S_n$ has crossed $2x$, it follows that

$$|R_x(x+A) - R(x+A)| \le \sup_{y \ge x} R((-y, -y+h]) \quad \text{for all } A \subset (0, h].$$

Hence, recalling that $R((-y, -y+h]) \to 0$ as $y \to \infty$,

$$\lim_{x\to\infty} \sup_{A \subset [0,h]} |R_x(x+A) - R(x+A)| \to 0. \tag{9}$$

Since $N^x - \tilde{N}^x$ is bounded above by the overall number of visits to $(x, x+h]$ after the first cycle, it also follows that

$$0 \le \mathbb{E}N^x - \mathbb{E}\tilde{N}^x \le R((x, x+h]) - R_x((x, x+h]),$$

so we also have

$$\lim_{x\to\infty} (\mathbb{E}N^x - \mathbb{E}\tilde{N}^x) = 0. \tag{10}$$

Let us first fix $x > 0$. By the ergodic theorem for regenerative processes (see, e.g. [1, p. 170]), the stationary distribution $\nu_x$ of $X_n^x$ is of the form $\nu_x(A) =$ expected number of points in $A$ in the first cycle divided by the expected cycle length, i.e. $\nu_x(A) = R_x(A)/c(x)$, where $c(x)$ is the (finite) expected cycle length of $X_n^x$.

Now, make the (Markov) process $(X_n^x)_{n\ge0}$ a stationary and ergodic sequence by starting it with $\nu_x$. Then let $V_1^x, V_2^x, \dots$ be the lengths of the consecutive passages of $X_n^x$ through $(x, x+h]$. From Lemma 1, as $n \to \infty$,

$$n^{-1} \sum_{i=1}^n V_i^x \to v_x =: (1 - \mathbb{P}(X_1^x \in (x, x+h] \mid X_0^x \in (x, x+h]))^{-1} \quad \text{almost surely.} \tag{11}$$

Let $Y \sim R_x(\cdot)/c(x)$ be independent of $X$ ($X \sim X_1$). Then, the conditional probability on the right-hand side of (11) can be written as

$$\begin{aligned} &\frac{\mathbb{P}(X_0^x \in (x, x+h],\ X_1^x \in (x, x+h])}{\mathbb{P}(X_0^x \in (x, x+h])} \\ &\quad = \frac{\mathbb{P}(Y \in (x, x+h],\ Y + X \in (x, x+h])}{\mathbb{P}(Y \in (x, x+h])} \\ &\quad = \frac{\mathbb{P}(x + X^- < Y \le x + h - X^+)}{\mathbb{P}(Y \in (x, x+h])} \\ &\quad = \frac{\mathbb{P}(x + X^- < Y \le x + h - X^+,\ X^- < h - X^+)}{\mathbb{P}(Y \in (x, x+h])} \end{aligned}$$

$$= \frac{\mathbb{P}(Y \in (x+X^-, x+h-X^+],\ |X|<h)}{\mathbb{P}(Y \in (x, x+h])}$$
$$= \frac{c(x)^{-1}\mathbb{E}R_x((x+X^-, x+h-X^+])\,\mathbf{1}_{\{|X|<h\}}}{c(x)^{-1}R_x((x, x+h])}$$
$$= \frac{\mathbb{E}R_x((x+X^-, x+h-X^+])\,\mathbf{1}_{\{|X|<h\}}}{R_x((x, x+h])}.$$

It is well known (and quite easy to show) that there are finite constants $a$ and $b$ such that $R((x, x+h]) \le ah + b$ for all (finite) $x, h > 0$ and, thus,

$$\begin{aligned} R_x((x+X^-, x+h-X^+])\,\mathbf{1}_{\{|X|<h\}} &\le R((x+X^-, x+h-X^+])\,\mathbf{1}_{\{|X|<h\}} \\ &\le a(h-|X|)^+ + b \\ &\le ah + b. \end{aligned}$$

