AXIOMATIZABILITY OF PROPOSITIONALLY QUANTIFIED MODAL LOGICS ON RELATIONAL FRAMES

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Abstract. Propositional modal logic over relational frames is naturally extended with propositional quantifiers by letting them range over arbitrary sets of worlds of the relevant frame. This is also known as second-order propositional modal logic. The propositionally quantified modal logic of a class of relational frames is often not axiomatizable, although there are known exceptions, most notably the case of frames validating the strong modal logic S5. Here, we develop new general methods with which many of the open questions in this area can be answered. We illustrate the usefulness of these methods by applying them to a range of examples, which provide a detailed picture of which normal modal logics define classes of relational frames whose propositionally quantified modal logic is axiomatizable. We also apply these methods to establish new results in the multimodal case.

§1. Introduction. The possible world structures of Kripke [32, 33] provide a very useful model theory for modal logics. Many propositional modal logics which arise naturally in alethic, temporal, epistemic, and deontic contexts can be shown to be sound and complete with respect to a class of relational (Kripke) frames. In many cases, this extends to multimodal logics, which include several modal operators, each of which is interpreted using its own accessibility relation on the set of worlds (points) of the frame. (The notions mentioned informally in this introductory section will be defined properly in Section 2.) While propositional logics allow the expression of many general intensional principles, these principles are in a sense all universal conditions. For example, it is possible to postulate axiomatically that generally, $\Box p$ only if p. Depending on the interpretation, this principle might say, e.g., that what is necessary is true, or that what is known is true. But it is impossible to postulate that this implication cannot be reversed, i.e., that there are cases in which p is true without $\Box p$ being true. Yet, it is equally natural to postulate that not every truth is known or necessary.

A simple extension of a given propositional modal language allows for a straightforward regimentation of the example just given. This is the extension by quantifiers $\forall$ and $\exists$ binding the atomic proposition letters $p, q, r, \dots$, which may then be conceived of as propositional *variables*. With these quantifiers, the relevant claim can be formulated as $\exists p(p \wedge \neg \Box p)$. Such extensions of modal logic were

already considered in the earliest works on symbolic modal logic, by Lewis and Langford [36], and their versatility in philosophical applications is evidenced in the work of Prior [43–47]. Propositional quantifiers are also straightforward to interpret on relational frames: when interpreting propositional formulas on such frames, each proposition letter is assigned a set of worlds—the set of worlds in which it is true. Thinking of propositional quantifiers as ranging over all possible interpretations of the variable which they bind, they are therefore straightforwardly interpreted as ranging over all sets of worlds. Such an interpretation of propositional quantifiers was already proposed by Kripke [32].

It is natural to ask whether well-known completeness results of propositional modal logics with respect to classes of relational frames are preserved under the addition of propositional quantifiers. Many important results on this question were established around 1970, in particular by Bull [7], Kaplan [27], and Fine [12]. To state their results, let $\Lambda\pi+$, for any normal modal logic Λ, be the propositionally quantified modal logic of Λ-frames, i.e., the set of formulas in the propositionally quantified modal language which are valid on the class of frames on which every theorem of Λ is valid. The results of Bull, Kaplan, and Fine show that $S5\pi+$ is axiomatizable; indeed, they show that $S5\pi+$ is decidable, and provide a natural and independent axiomatization. Fine shows that this cannot be extended to a number of common modal logics which are weaker than S5, as $K\pi+$, $T\pi+$, $K4\pi+$, $S4\pi+$, $S4.2\pi+$, and $B\pi+$ are all not axiomatizable.

A few further results on the axiomatizability of propositionally quantified modal logics interpreted over relational frames have been obtained since then. Kremer [30] notes that Fine's unaxiomatizability results can be strengthened to show that the relevant logics are not just not axiomatizable, but in fact recursively isomorphic to second-order logic (understood as the validities of a second-order predicate language on standard models). A proof of this result for S4.2 was presented by Kaminski and Tiomkin [26], which extends to any modal logic Λ weaker than S4.2. More recently, Ding [10] has extended the decidability result from the class of frames validating S5 to those validating KD4E (aka KD45). Multimodal logics have been considered as well: Antonelli and Thomason [1], Kuhn [34], and Belardinelli et al. [2] all show that the propositionally quantified bimodal logic on relational frames in which both modalities obey the principles of S5 is recursively isomorphic to second-order logic, and Fritz [16] extends this to the corresponding class of product frames, which validate certain interaction principles between the two modalities.

While these existing axiomatizability results cover many of the most well-known modal logics, the findings are at present somewhat unsystematic. This paper aims to improve this situation, by developing general methods by which a more complete picture can be obtained. First, as Ding [10, p. 1148] notes, it appears from the known results as if there is an "axiomatizability boundary" which divides normal modal logics according to strength, with a normal modal logic Λ being above the boundary if and only if $\Lambda\pi+$ is axiomatizable. We show that such a boundary does indeed exist among a wide class of normal modal logics, as in these cases, the complexity of $\Lambda\pi+$ decreases monotonically with the strength of Λ. We note that the result does not extend to all normal modal logics.

The current understanding is especially limited when it comes to axiomatizable logics: only two non-trivial normal modal logics—KD4E and S5—have been

$\mathcal{L}^{\square}$

Triv — Ver — KAlt$_1$

S5 — KAlt$_2$

K1.2 K3.1 KD4E K4.3W KAlt$_n$

S4.3.1 KE K4.3Z

K2.1 K3 B

K1.1 K2 S4.3 K4.2W

S4M S4.2 K4.2Z KW
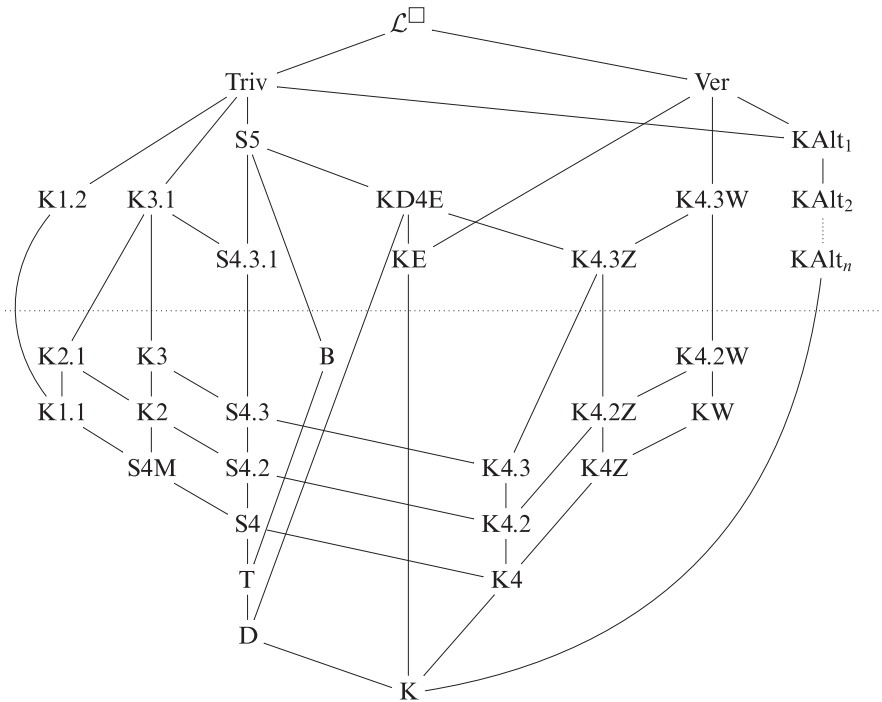
S4 K4.3 K4Z

T K4.2

D K4

K

FIGURE 1. The axiomatizability boundary (dotted) among some finitely axiomatized normal modal logics.

identified as defining classes of frames whose propositionally quantified modal logics are axiomatizable (indeed, decidable). We will develop a new technique for showing the decidability of $\Lambda\pi+$ which can be applied to many more strong normal modal logics Λ. Most notably, this includes KE and every one of its normal modal extension. We also simplify the construction presented in Kaminski and Tiomkin [26] which makes it applicable to a wider range of normal modal logics. With these two general methods, we are able to trace the axiomatizability boundary in fine detail both from above and from below. Figure 1 summarizes many of the central results: It presents a number of normal modal logics, ordered according to strength (inclusion). The figure indicates the axiomatizability boundary using a dotted line: for any logic Λ included in the diagram above this line, $\Lambda\pi+$ is axiomatizable (indeed, decidable), and for any logic Λ included in the diagram below this line, $\Lambda\pi+$ is not axiomatizable (and in most cases recursively isomorphic to second-order logic). The various modal logics included in Figure 1 will be defined below; see Figure 2. For reasons of uniformity, we follow the notational conventions for modal axioms and logics of Hughes and Cresswell [25, pp. 359–368] throughout, from which the arrangement of logics in Figure 1 is adapted as well. (The definition of K3 is missing from Hughes and Cresswell [25], but it can be found in Hughes and Cresswell [23, Appendix 3]. We have also changed some inclusion relations from the diagram of Hughes and Cresswell [25] to correct what appear to be oversights.)

