SOME NEW BOOKS ON LOGIC

A well-known Hungarian mathematician has conjectured that, as soon as man discovers a proof that $1 = 0$, the world will vanish. For fear of paradox intuitionists have been doing mathematics with one hand tied behind their back. Formalists still insist that metamathematics be done in this manner.

Having been brought up on the books by Quine, Rosenbloom, and Kleene, one is left with the impression that logic is there to talk about mathematics, and that it may only utilize the most rudimentary of constructive mathematics itself. Hence the emphasis on syntax, the science which deals with strings of symbols. Looking at the new books, one realizes that there has been a swing away from syntax to semantics, the science of meaning, and that the most powerful mathematical tools are now being used in logical investigations. To quote Fraissé: "One may say that semantics is to syntax as the theory of fields is to the methods for solving algebraic equations."

Cours de logique mathématique, tome 1; relation, formule logique, compacité, complétude (Collection de logique mathematique, série A. No. 23) by R. Fraissé. Gauthier-Villars, Paris, 1967. xii + 186.

This compact paperback begins with an instructive historical introduction. The first chapter deals with words and formulas, and contains a proof of the interesting but little known theorem that a string of symbols (in bracketless notation) is a formula if and only if it has zero "weight" but every initial segment has positive weight.

Important rôles in the treatment of the predicate calculus are played by a number of unusual concepts. A "multirelation" is a set with a finite family of finitary relations. "Local isomorphisms" are isomorphisms between restrictions of comparable multirelations. A multirelation N is "freely interpretable" in a multirelation M if every local automorphism of M is a local automorphism of N. One drawback of the book is that, when one opens it at random, unfamiliar terms like the above stare one in the face. However, these concepts are used in establishing theorems with familiar names: Skolem-Löwenheim, compactness, Henkin-Malcev, completeness (Gödel), interpolation (Craig). There is also a chapter dealing with finite axiomatizability, saturation and categoricity.

There are many non-routine exercises, a bibliography and an index.

Eléments de logique mathématique: théorie des modèles (Monographies de la société mathématique de France, 3) by G. Kreisel and J.L. Krivine. Dunod, Paris, 1967. viii + 212 pages.

This is the French version of the book reviewed below.

Elements of mathematical logic (model theory) by G. Kreisel and J.L. Krivine. North-Holland Publishing Company, Amsterdam, 1967. xi + 221 pages. U.S. $9.20.

This book has completely sacrificed syntax to semantics. In Chapter I, for example, the Propositional Calculus is studied as dealing with functions of the form $2^n \to 2$. The very first result proved is Craig's Interpolation Lemma, followed quickly by the Compactness Theorem. The theory is continued in the exercises, with answers given, which also provide interesting applications to group theory and graph theory.

Other chapters deal in the same spirit with successively more complicated systems: the predicate calculus, the predicate calculus with equality (among the exercises there is a proof of the existence of the algebraic closure of a field!), higher order logic. A highlight of the book is Chapter 4 on the elimination of quantifiers and the fact that this implies completeness. The elimination of quantifiers is carried out in detail for a number of special systems: various kinds of ordered sets, algebraically closed fields, real closed fields, certain boolean rings. Among the applications (developed in the exercises) there are Hilbert's Nullstellensatz and Artin's theorem on the representation of positive forms as sums of squares.

There are appendices on the axiomatic method, on set theoretic (semantic) foundations, on combinatorial (syntactic) foundations. They alternate between philosophical arguments and detailed mathematical reasoning.

The absence of bibliography and index is made up by an informative summary at the beginning of each chapter.

The reviewer found reading this book quite stimulating and feels that a detailed study of it would be most profitable.

Mathematical logic by J.R. Shoenfield. Addison-Wesley, Reading, Mass., 1967. vii + 344. U.S. $12.75.

This book is a remarkable production indeed. Not only does it contain almost everything one might conceivably wish to include in a treatise on logic, in spite of a modest disclaimer in the Preface, but it is extremely well written, neither too wordy nor too concise. Open the book on any page, and you will be able to read, with no strange symbolism jarring the eye. To prove this, I shall review the book backwards.

There is an index but, unfortunately, no bibliography.

The appendix contains a complete proof of Novikoff's Theorem: there is a finitely presented group which has an unsolvable word problem. This is the shortest such proof I have seen; it does not depend on the corresponding problem for semigroups, nor on the halting problem for Turing machines. Of course, it must depend on the existence of a recursively enumerable set which is not recursive. Indeed, let A be such a set, $G = [a, b]$ the free group in two generators. Form the amalgamated product of G with itself by equating the elements $a^n b a^{-n}$ of the two copies of G for each $n \in A$. This group has an unsolvable word problem and may be embedded into a finitely presented group using a theorem of Higman. (Here the absence of a bibliography proves embarrassing, as there are two Higmans in group theory.)

Chapter 9 presents Zermelo-Fraenkel set theory quite thoroughly from scratch and, assuming that this is consistent, the famous theorems by Gödel and and Cohen: the continuum hypothesis can neither be proved nor disproved. Gödel's proof depends on the model of "constructible" sets, and Cohen's proof, which has been much simplified here, on the idea of "forcing". It is shown that the continuum hypothesis cannot be settled even if it is assumed that there exist measurable cardinals in the sense of Ulam.

In Chapter 8 we find Gödel's famous observation that the formula of Peano arithmetic P which says that P is consistent is not a theorem of P, assuming of course that P is consistent. Nonetheless, there exist finitary consistency proofs of P, and one such proof, also due to Gödel, is given here. This chapter also contains a theorem by Kreisel which says roughly that a statement in P is a theorem if and only if there are no "counterexamples". There is also a discussion of second-order arithmetic.

Chapter 7 deals with advanced recursion theory, quite beyond what one finds in the books by Kleene, Peter and Grzegorczyk. While it begins with partial recursive functions, it soon proceeds to recursive functionals, the arithmetic hierarchy, Post's theorem, the Friedberg-Muchnik theorem, and many other things which are new to this reviewer.

Chapter 6 gives an amazingly brief description of the classical theory of recursive functions, using it to obtain the Gödel-Rosser theorem that if formalized arithmetic is consistent, then it is not complete.

Chapter 5 deals with model theory. The compactness theorem exploits the fact that only a finite number of axioms can enter any one proof. There are also many other theorems, only some of them bearing familiar names such as Löwenheim-Skolem, Craig interpolation theorem, etc., several about the concept of "categoricity". (A theory is categorical if any two models are isomorphic.)

In Chapter 4 we find Gödel's completeness theorem which says that a theory is consistent if and only if it has a model, the proof following Henkin; also theorems by Hilbert-Aikermann and Herbrand.

Finally, the first three chapters serve to introduce first order languages, which here may have function symbols as well as predicates, and must have a sign for equality. There is the usual discussion of tautologies, quantification, the deduction theorem, and prenex form.

There are many problems, with generous hints, which aim to extend the methods of the text and to present further results.

It is difficult to restrain one's enthusiasm for this book, which will surely serve not only graduate students, but also mature mathematicians for many years to come.

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