Thus, by dominated convergence, (9), and the generalized renewal theorem, it follows that, as $x \to \infty$,

$$\begin{aligned} \frac{\mathbb{E}R_x((x+X^-, x+h-X^+])\,\mathbf{1}_{\{|X|<h\}}}{R_x((x, x+h])} &\to \frac{\mathbb{E}(h-|X|)^+/\mu}{h/\mu} \\ &= \frac{\mathbb{E}(h-|X|)^+}{h} \\ &= 1 - \frac{\mathbb{E}\min(|X|, h)}{h}, \end{aligned}$$

and so, recalling (11), we have, as $x \to \infty$,

$$v_x \to \frac{h}{\mathbb{E}\min(|X|, h)}.$$

Next let $\tilde{N}_j^x$ be the number of passages through $(x, x+h]$ in the $j$th cycle, and let $V_{i,j}^x$ be the length of the $i$th passage through $(x, x+h]$ in the $j$th cycle. Then

$$\begin{aligned} v_x &= \lim_{k\to\infty} \frac{\sum_{j=1}^k \sum_{i=1}^{\tilde{N}_i^x} V_{i,j}^x}{\sum_{j=1}^k \tilde{N}_j^x} \\ &= \frac{\lim_{k\to\infty} k^{-1} \sum_{j=1}^k \sum_{i=1}^{\tilde{N}_i^x} V_{i,j}^x}{\lim_{k\to\infty} k^{-1} \sum_{j=1}^k \tilde{N}_j^x} \\ &= \frac{R_x((x, x+h])}{\mathbb{E}\tilde{N}^x} \quad \text{almost surely.} \end{aligned}$$

The last equality follows since we have the moment estimator from an i.i.d. sample of size $k$ of the expected number of points of $S_n$ in $(x, x+h]$ before exceeding $2x$ in the numerator, and the corresponding moment estimator of $\mathbb{E}\tilde{N}^x$ in the denominator. Thus, we have, as $x \to \infty$,

$$\mathbb{E}\tilde{N}^x = R((x, x+h])v_x^{-1} \to \frac{h}{\mu}\frac{\mathbb{E}\min(|X|, h)}{h} = \frac{\mathbb{E}\min(|X|, h)}{\mu}, \tag{12}$$

and, finally, from (10), the desired limit is achieved.

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### 3.2. Proof of Theorem 2: $h = \infty$

We first note that clearly every passage above $x$ can be matched with a passage below $x$ and, thus, for $x > 0$, the number of passages through $(x, \infty)$ is the same as the number of passages through $(-\infty, x]$, provided that the terminal passage that starts above $x$ and never ends is also counted as one passage and the same holds for the first passage under $x$ that starts at 0. The proof, therefore, follows the same procedure as before but with $N^x$ denoting the number of passages below (and, thus, above) $x$. The only difference is the following computation:

$$\begin{aligned} v_x^{-1} &= 1 - \frac{\mathbb{P}(X_0^x \le x,\ X_1^x \le x)}{\mathbb{P}(X_0^x \le x)} \\ &= 1 - \frac{\mathbb{P}(Y \le x,\ Y + X \le x)}{\mathbb{P}(Y \le x)} \\ &= 1 - \frac{\mathbb{P}(Y \le x - X^+)}{\mathbb{P}(Y \le x)} \\ &= 1 - \frac{c(x)^{-1}\mathbb{E}R_x((-\infty, x - X^+])}{c^{-1}(x)R_x((-\infty, x])} \\ &= 1 - \frac{\mathbb{E}R_x((-\infty, x - X^+])}{R_x((-\infty, x])} \\ &= \frac{\mathbb{E}R_x((x - X^+, x])}{R_x((-\infty, x])}. \end{aligned}$$

Since

$$R_x((x - X^+, x]) \le R((x - X^+, x]) \le aX^+ + b,$$

and $\mathbb{E}|X| < \infty$, we can conclude as in (12), using dominated convergence and applying the same arguments as in the proof of Theorem 1, that

$$\mathbb{E}\tilde{N}^x = R_x(-\infty, x]v_x^{-1} = \mathbb{E}R_x((x - X^+, x]) \to \frac{\mathbb{E}X^+}{\mu}$$

as $x \to \infty$ and, hence, also that

$$\mathbb{E}N^x \to \frac{\mathbb{E}X^+}{\mu},$$

as required.

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