Ver: $\Box p$ **G_0**: $\Diamond(p \wedge \Box q) \to \Box(p \vee \Diamond q)$
Triv: $p \leftrightarrow \Box p$ **H1**: $p \to \Box(\Diamond p \to p)$
D: $\Box p \to \Diamond p$ **W (G, L)**: $\Box(\Box p \to p) \to \Box p$
T: $\Box p \to p$ **Z**: $\Box(\Box p \to p) \to (\Diamond \Box p \to \Box p)$
4: $\Box p \to \Box \Box p$ **J1 (Grz)**: $\Box(\Box(p \to \Box p) \to p) \to p$
B: $p \to \Box \Diamond p$ **N1 (Dum)**: $\Box(\Box(p \to \Box p) \to p) \to (\Diamond \Box p \to p)$
E (5): $\Diamond p \to \Box \Diamond p$ **D1 (Lem)**: $\Box(\Box p \to q) \vee \Box(\Box q \to p)$
M: $\Box \Diamond p \to \Diamond \Box p$ **Lem$_0$**: $\Box((p \wedge \Box p) \to q) \vee \Box((q \wedge \Box q) \to p)$
G1: $\Diamond \Box p \to \Box \Diamond p$ **Alt$_n$** $(n \in \mathbb{N})$: $\bigvee_{i=0}^{n} \Box(\bigwedge_{j=0}^{i} p_j \to p_{i+1})$

D: K + **D**	B: T + **B**	KD4E: D + **4** + **E**	Triv: K + **Triv**
T: K + **T**	KE: K + **E**	S5: T + **E**	Ver: K + **Ver**
S4: T + **4**	K4: K + **4**	K4Z: K4 + **Z**	KW (GL): K + **W**
S4.2: S4 + **G1**	K4.2: K4 + **G$_0$**	K4.2Z: K4.2 + **Z**	K4.2W: K4.2 + **W**
S4.3: S4 + **D1**	K4.3: K4 + **Lem$_0$**	K4.3Z: K4.3 + **Z**	K4.3W: K4.3 + **W**
S4.3.1: S4.3 + **N1**	K1.1: S4 + **J1**	K2: S4.2 + **M**	K3: S4.3 + **M**
S4M: S4 + **M**	K1.2: S4M + **H1**	K2.1: K2 + **J1**	K3.1: S4.3 + **J1**
KAlt$_n$: K + **Alt$_n$**			

FIGURE 2. Propositional unimodal axioms and logics.

Figure 1 only summarizes the results of this paper for a selection of unimodal logics discussed in the literature. The techniques developed and improved here are applicable more widely, and we establish results which are in several ways stronger than what is conveyed in the figure. For example, in many cases, establishing the unaxiomatizability of a given logic extends to all of its weakenings, and in some cases establishing the axiomatizability of a given logic extends to all of its strengthenings. The techniques used to establish these results are also applicable to multimodal logics. This is especially interesting in the case of the decidability results, as there are at present no decidability results for propositionally quantified multimodal logics. As an example, we will show the decidability of the propositionally quantified modal logic of bimodal relational frames in which both accessibility relations are equivalence relations, one of which is a refinement of the other.

The paper is organized as follows: Section 2 defines the basic concepts of propositionally quantified modal logics and relational frames. Section 3 develops a new method for establishing the decidability of $\Lambda\pi+$, and applies it to a range of unimodal examples. Section 4 presents the simplified construction for establishing that $\Lambda\pi+$ is recursively isomorphic to second-order logic, and applies it to a range of unimodal examples. Section 5 considers the special case of extensions of K4.3, which requires different methods. Section 6 considers multimodal logics. The final Section 7 concludes with some open problems.

§2. Basic concepts. This section sets up more rigorously the questions to be answered below and some common logical concepts to be used in developing the answers. We recall only briefly standard definitions of normal modal logics and relational frames; for fuller developments, see a standard textbook such as [25]. For the required recursion-theoretic notions, see [4, Appendix C].

2.1. Normal modal logics and relational frames. We start with the syntax of the propositionally quantified languages. In order to accommodate multimodal system, we define the language relative to a finite or countably infinite set of operators O. The relativity to this set of operators will mostly be assumed tacitly.

DEFINITION 2.1. For any finite or countably infinite set O, $\mathcal{L}^O$ is the language whose formulas are specified in Backus–Naur form (BNF) as follows:

$$\varphi ::= p \mid \bot \mid (\varphi \rightarrow \varphi) \mid \Box\varphi \mid \forall p\varphi,$$

where p indicates any member of a countably infinite set of propositional variables Φ, and $\Box$ indicates any member of O. $\mathcal{L}^O_{\mathrm{qf}}$ is the set of quantifier-free formulas of $\mathcal{L}^O$.

As usual, we treat other Boolean operators like $\neg$, $\wedge$, $\vee$, $\leftrightarrow$, and $\top$ as syntactic abbreviations; e.g., we use $\neg\varphi$ to abbreviate $\varphi \rightarrow \bot$, and $\top$ to abbreviate $\neg\bot$. Similarly, $\Diamond$ and $\exists p$ abbreviate the duals of $\Box$ and $\forall p$, respectively. We use $\bigwedge$ and $\bigvee$ for finite conjunctions and disjunctions, writing, e.g., $\bigwedge_{i=1}^{n} p_i$ for $p_1 \wedge \cdots \wedge p_n$. We also introduce some less familiar abbreviations: For any formula φ, $\bar{\forall}\varphi$ abbreviates $\forall p_1 \ldots \forall p_n\varphi$, where $p_1, \ldots, p_n$ are the free variables in φ. Further, if O is finite, then for any $n \in \mathbb{N}$ and formula φ, we define $O^n\varphi$ and $O^{\leq n}\varphi$ using the following induction:

$$
\begin{aligned}
O^0\varphi &:= \varphi, & O^{\leq 0}\varphi &:= \varphi, \\
O^{n+1}\varphi &:= \bigwedge_{\Box \in O} \Box O^n\varphi, & O^{\leq n+1}\varphi &:= \varphi \wedge \bigwedge_{\Box \in O} \Box O^{\leq n}\varphi.
\end{aligned}
$$

We further shorten $\{\Box\}^n\varphi$ and $\{\Box\}^{\leq n}\varphi$ to $\Box^n\varphi$ and $\Box^{\leq n}\varphi$, respectively.

Normal modal logics can be understood as sets of quantifier-free formulas containing certain axioms and closed under certain rules. In order to state one of these rules, let $\varphi[\psi/p]$ be the result of replacing every occurrence of the propositional variable p in the formula φ by the formula ψ. In cases in which φ contains quantifiers, this is only defined if ψ is free for p in φ, i.e., if every free occurrence of every variable q in ψ remains free in $\varphi[\psi/p]$.

DEFINITION 2.2. A *normal modal logic* is a subset of $\mathcal{L}^O_{\mathrm{qf}}$ which, for every modality $\Box \in O$, contains every classical tautology and the distributivity axiom $\mathrm{K}^\Box$: $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$, and is closed under the rules of modus MP: $\varphi, \varphi \rightarrow \psi/\psi$, necessitation Nec: $\varphi/\Box\varphi$, and uniform substitution US: $\varphi/\varphi[\psi/p]$.

K is the smallest normal modal logic. If Λ is a normal modal logic and $\Gamma \subseteq \mathcal{L}^O_{\mathrm{qf}}$, then $\Lambda + \Gamma$ is the smallest normal modal logic including both Λ and Γ. If $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$, we also write this logic as $\Lambda + \gamma_1 + \cdots + \gamma_n$.

This definition presupposes that there is always a unique smallest normal modal logic including a given set. This claim is easily seen to be witnessed by the intersection of all normal modal logics including the relevant set. Having defined the general notion of a normal modal logic, we can now define the various unimodal logics included in Figure 1. They will be discussed in more detail below, but for ease of reference, we present them all together in Figure 2. To forestall a likely source of confusion, note that whereas K4 is the extension of K by the axiom **4**, the names for K1.1/2, K2(.1), and K3.1, which go back to Sobociński [55], have no such origin. Instead, they appear to derive merely from an enumeration of certain extensions of S4M, which Sobociński labels K1.

We say that a normal modal logic is *axiomatizable* if it is a recursively enumerable set. In fact, we will call any set of formulas of any of the languages considered here *axiomatizable* just in case it is recursively enumerable.

Turning from the syntactic to the model-theoretic side, we define the well-known concept of a relational frame, and the evaluation of a formula relative to a world of the frame and an assignment function mapping each propositional variable to the set of worlds in which it is to be evaluated as true. The usual concept of a relational (Kripke) model can be understood as consisting of a relational frame and an assignment function. We make use of two items of notation to state the clauses for modal operators and quantifiers, respectively: First, we write $R[w]$ for the set of elements R-accessible from w, i.e., $\{v : wRv\}$. For later uses, we also extend this notation to any set x, writing $R[x]$ for $\{v : wRv \ \& \ w \in x\}$. Second, we write $a[x/p]$ for the function mapping p to x and every other $q \in \Phi$ to $a(q)$.

DEFINITION 2.3.   A *relational frame (for O)* is a structure $\mathfrak{F} = \langle W, R_\square \rangle_{\square \in O}$ such that W is a set, and for every $\square \in O$, $R_\square$ is a binary relation on W. For every $w \in W$ and *assignment function* $a : \Phi \to \mathcal{P}(W)$, φ being *true* relative to w and a is recursively defined as follows:

$\mathfrak{F}, w, a \Vdash p$ iff $w \in a(p)$;

$\mathfrak{F}, w, a \nVdash \bot$;

$\mathfrak{F}, w, a \Vdash \varphi \to \psi$ iff $\mathfrak{F}, w, a \Vdash \varphi$ only if $\mathfrak{F}, w, a \Vdash \psi$;

$\mathfrak{F}, w, a \Vdash \square\varphi$ iff $\mathfrak{F}, v, a \Vdash \varphi$ for all $v \in R_\square[w]$;

$\mathfrak{F}, w, a \Vdash \forall p\varphi$ iff $\mathfrak{F}, w, a[x/p] \Vdash \varphi$ for all $x \subseteq W$;

φ is *valid in w*, written $\mathfrak{F}, w \Vdash \varphi$, if $\mathfrak{F}, w, a \Vdash \varphi$ for every assignment function a. φ is *valid in* $\mathfrak{F}$, written $\mathfrak{F} \Vdash \varphi$, if $\mathfrak{F}, w \Vdash \varphi$ for every $w \in W$. φ is *valid on* a class of relational frames C, written $\mathsf{C} \Vdash \varphi$, if $\mathfrak{F} \Vdash \varphi$ for every $\mathfrak{F}$ in C.

In the unimodal case, we omit the index $\square$, and write $\langle W, R \rangle$ for a frame. The different notions of validity are extended to sets of formulas: a set of formulas is valid just in case all of its members are valid in the relevant sense. Satisfiability is dual to validity: a formula is satisfiable (in a world, in a frame, or on a class) if its negation is not valid (in the same sense). We consider the *size* of a frame $\langle W, R_\square \rangle_{\square \in O}$ to be $|W|$, the cardinality of the set of worlds.

The notion of being valid in a relational frame gives rise to two crucial notions connecting sets of formulas and classes of relational frames: for any set of formulas Γ, there is the class of relational frames defined by Γ, written $\mathsf{R}(\Gamma)$, and for any class of relational frames C, there is the propositionally quantified modal logic of C, written $\mathcal{L}(\mathsf{C})$. With this, the propositionally quantified extension $\Lambda\pi+$ of a given normal modal logic Λ can be defined:

DEFINITION 2.4.   Let $\Gamma \subseteq \mathcal{L}^O$, C a class of relational frames, and Λ a normal modal logic. Then:

$\mathsf{R}(\Gamma)$ is the class of relational frames $\mathfrak{F}$ such that $\mathfrak{F} \Vdash \gamma$ for all $\gamma \in \Gamma$.

$\mathcal{L}(\mathsf{C}) := \{\varphi \in \mathcal{L}^O : \mathsf{C} \Vdash \varphi\}$.

$\Lambda\pi+ := \mathcal{L}(\mathsf{R}(\Lambda))$.

The validities on any relational frame contain the axioms used in the definition of normal modal logics, and are closed under the rules used in this definition. Consequently, for any $\Gamma \subseteq \mathcal{L}_{qf}^O$, $R(K\Gamma)$ is $R(\Gamma)$, and $\mathcal{L}(R(\Gamma))$ is $K\Gamma\pi+$. It is also worth noting that $\Lambda\pi+$ is consistent just in case Λ is consistent; this follows from Makinson's [38] result that every consistent normal modal logic is valid in some one-element frame.

2.2. Generated subframes. The addition of propositional quantifiers preserves an important feature of modal languages interpreted on relational frames: the truth of a formula φ in a world w is dependent only on the subframe consisting of those worlds which are accessible from w by a finite sequence of steps along the accessibility relations. More carefully, if φ contains at most n nested modal operators, only those worlds accessible by n steps along the accessibility relations are relevant. This fact will be appealed to repeatedly in the following, so we introduce the necessary definitions and results, even though they are essentially unchanged from the familiar quantifier-free setting; compare Blackburn et al. [4, p. 140].

First, we introduce the notion of the modal depth of a formula φ as the maximal number of nested modal operators in φ. Alongside this, we introduce the corresponding notion of *quantifier depth* for quantifiers.

DEFINITION 2.5. md and qd are the functions from $\mathcal{L}^O$ to $\mathbb{N}$ defined using the recursive clauses specified by the following table:

φ	$\mathrm{md}(\varphi)$	$\mathrm{qd}(\varphi)$
p	0	0
$\bot$	0	0
$\varphi \rightarrow \psi$	$\max(\mathrm{md}(\varphi), \mathrm{md}(\psi))$	$\max(\mathrm{qd}(\varphi), \mathrm{qd}(\psi))$
$\forall p\varphi$	$\mathrm{md}(\varphi)$	$\mathrm{qd}(\varphi) + 1$
$\Box\varphi$	$\mathrm{md}(\varphi) + 1$	$\mathrm{qd}(\varphi)$

$\mathrm{md}(\varphi)$ is the *modal depth of* φ; $\mathrm{qd}(\varphi)$ is the *quantifier depth of* φ.

Next, we define the notion of an ($(n$-step) generated) subframe. To do so, we restrict any binary relation R to a set x by writing $R|x$ for $\{\langle w, v \rangle : wRv \,\&\, w, v \in x\}$. Later on, we similarly write $f|x$ for the restriction of a function f to a subset x of its domain.

DEFINITION 2.6. Let $\mathfrak{F} = \langle W, R_\Box \rangle_{\Box \in O}$ be a relational frame. $\mathfrak{F}' = \langle W', R'_\Box \rangle_{\Box \in O}$ is a *subframe of* $\mathfrak{F}$ if $W' \subseteq W$ and $R'_\Box = R_\Box|W'$ for all $\Box \in O$. For any $x \subseteq W$ and $n \in \mathbb{N}$,

$$W_x^0 := x,$$
$$W_x^{n+1} := W_x^n \cup \bigcup\nolimits_{\Box \in O} R_\Box[W_x^n], \quad \mathfrak{F}_x^n := \langle W_x^n, R_\Box|W_x^n \rangle_{\Box \in O},$$
$$W_x := \bigcup\nolimits_{n \in \mathbb{N}} W_x^n, \quad\quad\quad\quad \mathfrak{F}_x := \langle W_x, R_\Box|W_x \rangle_{\Box \in O}.$$

$\mathfrak{F}_x^{(n)}$ is the (n-step) subframe of $\mathfrak{F}$ generated by x. If x is a singleton $\{w\}$, we call the subframes *point-generated*, and omit set brackets, writing $\mathfrak{F}_w^{(n)}$ for $\mathfrak{F}_{\{w\}}^{(n)}$.

We can now show that a formula φ is valid in a world w of a frame just in case it is valid in w in a subframe generated by a set containing w, as well as in the corresponding n-step generated subframes, as long as n is at least as great as the modal depth of φ.

PROPOSITION 2.7. *Let $\mathfrak{F}$ be a relational frame, x a set of worlds, $w \in x$, $\varphi \in \mathcal{L}^O$, and $n \geq \mathrm{md}(\varphi)$. Then,*

$$\mathfrak{F}, w \Vdash \varphi \quad \textit{iff} \quad \mathfrak{F}_x, w \Vdash \varphi \quad \textit{iff} \quad \mathfrak{F}_x^n, w \Vdash \varphi.$$

PROOF. Routine, by an induction on the complexity of φ relative to corresponding assignment functions. ⊣

2.3. The axiomatizability boundary. Using generated subframes, we can now establish the (partial) axiomatizability boundary mentioned above. The crucial observation is that among certain sets of formulas $\Gamma \subseteq \mathcal{L}^O$, the complexity of $\mathcal{L}(\mathsf{R}(\Gamma))$ decreases monotonically with the strength of Γ. That is, if $\Gamma \subseteq \Delta$, then $\mathcal{L}(\mathsf{R}(\Delta))$ is reducible to $\mathcal{L}(\mathsf{R}(\Gamma))$ in the sense that there is a computable function $f : \mathcal{L}^O \to \mathcal{L}^O$ such that $\varphi \in \mathcal{L}(\mathsf{R}(\Delta))$ iff $f(\varphi) \in \mathcal{L}(\mathsf{R}(\Gamma))$. We will establish such monotonicity for sets which satisfy two requirements: First, both O and $\Delta \backslash \Gamma$ are finite. Second, $O^{\leq n} p \to O^{n+1} p \in \Delta$ for some natural number n.

PROPOSITION 2.8. *If $\Gamma \subseteq \Delta \subseteq \mathcal{L}^O$, both O and $\Delta \backslash \Gamma$ are finite, and $O^{\leq n} p \to O^{n+1} p \in \Delta$ for some $n \in \mathbb{N}$, then $\mathcal{L}(\mathsf{R}(\Delta))$ is reducible to $\mathcal{L}(\mathsf{R}(\Gamma))$.*

PROOF. The required reduction can be defined as mapping every $\varphi \in \mathcal{L}^O$ to

$$\varphi^* := \bar{\forall}(O^{\leq n} \bigwedge \Delta \backslash \Gamma \to \varphi).$$

We show that $\varphi \in \mathcal{L}(\mathsf{R}(\Delta))$ iff $\varphi^* \in \mathcal{L}(\mathsf{R}(\Gamma))$.

The "if" direction is immediate. So assume that $\varphi^* \notin \mathcal{L}(\mathsf{R}(\Gamma))$. Then there is a frame $\mathfrak{F}$ in $\mathsf{R}(\Gamma)$ and w such that $\mathfrak{F}, w \nVdash \varphi^*$. Letting $\mathfrak{F}_w = \langle W, R \rangle$, note that $\mathfrak{F}_w = \mathfrak{F}_W$. Thus by Proposition 2.7, $\mathfrak{F}_w$ is in $\mathsf{R}(\Gamma)$ as well. By Proposition 2.7, it also follows that $\mathfrak{F}_w, w \nVdash \varphi^*$. So, first, $\mathfrak{F}_w, w \Vdash O^{\leq n} \bigwedge \Delta \backslash \Gamma$, whence with $O^{\leq n} p \to O^{n+1} p \in \Delta$, it follows that $\mathfrak{F}_w \Vdash \bigwedge \Delta \backslash \Gamma$, and so that $\mathfrak{F}_w \in \mathsf{R}(\Delta)$. Second, $\mathfrak{F}_w, w \nVdash \varphi$, whence $\varphi \notin \mathcal{L}(\mathsf{R}(\Delta))$. ⊣

From this proposition, we obtain the partial axiomatizability boundary among unimodal logics mentioned above: If Λ is a normal unimodal logic and Γ is a finite set of $\mathcal{L}_{\mathrm{qf}}^{\Box}$-formulas such that $\Box^{\leq n} p \to \Box^{n+1} p \in \Lambda + \Gamma$, then $\Lambda \pi+$ is recursively enumerable (decidable) only if $(\Lambda + \Gamma)\pi+$ is recursively enumerable (decidable) as well. This includes the majority of logics included in Figure 1; note in particular that every extension of K4 contains $\Box p \to \Box\Box p$, and every extension of KE contains $\Box\Box p \to \Box\Box\Box p$, as is easy to see using the completeness of KE with respect to relational frames with a Euclidean accessibility relation.

It is natural to ask whether the two constraints on sets of formulas in Proposition 2.8 are essential. We will later see that the requirement of finiteness of $\Delta \backslash \Gamma$ cannot be dropped completely, or weakened to recursively enumerable sets of axioms, even among sets of quantifier-free formulas; see Proposition 5.9. We leave open the question whether the requirement of finiteness can be weakened

to decidability, as well as the question whether the assumption of containing $O^{\leq n}p \to O^{n+1}p \in \Delta$ for some $n \in \mathbb{N}$ can be dropped or weakened.

2.4. Second-order logic. Most of the complexity results to be established proceed by finding a reduction between $\mathcal{L}^O$ over a class of relational frames and a suitable form of second-order logic over such frames. We assume familiarity with second-order logic and its standard semantics; for a detailed exposition, see [53].

The most basic case is that of monadic second-order logic. Given a countably infinite set of first-order variables Ψ and a disjoint set of second-order variables Ω, the language $\mathcal{M}^O$ of monadic second-order logic (over O) is defined in BNF as follows:

$$\varphi ::= Xx \mid R_\square xx \mid \bot \mid (\varphi \to \varphi) \mid \forall x \varphi \mid \forall X \varphi,$$

where $x \in \Psi$ and $X \in \Omega$. Formulas of φ can straightforwardly be interpreted over a relational frame $\mathfrak{F}$, with individual quantifiers ranging over worlds, and monadic second-order quantifiers ranging over sets of worlds. We adapt the notation used above for $\mathcal{L}^O$, writing $\mathfrak{F}, a \Vdash \varphi$ for the truth of φ in $\mathfrak{F}$ relative to an assignment function a (on $\Psi \cup \Omega$), and $\mathfrak{F} \Vdash \varphi$ for φ being valid on $\mathfrak{F}$, i.e., true relative to every assignment function. Similarly, for a set $\Gamma \subseteq \mathcal{M}^O$, we write $\mathsf{R}(\Gamma)$ for the class of relational frames validating every member of Γ, and $\mathcal{M}(\mathsf{C})$ for the set of $\mathcal{M}^O$-formulas valid on every frame in C. The analogous conventions will be adopted for the other versions of second-order logic discussed below; we won't remark on this separately.

Monadic second-order logic arises naturally when one extends the well-known standard translation of propositional modal logic to propositional quantifiers. For a discussion of the standard translation in the quantifier-free setting, see [4, Section 2.4]. The standard translation maps every propositional modal formula φ to a corresponding first-order formula φ^x in which no first-order variable other than x is free. This is can be extended to propositional quantifiers by translating them as monadic second-order quantifiers. Assuming that for every propositional variable p, there is a distinct monadic second-order variable X_p, this leads to the recursive mapping from $\mathcal{L}^O$ to $\mathcal{M}^O$ whose only non-trivial clauses are the following:

$$p^x := X_p x,$$
$$(\forall p \varphi)^x := \forall X_p(\varphi^x),$$
$$(\square \varphi)^x := \forall y (R_\square xy \to \varphi^y).$$

As in the case without propositional quantifiers, a formula $\varphi \in \mathcal{L}^O$ is true in a world of a frame, relative to an assignment function, just in case φ^x is true in that frame relative to a corresponding assignment function which interprets the first-order variable x as w. We can then state the fundamental result on the standard translation as follows:

PROPOSITION 2.9. *Let $\mathfrak{F}$ be a relational frame, w a world, a an assignment function on propositional variables, and b an assignment function on first- and monadic second-order variables such that $b(X_p) = a(p)$ for all $i \in \mathbb{N}$. Then,*

$$\mathfrak{F}, w, a \Vdash \varphi \quad \textit{iff} \quad \mathfrak{F}, b[w/x] \Vdash \varphi^x.$$

PROOF. By a routine induction on the complexity of φ. ⊣

The standard translation brings out that propositionally quantified modal logic interpreted over relational frames can be thought of as a fragment of monadic second-order logic with binary relational constants. This is why it is sometimes called *second-order propositional modal logic*, e.g., by Kaminski and Tiomkin [26]. In fact, already the study of propositional modal logic without propositional quantifiers over classes of relational frames involves a monadic second-order element: A quantifier-free formula φ is valid in a world of a frame just in case its universal closure $\bar{\forall}\varphi$ is true in that world (independently of the assignment function). Because of this implicit second-order aspect, various sets $\Gamma \subseteq \mathcal{L}_{\mathrm{qf}}^{\square}$, such as the singletons of **M** and **W**, define classes of relational frames $\mathsf{R}(\Gamma)$ which are not first-order definable. This aspect of propositional modal logic is further discussed by Hughes and Cresswell [25, pp. 188–189] and Blackburn et al. [4, Section 3.2].

An immediate consequence of Proposition 2.9 is that for any class of relational frames C, $\mathcal{L}(\mathsf{C})$ is reducible to $\mathcal{M}(\mathsf{C})$. Thus, the decidability of the former follows from the decidability of the latter. This connection will form the basis of many of the decidability results to be presented. (This is a common approach to decidability results in modal logic; see [4, Section 6.3] for examples in the quantifier-free setting, and [58] for an example including propositional quantifiers.) In some cases, we can simply appeal to known results on the decidability of monadic second-order logic over classes of relational frames (see [19] for an overview of results in this area). In other cases, we will establish some new limitative results concerning monadic second-order logic.

The new limitative results can be simplified by using an obvious notational variant of monadic second-order logic which dispenses with first-order quantifiers. First-order quantifiers can be simulated using monadic second-order quantification over singleton sets if we admit two new kinds of atomic formulas. (See, e.g., [19, p. 482] for a similar variant; the present version is chosen so as to harmonize with $\mathcal{L}^O$, but the differences are immaterial.) The variant monadic second-order language $\mathcal{N}^O$ is defined in BNF as follows:

$$\varphi ::= X \subseteq X \mid R_\square XX \mid \bot \mid (\varphi \to \varphi) \mid \forall X\varphi,$$

where $X \in \Omega$. The truth-conditions for the new (non-standard) atomic formulas are as follows:

$\mathfrak{F}, a \Vdash X \subseteq Y$ iff $a(X) \subseteq a(Y)$,

$\mathfrak{F}, a \Vdash R_\square XY$ iff for all $x \in a(X)$ there is some $y \in a(Y)$ such that $xR_\square y$.

$\mathcal{M}^O$ and $\mathcal{N}^O$ are intertranslatable: Assume that for every variable $\xi \in \Psi \cup \Omega$, X_ξ is a distinct variable of Ω, and consider the recursive mappings $v : \mathcal{M}^O \to \mathcal{N}^O$ and $\mu : \mathcal{N}^O \to \mathcal{M}^O$ whose only non-trivial clauses are the following:

$v(Xx) := X_x \subseteq X_X,$ $\mu(X \subseteq Y) := \forall x(Xx \to Yx),$

$v(R_\square xy) := R_\square X_x X_y,$ $\mu(R_\square XY) := \forall x(Xx \to \exists y(Yy \land R_\square xy)),$

$v(\forall x\varphi) := \forall X_x(\mathrm{sing}(X_x) \to v(\varphi)),$

where $\mathrm{sing}(X)$ states that X is a singleton, which can be defined as follows:

$$\mathrm{empty}(X) := \forall Y(X \subseteq Y),$$
$$\mathrm{sing}(X) := \neg\mathrm{empty}(X) \wedge \forall Y(Y \subseteq X \rightarrow \mathrm{empty}(Y) \vee X \subseteq Y).$$

It is easy to see that along a result analogous to Proposition 2.9, $\mathcal{M}(\mathsf{C})$ is recursively isomorphic to $\mathcal{N}(\mathsf{C})$, for any class of relational frames. For present purposes, we can therefore treat $\mathcal{M}^O$ and $\mathcal{N}^O$ as notational variants of monadic second-order logic, and so interchangeable.

Monadic second-order logic will also play a role in establishing unaxiomatizability results. For this, we now note that on certain classes of relational frames, the standard translation can be reversed, and monadic second-order logic can conversely be thought of as a fragment of propositionally quantified modal logic. We establish this for point-generated unimodal frames with a transitive accessibility relation. The observation will be especially easy to show using the variant monadic second-order language $\mathcal{N}^\Box$. We call the relevant function the "backwards translation." Assuming that for every propositional variable $p \in \Phi$, there is a distinct second-order variable $X_p \in \Omega$, it is the recursive mapping $\cdot^\leftarrow : \mathcal{N}^\Box \rightarrow \mathcal{L}^\Box$ whose only non-trivial clauses are the following:

$$(X \subseteq Y)^\leftarrow := \Box(p_X \rightarrow p_Y),$$
$$(RXY)^\leftarrow := \Box(p_X \rightarrow \Diamond p_Y),$$
$$(\forall X\varphi)^\leftarrow := \forall p_X(\varphi^\leftarrow).$$

Analogous to Proposition 2.9, we can show:

PROPOSITION 2.10.   *Let $\mathfrak{F} = \langle W, R \rangle$ be a relational frame, R be transitive, $w \in W$, and $\varphi \in \mathcal{N}^\Box$. For every second-order assignment function a on $\mathfrak{F}_w$,*

$$\mathfrak{F}_w, a \Vdash \varphi \quad \text{iff} \quad \mathfrak{F}_w, w, b \Vdash \varphi^\leftarrow,$$

where $b(p_X) = a(X)$ for all $X \in \Omega$.

PROOF.   By induction on the complexity of φ.   ⊣

Thus, if C is a class of point-generated unimodal frames with a transitive accessibility relation, then $\mathcal{N}(\mathsf{C})$, and so also $\mathcal{M}(\mathsf{C})$, is reducible to $\mathcal{L}(\mathsf{C})$. We can therefore show that $\mathcal{L}(\mathsf{C})$ is not axiomatizable by showing that $\mathcal{M}(\mathsf{C})$ is not axiomatizable. We will employ this method for certain classes of frames. For other classes of frames, we devise a more *ad hoc* translation from $\mathcal{L}(\mathsf{C})$, not to *monadic* second-order logic over C, but to *full* second-order logic (which we also simply call *second-order logic*). In its usual presentation, (full) second-order logic provides n-ary variables for all $n \in \mathbb{N}$, which are interpreted using n-ary relations. However, similar to the case of $\mathcal{M}$ and $\mathcal{N}$, second-order logic can be shown to be intertranslatable with a seemingly more restricted language, and it will greatly simplify some of the following exposition to work with this latter language. The language, which we call $\mathcal{S}$, employs only binary second-order variables, and no non-logical constants; its syntax is specified in BNF as follows:

$$\varphi ::= Xxx \mid \bot \mid (\varphi \rightarrow \varphi) \mid \forall x\varphi \mid \forall X\varphi,$$

where $x \in \Psi$ and $X \in \Omega$. (While elements of Ω serve as monadic second-order variables in $\mathcal{M}$ and $\mathcal{N}$, they serve as binary second-order variables in $\mathcal{S}$. Below, it will always be clear which of these languages is used, and so it will always be clear whether a given member of Ω is used as a monadic or as a binary variable.)

Crucially, the variant second-order language $\mathcal{S}$ is not only syntactically restricted, but also interprets binary second-order quantifiers as restricted to *symmetric* relations. To emphasize this, we state the truth-conditions of such quantifiers explicitly, for any set D (since $\mathcal{S}$ has no non-logical constants, its models are simply sets):

$$D, a \Vdash \forall X \varphi \text{ iff } D, a[S/X] \Vdash \varphi \text{ for all symmetric relations } S \subseteq D^2.$$

The more general quantifiers of second-order logic as standardly defined can be simulated using this more restricted form of binary second-order quantification, as shown by Rabin and Scott in work presented by Nerode and Shore [41]. For present purposes, we can therefore treat $\mathcal{S}$ as a notational variant of second-order logic. To emphasize the restricted interpretation of second-order quantifiers, we call it *symmetric second-order logic*, and write SSOL for the set of its validities, i.e., the formulas of $\mathcal{S}$ valid on all sets.

§3. Decidability. The main existing decidability result for propositionally quantified modal logics is the decidability of S5π+, which was established by Kaplan [27] and Fine [12]. Fine notes two ways of obtaining this result: The first uses a quantifier-elimination result for S5π+, a version of which is also established by Kaplan. This technique also plays a role in the proof of the decidability of KD4Eπ+ by Ding [10]. The other way of establishing the decidability of S5π+, discussed by both Kaplan [27] and Fine [12], proceeds by reducing S5π+ to monadic second-order logic (without non-logical constants), which is known to be decidable. The more general technique developed in this section is inspired by this second method.

3.1. Frames of finitely bounded size. With the basic concepts introduced in the previous section, the decidability of S5π+ is straightforward: R(S5) is the class of unimodal frames whose accessibility relation is an equivalence relation. The accessibility relation of any point-generated subframe of such a frame is universal, so by Proposition 2.7, S5π+ is the propositionally quantified modal logic of relational frames with a universal accessibility relation. Using the standard translation and Proposition 2.9, we can think of this logic as a fragment of monadic second-order logic over such frames. But on such frames, atomic formulas of the form Rxy are trivially true. Monadic second-order logic over such frames is therefore recursively isomorphic to monadic second-order logic without non-logical constants, models of which are simply sets. And the latter logic is well-known to be decidable; in fact, its decidability follows from results already obtained by Löwenheim [37].

It will suffice to give just a sketch of a proof of this result: For simplicity, we work with the variant monadic second-order language $\mathcal{N}$. First, we note that it suffices to show that every satisfiable monadic second-order formula without non-logical constants is satisfiable on a model whose size is bounded by some finite number which is computable in terms of the complexity of the formula: with this, the validity of a

formula φ can be determined by a finite computation, considering just the models of such bounded size, which are finite in number up to isomorphism. This finite bound on models is obtained in terms of the quantifier depth $qd(\varphi)$ of φ, where the definition of quantifier depth for formulas of $\mathcal{N}^O$ is straightforwardly adapted from Definition 2.5. The crucial insight is the following: As noted, a model for monadic second-order logic without non-logical constants is simply a set; therefore, in assigning values to variables, it only matters how these values are related by the subset relation. And given n variables, any such constellation of subset relationships can be exhibited in a set of size 2^n. (Related considerations were also used by Parry [42] to show the decidability of S5.)

This idea can be turned into a more rigorous proof using the technique of Ehrenfeucht–Fraïssé games, as noted by Väänänen [56, Section 8], or their variant presentation as back and forth systems. These techniques allow us to show that a formula up to certain quantifier depth n is true in one model if and only if it is true in another model, connected to the first by a back-and-forth system; in the present case, a model (i.e., set) of size greater than 2^n and a model (set) of size 2^n. Below, we therefore adapt back and forth systems to the variant monadic second-order language $\mathcal{N}^O$.

In the relational frames we will be interested in, two worlds will not in general be interchangeable. But using back and forth systems, we will at least be able to reduce any cluster of duplicate worlds to a finite number determined by the quantifier depth of the formula under consideration. Combined with taking point-generated subframes, this gives us a flexible method for reducing the size of relational frames in a given class without changing the satisfiability of any propositionally quantified modal formula of a given complexity. In many interesting cases, we will be able to impose a finite bound on the size of the resulting frame which is computable in terms of this complexity, and this allows us to prove decidability of the relevant propositionally quantified modal logic.

We first make precise how such a bound on the size of relational frames entails decidability. To start, we introduce the notion of two frames validating the same *closed* monadic second-order formulas up to quantifier depth n, which we call being *n-equivalent*. We will be interested in classes of frames which contain, for every point-generated subframe of one of their frames, an n-equivalent frame of a size that is computably finitely bounded in terms of n; we call these computably reducible.

DEFINITION 3.1. For any $n \in \mathbb{N}$, relational frames $\mathfrak{F}$ and $\mathfrak{F}'$ are *n-equivalent*, written $\mathfrak{F} \equiv_n \mathfrak{F}'$, if for all closed formulas $\varphi \in \mathcal{N}^O$ of quantifier depth $\leq n$, $\mathfrak{F} \Vdash \varphi$ iff $\mathfrak{F}' \Vdash \varphi$.

Let C be a class of relational frames.

- For any $r : \mathbb{N} \to \mathbb{N}$, C is *r-reducible* if for every $n \in \mathbb{N}$, every point-generated subframe of a frame in C is n-equivalent to some frame in C of size $\leq r(n)$.
- C is *computably reducible* if C is r-reducible for some computable function $r : \mathbb{N} \to \mathbb{N}$.

We can now show that computably reducible classes of frames defined by finite sets of formulas give rise to decidable propositionally quantified modal logics. The proof is an instance of a standard type of decidability argument; for further details on this

type of argument in the quantifier-free case, see [4, Section 6.2, esp. Theorem 6.7 on p. 342].

LEMMA 3.2.   *If $\Gamma \subseteq \mathcal{L}^O$ is finite and $R(\Gamma)$ is computably reducible, then $\mathcal{L}(R(\Gamma))$ is decidable.*

PROOF.   Assume $R(\Gamma)$ is r-reducible for some computable function $r : \mathbb{N} \to \mathbb{N}$. If $\varphi \in \mathcal{L}^O$ is satisfiable in a frame in $R(\Gamma)$, then by Proposition 2.7, it is satisfiable in one of its point-generated subframes $\mathfrak{F}$. Let n be the quantifier depth of $v(\varphi^x)$, the translation of φ in $\mathcal{N}^O$. Then there is a frame $\mathfrak{F}'$ in $R(\Gamma)$ of size $\leq r(n)$ which is n-equivalent to $\mathfrak{F}$, whence φ is satisfiable on $\mathfrak{F}'$.

Consequently, the satisfiability (or, correspondingly, validity) of any formula φ can be determined by considering only frames in $R(\Gamma)$ up to the finite size $r(\mathrm{qd}(v(\varphi^x)))$. Since r is a computable function, this bound is computable in terms of φ. Up to isomorphism, there are only finitely many such frames. Since Γ is finite, and the validity of a formula on a finite frame is decidable, the finite frames in $R(\Gamma)$ are recursively enumerable. It follows that $\mathcal{L}(R(\Gamma))$ is decidable.   ⊣

3.2. Diversity. In this section, we make precise the notion of worlds of a relational frame being duplicates: worlds w and v are duplicates if the structure of the frame is invariant under interchanging w and v, i.e., if the transposition (wv) (the permutation of worlds mapping w and v to each other, and every other world to itself) is an automorphism of the frame (i.e., an isomorphism from the frame to itself). The relation of being a duplicate is an equivalence relation, dividing the frame into classes of interchangeable worlds. Without changing the validity of a given formula of a given syntactic complexity, each of these classes can be reduced to a certain finite number of worlds. Therefore, the number of these classes will play an important role; we call it the frame's diversity.

DEFINITION 3.3.   Let $\mathfrak{F} = \langle W, R_\square \rangle_{\square \in O}$ be a relational frame. $w \in W$ is a *duplicate of* $v \in W$ if (wv) is an automorphism of $\mathfrak{F}$. $\triangle_\mathfrak{F}$ is the relation of being a duplicate on $\mathfrak{F}$, i.e.,

$$w \triangle_\mathfrak{F} v \text{ iff } (wv) \in \mathrm{aut}(\mathfrak{F}).$$

A *duplicate class* of $\mathfrak{F}$ is an equivalence class of $\triangle_\mathfrak{F}$. The *diversity of* $\mathfrak{F}$, written $\Delta(\mathfrak{F})$ is the number of duplicate classes of $\mathfrak{F}$:

$$\Delta(\mathfrak{F}) := |W/\triangle_\mathfrak{F}|.$$

The diversity of a class of relational frames C, written $\Delta(C)$, is the supremum of diversities of point-generated subframes of frames in C, if there is such a cardinal, and undefined otherwise.

Below, we will show that for any class C of frames which is defined by a finite set of propositionally quantified modal formulas and which has finite diversity, $\mathcal{L}(C)$ is decidable; see Theorem 3.13. From this, many new results about the decidability of propositionally quantified modal logics can be derived, and we present a number of examples in Section 3.5.

In the following, we will introduce a generalization of the notion of a duplicate, and consider equivalence relations which are refinements of the duplicate relation,

i.e., which only relate duplicates. While not strictly necessary, it is worth noting that for the simple notion of duplicates stated in Definition 3.3, there is a very natural alternative characterization of such equivalence relations. We discuss this in the remainder of this section; readers may skip ahead to Section 3.3 without loss of continuity.

Below, we will limit the number of worlds in a frame by reducing the number of duplicates of any world to a certain finite number. This is most straightforwardly done by choosing a suitable subframe, simply omitting any extraneous duplicates. But in the present simple case, an alternative approach is possible: instead of identifying a subset of the set of worlds, and considering the relevant subframe, we can also identify a suitable equivalence relation which is a refinement (subset) of the relation of being a duplicate, and consider the corresponding quotient frame, whose worlds are the equivalence relations of the original frame. The accessibility relations are lifted from the individual worlds to the equivalence classes. For it to be possible to define this lift in a suitable way, one might suppose that the equivalence relation must be a congruence with respect to the accessibility relations. But a weaker requirement suffices. Consider the case of an equivalence relation on a unimodal frame in which all worlds can access themselves but no other worlds. If the equivalence relation is not identity, it is not a congruence. But it is still clear how to define the quotient frame: every equivalence class should be able to access itself, and nothing else. What is needed for it to be possible to define the quotient frame is what we will call a weak congruence, which imposes the usual congruence constraint only on two pairs of equivalent worlds if their first coordinates are identical iff their second coordinates are identical. For such weak congruences, the accessibility relation of the quotient frame is defined as usual, except that an equivalence class can access itself just in case each/one of its members can access itself.

DEFINITION 3.4. Let $\mathfrak{F} = \langle W, R_\square \rangle_{\square \in O}$ be a relational frame and $\sim$ an equivalence relation on W. $\sim$ is a *weak congruence on* $\mathfrak{F}$ if for all $w \sim w', v \sim v' \in W$ and $\square \in O$,

if $w = v$ iff $w' = v'$, then $wR_\square v$ iff $w'R_\square v'$.

In this case, $\mathfrak{F}/\sim$ is the relational frame $\langle W/\sim, R_\square/\sim \rangle_{\square \in O}$ where

$[w]_\sim R_\square/\sim[v]_\sim$ iff $w \nsim v$ and $wR_\square v$, or $w \sim v$ and $wR_\square w$.

It is routine to show that the constraints on weak congruences suffice to ensure that $\mathfrak{F}/\sim$ is well-defined. It is also easy to verify that an equivalence relation is a weak congruence just in case it relates only duplicates; i.e., $\sim$ is a weak congruence on $\mathfrak{F}$ just in case it is a refinement of $\triangle_\mathfrak{F}$. Finally, it can be shown that the result of reducing the worlds of any duplicate class in $\mathfrak{F}$ to n members, which we will define in Definition 3.10 as $\mathfrak{F}_{id}^n$, is isomorphic to the quotient frame $\mathfrak{F}/\sim$, where $\sim$ is any weak congruence which divides every duplicate class d of $\mathfrak{F}$ into $\min(|d|, n)$ equivalence classes.

3.3. Back and forth systems. This section introduces back and forth systems for $\mathcal{N}^O$. Such systems have been developed for a wide range of logics, including propositional modal logic (see the notion of bisimulation in [4, Section 2.2]), first-order logic ([29] and [22, Section 3.2]), second-order logic [56, Section 3.1], and

various quantified modal logics [3, p. 123] and [14] including higher-order and/or infinitary quantified modal logics [15, 17].

DEFINITION 3.5. Let $\mathfrak{F} = \langle W, R_\Box \rangle_{\Box \in O}$ and $\mathfrak{F}' = \langle W', R'_\Box \rangle_{\Box \in O}$ be relational frames.

A *partial isomorphism from $\mathfrak{F}$ to $\mathfrak{F}'$* is a partial injection i from $\mathcal{P}(W)$ to $\mathcal{P}(W')$ such that for all $x, y \in \mathrm{dom}(i)$ and $\Box \in O$:

- $x \subseteq y$ iff $i(x) \subseteq i(y)$, and
- for all $w \in x$ there is a $v \in y$ such that $wR_\Box v$ iff for all $w' \in i(x)$ there is a $v' \in i(y)$ such that $w'R'_\Box v'$.

A *back and forth system from $\mathfrak{F}$ to $\mathfrak{F}'$* is a function I mapping every $n \in \mathbb{N}$ to a set of partial isomorphisms, such that for all $n \in \mathbb{N}$ and $i \in I_{n+1}$:

- for all $x \subseteq W$, there is a $j \supseteq i$ in I_n such that $x \in \mathrm{dom}(j)$, and
- for all $x' \subseteq W'$, there is a $j \supseteq i$ in I_n such that $x' \in \mathrm{im}(j)$.

I_n is the nth *stage* of I.

This definition is set up just to allow for the following result, which shows that worlds of two frames connected by a partial isomorphism in the nth stage of a back and forth system validate the same formulas up to quantifier depth n, relative to suitable assignment functions:

PROPOSITION 3.6. *Let $\mathfrak{F}$ and $\mathfrak{F}'$ be relational frames, I a back and forth system from $\mathfrak{F}$ to $\mathfrak{F}'$, $\varphi \in \mathcal{N}^O$, $n \geq \mathrm{qd}(\varphi)$, $i \in I_n$, and a an assignment function for $\mathfrak{F}$ such that $\mathrm{im}(a) \subseteq \mathrm{dom}(i)$. Then,*

$$\mathfrak{F}, a \Vdash \varphi \text{ iff } \mathfrak{F}', i \circ a \Vdash \varphi.$$

PROOF. By induction on the complexity of φ. ⊣

Using the notion of n-equivalence, we can sum up to the consequences of this lemma for closed formulas more concisely as follows:

COROLLARY 3.7. *If $\mathfrak{F}$ and $\mathfrak{F}'$ are relational frames, I is a back and forth system from $\mathfrak{F}$ to $\mathfrak{F}'$, $n \geq \mathrm{qd}(\varphi)$, and I_n is non-empty, then $\mathfrak{F}$ and $\mathfrak{F}'$ are n-equivalent.*

3.4. Reducing frames. We will now use back and forth systems to show that for any relational frame, an n-equivalent subframe can be obtained by limiting the number of duplicates of any given world to 2^n. However, for applications in the multimodal case in Section 6.2, it will be useful to develop this idea in a more general form.

The simplest multimodal application which motivates this generalization is the case of bimodal frames in which both accessibility relations are equivalence relations, one of which is a refinement (subset) of the other. Let these relations be $R_\boxtimes \subseteq R_\Box$. In a point-generated subframe of such a frame, $R_\Box$ is the universal relation. Any members of an $R_\boxtimes$-equivalence class are therefore duplicates. By reducing the number of duplicate worlds, these equivalence classes can therefore be shrunk to a size no greater than 2^n, while retaining an n-equivalent subframe. The resulting subframe may still be infinite, since there may still be infinitely many $R_\boxtimes$-equivalence classes. However, the size of these equivalence classes is bounded by 2^n, and there is no way of distinguishing two equivalence classes structurally other

than in terms of their size. We can therefore apply the idea of reducing numbers of duplicates again, but on the level of equivalence classes of worlds instead of the worlds themselves, by reducing the duplicates of any $R_{\boxtimes}$-equivalence class to a finite number. It turns out that we can obtain an n-equivalent subframe by limiting the number of duplicates of $R_{\boxtimes}$-equivalence classes to $2^{2^n n}$.

In anticipation of this generalization, we first generalize the notion of a duplicate from worlds to equivalence classes of worlds, relative to an arbitrary equivalence relation. It will also be useful to relativize the notion of a duplicate in a frame to a finite sequence of sets of worlds, which are understood as imposing additional structure on the frame under consideration. These sets of worlds will later be used to account for free variables, when considering open formulas. We therefore introduce the notion of an augmented frame, which adds to a relational frame $\mathfrak{F}$ an equivalence relation $\sim$ and a finite sequence of sets of worlds α, and define what it takes for equivalence classes x and y to be duplicates in such an augmented frame: there must be an isomorphism between the subframes based on x and y, and this isomorphism must constitute an automorphism of the whole frame. Here, the notion of a subframe of an augmented frame is defined in the obvious way: $\langle \mathfrak{F}, \sim, \alpha \rangle | x$ is $\langle \mathfrak{F} | x, \sim \cap x^2, \alpha' \rangle$, where $\alpha'(m) = \alpha(m) \cap x$ for all $m \in \mathrm{dom}(\alpha)$.

DEFINITION 3.8. An *augmented frame* is a structure $\mathfrak{A} = \langle \mathfrak{F}, \sim, \alpha \rangle$, where $\mathfrak{F} = \langle W, R_{\square} \rangle_{\square \in O}$ is a relational frame, $\sim$ is an equivalence relation on W, and α is a sequence of pairwise distinct elements of $\mathcal{P}(W)$ of some finite length $k \in \mathbb{N}$.

$x, y \in W/\sim$ are *duplicates in* $\mathfrak{A}$ if there is an isomorphism f from $\mathfrak{A}|x$ to $\mathfrak{A}|y$, and $\hat{f}$ is an automorphism of $\mathfrak{A}$, where $\hat{f}$ is the function on W such that for all $w \in W$,

$$\hat{f}(w) = \begin{cases} f(w), & \text{if } w \in x, \\ f^{-1}(w), & \text{if } w \in y, \\ w, & \text{otherwise.} \end{cases}$$

$\triangle_{\mathfrak{A}}$ is the relation of being a duplicate in $\mathfrak{A}$. A *duplicate congruence on* $\mathfrak{A}$ is an equivalence relation on $W/\sim$ which relates only duplicates in $\mathfrak{A}$.

Since $\triangle_{\mathfrak{A}}$ is an equivalence relation on $W/\sim$, it is the coarsest duplicate congruence on $\mathfrak{A}$. It is also easy to see that the original notion of duplicate worlds in a relational frame $\mathfrak{F}$ can be recovered from the notion of duplicate equivalence classes by augmenting $\mathfrak{F}$ with the equivalence relation id of identity and the empty sequence $\langle \rangle$, and considering any world to be represented by its singleton set: $w \triangle_{\mathfrak{F}} v$ is equivalent to $\{w\} \triangle_{\langle \mathfrak{F}, \mathrm{id}, \langle \rangle \rangle} \{v\}$.

We can now define the notion of a subframe obtained by reducing the number of duplicate equivalence classes to n: We first index duplicate equivalence classes by ordinals, and then take the subframe based on those worlds which are in equivalence classes with index $< n$. Strictly speaking, the resulting subframe will be depended on the choice of the indexing function, but as such subframes are unique up to isomorphism, we can ignore this dependence.

DEFINITION 3.9. Assume $\mathfrak{F} = \langle W, R_{\square} \rangle_{\square \in O}$ is a relational frame, $\sim$ an equivalence relation on W, and $n > 0$. Let $\mathfrak{A} = \langle \mathfrak{F}, \sim, \langle \rangle \rangle$. Consider any function δ mapping every $x \in W/\sim$ to an ordinal $\delta(x)$ such that for every duplicate class $D \in W/\sim/\triangle_{\mathfrak{A}}, \delta|D$ is a bijection from D to $|D|$.

$\mathfrak{F}_{\sim}^{n}$ is the subframe of $\mathfrak{F}$ based on the set W' such that $w \in W'$ iff $\delta([w]_{\sim}) < n$, for all $w \in W$.

As the next step, we will connect $\mathfrak{F}$ and $\mathfrak{F}_{\sim}^{n}$ by a back and forth system. We do so by first defining a way of connecting any two relational frames, with associated equivalence relations, by a generic back and forth system. Of course, if the underlying relational frames are insufficiently similar, then this generic back and forth system will not be very useful, since most of its stages will be empty. The central idea behind the construction of the generic back and forth system is that a partial isomorphism between the relational frames is determined by any bijection f between the equivalence classes of duplicate congruences of the augmented frames satisfying certain additional conditions. Most importantly, if f maps equivalence class x to x', then the subframes based on x and x' must be isomorphic. Further, if f also maps y to y', then corresponding worlds in x and x' must relate to corresponding worlds in y and y' in corresponding ways in terms of the accessibility relations. Effectively, such a bijection f divides the equivalence classes of worlds of the two frames into clusters of duplicates, with the two frames being alike except that the relevant clusters may contain different numbers of duplicates. These numbers of duplicates will be important, and we call f an n-correspondence if it only connects clusters whose size is either both at least n, or identical. The generic back and forth system can now be defined, by including in its nth stage the partial isomorphisms determined by 2^{ln}-correspondences, where l bounds the sizes of equivalence classes in the augmented frames. Given such a correspondence f, the relevant partial isomorphism can be read off just from the augmented frames connected by f, in particular their third components α and α': the relevant partial isomorphism maps $\alpha(m)$ to $\alpha'(m)$, whenever these are defined. We therefore also require these sequences to have the same length, as part of the definition of an n-correspondence.

DEFINITION 3.10. Let $\mathfrak{A} = \langle \mathfrak{F}, \sim, \alpha \rangle$ and $\mathfrak{A}' = \langle \mathfrak{F}', \sim', \alpha' \rangle$ be augmented frames, with $\mathfrak{F} = \langle W, R_{\square} \rangle_{\square \in O}$ and $\mathfrak{F}' = \langle W', R'_{\square} \rangle_{\square \in O}$.

For any $n > 1$, an n-correspondence from $\mathfrak{A}$ to $\mathfrak{A}'$ is a function f satisfying all of the following conditions:

(i) f is a bijection from $W/\sim/\approx$ to $W'/\sim'/\approx'$, where $\approx$ is a duplicate congruence on $\mathfrak{A}$ and $\approx'$ is a duplicate congruence on $\mathfrak{A}'$.

(ii) α and α' have the same length k.

(iii) For any $x \in W/\sim$ and $x' \in f([x]_{\approx})$, there is an isomorphism g from $\mathfrak{A}|x$ to $\mathfrak{A}'|x'$.

(iv) For any $x, y \in W/\sim$, $x' \in f([x]_{\approx})$, $y' \in f([y]_{\approx})$, isomorphisms g from $\mathfrak{A}|x$ to $\mathfrak{A}'|x'$ and h from $\mathfrak{A}|y$ to $\mathfrak{A}'|y'$, $w \in x$, $v \in y$, and $\square \in O$:

$$wR_{\square}v \text{ iff } g(w)R'_{\square}h(v).$$

(v) For all $X \in \text{dom}(f)$, either $|X| \geq n$ and $|f(X)| \geq n$, or $|X| = |f(X)|$.

Assuming α and α' have the same length k, $i_{\mathfrak{A}}^{\mathfrak{A}'}$ is the partial function from $\mathcal{P}(W)$ to $\mathcal{P}(W')$ such that

$i_{\mathfrak{A}}^{\mathfrak{A}'}(x) = x'$ whenever $x = \alpha(m)$ and $x' = \alpha'(m)$, for some $m \leq k$.

Given relational frames $\mathfrak{F}$ and $\mathfrak{F}'$, equivalence relations $\sim$ and $\sim'$, and $n \in \mathbb{N}$, let I_n be the set of functions $i_{\mathfrak{A}}^{\mathfrak{A}'}$ for $\mathfrak{A}$ an augmented frame based on $\mathfrak{F}$ and $\sim$ and $\mathfrak{A}'$ an augmented frame based on $\mathfrak{F}'$ and $\sim'$ such that there exists a 2^{ln}-correspondence from $\mathfrak{A}$ to $\mathfrak{A}'$, where $l = \sup\{|x| : x \in W/\sim\}$. $I_{\mathfrak{F},\sim}^{\mathfrak{F}',\sim'}$ is the function mapping any $n \in \mathbb{N}$ to I_n.

The next lemma shows that this construction succeeds, at least insofar as $I_{\mathfrak{F},\sim}^{\mathfrak{F}',\sim'}$ is indeed a back and forth system. Note that it is immediate by construction that if $m \le n$, then $I_m \supseteq I_n$. In the proof, we write $\alpha^\frown x$ for the result of appending x to the end of a sequence α.

LEMMA 3.11. *For any relational frames $\mathfrak{F} = \langle W, R_\square \rangle_{\square \in O}$ and $\mathfrak{F}' = \langle W', R'_\square \rangle_{\square \in O}$ and equivalence relations $\sim$ and $\sim'$ on W and W', respectively, $I_{\mathfrak{F},\sim}^{\mathfrak{F}',\sim'}$ is a back and forth system from $\mathfrak{F}$ to $\mathfrak{F}'$.*

PROOF. It is routine to show that for every $n \in \mathbb{N}$, I_n is a set of partial isomorphisms. So consider any $n \in \mathbb{N}$ and $i \in I_{n+1}$. By symmetry, it suffices to consider only the first condition. So, let $x \subseteq W$. If $x \in \text{dom}(i)$, the claim is immediate, so assume otherwise.

Let $\mathfrak{A} = \langle \mathfrak{F}, \sim, \alpha \rangle$ and $\mathfrak{A}' = \langle \mathfrak{F}', \sim', \alpha' \rangle$ be augmented frames determining i, and f be the required $2^{l(n+1)}$-correspondence, where $l = \sup\{|s| : s \in W/\sim\}$. Consequently, f is a bijection from $W/\sim/\approx$ to $W'/\sim'/\approx'$, where $\approx$ is a duplicate congruence on $\mathfrak{A}$ and $\approx'$ is duplicate congruence on $\mathfrak{A}'$. Let $\overline{\mathfrak{A}} = \langle \mathfrak{F}, \sim, \alpha^\frown x \rangle$, and $\overline{\approx}$ the refinement of $\approx$ which relates $y \approx z$ just in case y and z are duplicates in $\overline{\mathfrak{A}}$. Note that $\overline{\approx}$ divides every equivalence class of $\approx$ into at most 2^l classes. So, for cardinality reasons, there is a refinement $\overline{\approx}'$ of $\approx'$ and bijection g from $W/\sim/\overline{\approx}$ to $W'/\sim'/\overline{\approx}'$ which refines f in the sense that if $g([y]_{\overline{\approx}}) = [y']_{\overline{\approx}'}$ then $f([y]_{\approx}) = [y']_{\approx'}$, such that for every $X \in W/\sim/\overline{\approx}$, either $|X| \ge 2^{ln}$ and $|g(X)| \ge 2^{ln}$, or $|X| = |g(X)|$.

For every $X' \in W/\sim'/\overline{\approx}'$, choose some representative $r_{X'} \in X'$, $y_{X'} \in g^{-1}(X')$, and isomorphism $h_{X'}$ from $\mathfrak{A}'|r_{X'}$ to $\mathfrak{A}|y_{X'}$. For every $y' \in W/\sim'$, choose some isomorphism $h_{y'}$ witnessing that y' and $r_{[y']_{\overline{\approx}'}}$ are duplicates in $\mathfrak{A}'$. Define $y'_x = \{w' \in y' : h_{[y']_{\overline{\approx}'}} \circ h_{y'}(w') \in x\}$. Let $x' = \bigcup\{y'_x : y' \in W'/\sim'\}$ and $\overline{\mathfrak{A}}' = \langle \mathfrak{F}', \sim', \alpha'^\frown x' \rangle$. Since x is not among the sets indexed by α, it follows by the construction that x' is not among the sets indexed by α', so $\overline{\mathfrak{A}}'$ is an augmented frame. It is routine to verify that g is a 2^{ln}-correspondence from $\overline{\mathfrak{A}}$ to $\overline{\mathfrak{A}}'$. From this, it follows that $j := i_{\overline{\mathfrak{A}}}^{\overline{\mathfrak{A}}'} \in I_n$. It is also clear that $j \supseteq i$ and $x \in \text{dom}(j)$, as required.   ⊣

We are now ready to put together the pieces developed in this section: $\mathfrak{F}_\sim^m$ is obtained from $\mathfrak{F}$ by reducing the number of duplicate equivalence classes to at most m. Letting $\approx$ be the duplicate relation, and $\approx'$ its restriction for $\mathfrak{F}_\sim^m$, there is a straightforward m-correspondence, mapping every duplicate cluster to its reduction. We have seen that the corresponding partial isomorphism is contained in a stage of the generic back and forth system, the index of which is determined by m and the limit of the sizes of $\sim$-equivalence classes. From this, it follows that $\mathfrak{F}$ and $\mathfrak{F}_\sim^m$ satisfy the same closed formulas up to a certain complexity.

PROPOSITION 3.12. *Assume* $\mathfrak{F} = \langle W, R_\square \rangle_{\square \in O}$ *is a relational frame,* $\sim$ *an equivalence relation on* W, *and* $n \in \mathbb{N}$. *Let* $l = \sup\{|x| : x \in W/\sim\}$. *Then* $\mathfrak{F}$ *and* $\mathfrak{F}^{2^{ln}}_\sim$ *are* n-*equivalent.*

PROOF. Let $\mathfrak{F}' = \mathfrak{F}^{2^{ln}}_\sim$, W' the set of worlds of $\mathfrak{F}'$, and $\sim' = \sim|W'$. Let $\mathfrak{A} = \langle \mathfrak{F}, \sim, \langle \rangle \rangle$, $\mathfrak{A}' = \langle \mathfrak{F}', \sim', \langle \rangle \rangle$, $\approx = \triangle_{\mathfrak{A}}$, and $\approx' = \approx|(W'/\sim')$. Let f be the function from $W/\sim/\approx$ to $W'/\sim'/\approx'$ such that $f : X \mapsto X|(W'/\sim')$. Then f is a 2^{ln}-correspondence from $\mathfrak{A}$ to $\mathfrak{A}'$. So $i^{\mathfrak{A}'}_{\mathfrak{A}}$ is a member of the nth stage of $I^{\mathfrak{F}',\sim'}_{\mathfrak{F},\sim}$. By Lemma 3.11 and Corollary 3.7, it follows that $\mathfrak{F}$ and $\mathfrak{F}'$ are n-equivalent. ⊣

3.5. Decidability via finite diversity. In this section, we turn to the first application of reducing frames, and prove the decidability of several unimodal logics. First, we show generally that any class of relational frames defined by a finite set of formulas which has finite diversity gives rise to a decidable propositionally quantified modal logic. For this relatively simple result, much of the complexity involved above can be ignored, as we may work just with reductions in which the equivalence relation on worlds is identity. Using Proposition 3.12, we can thus note that any frame of diversity n can be reduced to one whose size is bounded in terms of n and the syntactic complexity of the formulas whose truth we wish to remain invariant. This allows us to show that any finitely defined class of frames with finite diversity is computably reducible, from which decidability follows by Lemma 3.2.

THEOREM 3.13. *If* $\Gamma \subseteq \mathcal{L}^O$ *is finite and* $\mathsf{R}(\Gamma)$ *has finite diversity, then* $\mathcal{L}(\mathsf{R}(\Gamma))$ *is decidable.*

PROOF. Assume $\Gamma \subseteq \mathcal{L}^O$ is finite and $\mathsf{R}(\Gamma)$ has diversity $n \in \mathbb{N}$. Let $l := \max\{\mathrm{qd}(v(\gamma^x)) : \gamma \in \Gamma\}$. Let $r : \mathbb{N} \to \mathbb{N}$ such that $r : m \mapsto n \cdot 2^{\max(m,l)}$. We show that $\mathsf{R}(\Gamma)$ is r-reducible; the claim follows by Lemma 3.2. So consider any $m \in \mathbb{N}$ and point-generated subframe $\mathfrak{F}$ of a frame in $\mathsf{R}(\Gamma)$. Let $\mathfrak{F}'$ be $\mathfrak{F}^{2^{\max(m,l)}}_{\mathrm{id}}$. By Proposition 3.12, $\mathfrak{F}$ and $\mathfrak{F}'$ are $\max(m,l)$-equivalent (and so also m-equivalent). Thus $\mathfrak{F}'$ validates Γ, whence $\mathfrak{F}'$ is in $\mathsf{R}(\Gamma)$. By construction, the size of $\mathfrak{F}'$ is limited by $r(m)$. ⊣

As a first illustration of how Theorem 3.13 can be applied, we note that Fine and Kaplan's result on the decidability of S5π+ follows immediately:

EXAMPLE 3.14. $\mathsf{R}(\mathrm{S5})$ has diversity 1, and so S5π+ is decidable.

PROOF. $\mathsf{R}(\mathrm{S5})$ is the class of unimodal frames with an equivalence relation. Point-generated subframes of such frames have a universal relation, and so diversity 1. The claim follows by Theorem 3.13. ⊣

More interestingly, the theorem can also be applied to the weaker logic KE (aka K5), which establishes our first new decidability result:

EXAMPLE 3.15. $\mathsf{R}(\mathrm{KE})$ has diversity 3, and so KEπ+ is decidable.

PROOF. $\mathsf{R}(\mathrm{KE})$ is the class of relational frames with a Euclidean relation. Let $\mathfrak{F}$ be such a frame, w one of its worlds, and $\mathfrak{F}_w = \langle W, R \rangle$. It is routine to show that then, either R is universal on W, or there is a set $x \subseteq W \setminus \{w\}$ such that vRu iff either $v = w$ and $u \in x$, or $v \neq w$ and $u \neq w$. Thus, the diversity of $\mathfrak{F}_w$ is bounded by 3. The claim follows by Theorem 3.13. ⊣

With Proposition 2.8 and the fact that $\Box\Box p \to \Box\Box\Box p \in$ KE, it follows that $\Lambda\pi+$ is decidable, for any finite axiomatic extension Λ of KE. We therefore obtain Example 3.14 as a corollary, as well as the result of Ding [10] on the decidability of KD4E$\pi+$. In fact, every normal extension of KE is finitely axiomatizable, as shown by Nagle and Thomason [40]. We conclude:

COROLLARY 3.16. *If Λ is a normal extension of* KE, *then $\Lambda\pi+$ is decidable.*

To illustrate the versatility of our decidability argument, and to delineate the axiomatizability boundary further from above, we consider another, more exotic, example of a unimodal logic.

EXAMPLE 3.17. R(K1.2) has diversity 2, and so K1.2$\pi+$ is decidable.

PROOF. The argument is analogous to the above cases, using the following observation: If $\mathfrak{F}$ validates K1.2, then in any point-generated subframe $\mathfrak{F}_w$, w can access only itself and worlds which can access only themselves. So the diversity of $\mathfrak{F}_w$ is bounded by 2.   ⊣

So far, we have considered finitely axiomatized normal modal logics and the classes of relational frames they define. But the method developed here applies as well to finite sets of formulas involving propositional quantifiers and the classes of relational frames they define. By way of illustration, we consider the logic of "elsewhere" discussed by von Wright [57], which is the propositional modal logic of the class of frames in which every world can access every other world, but not itself. Segerberg [52] shows that this is the normal modal logic axiomatized by the axioms **B** and $p \to (\Box p \to \Box\Box p)$. Since these axioms are valid on frames with a universal relations, it follows that they fail to impose the requirement of irreflexivity on the class of relational frames they define. However, using propositional quantifiers, irreflexivity is easily captured by the negation of the universal closure of the axiom **T**. Adding this to the other two axioms, we define the class of frames whose point-generated subframes are in the intended class of frames. By Proposition 2.7, these two classes of frames have the same propositionally quantified modal logic. We can thus reason as follows:

EXAMPLE 3.18. Let E be the class of unimodal frames in which every world can access every other world, but not itself. $\mathcal{L}(\mathsf{E})$ is decidable.

PROOF. Let $\Gamma = \{p \to \Box\Diamond p, p \to (\Box p \to \Box\Box p), \neg\forall p(\Box p \to p)\}$. The frames of E are exactly the point-generated subframes of frames in R(Γ). Since the frames of E have diversity 1, R(Γ) has diversity 1. It follows by Theorem 3.13 that $\mathcal{L}(\mathsf{R}(\Gamma)) = \mathcal{L}(\mathsf{E})$ is decidable.   ⊣

We postpone the discussion of decidability results for extensions of K4.3 and multimodal logics; they will be discussed in Sections 5.2 and 6, respectively. Instead, the next section turns to the main unaxiomatizability results.

§4. Recursive isomorphism to second-order logic. Fine [12] already noted that for several well-known normal modal logics Λ, $\Lambda\pi+$ is not axiomatizable, by sketching a reduction of second-order arithmetic to these logics. The relevant logics consisted of a few logics included in S4.2, as well as B. A more detailed exposition of the relevant reduction was eventually presented by Garson [18]. Kremer [30] notes that these

results can be strengthened to showing that the relevant logics $\Lambda\pi+$ are recursively isomorphic to second-order logic, using the techniques applied to propositionally quantified relevant logic in his paper. According to Kremer [31, p. 530], Fine and Kripke had already proved the stronger claim of recursive isomorphism to second-order logic in unpublished work "shortly after the publication of" Fine [12]. Kaminski and Tiomkin [26] present a more detailed construction showing that for every normal modal logic Λ included in S4.2, second-order logic is reducible to $\Lambda\pi+$.

In this section, we will present a variation on the construction of Kaminski and Tiomkin [26]. By making use of the result of Rabin and Scott and the symmetric second-order language $\mathcal{S}$ discussed in Section 2.4, we will be able to provide a construction which is both simpler and more versatile. Aside from providing a more accessible presentation, we show that we easily obtain, as corollaries, analogous results for a number of further logics, including B and K4.2W. We work with unimodal logics throughout this section.

4.1. Fully representing class of frames. Key to showing the recursive isomorphism of the relevant logic $\Lambda\pi+$ is proving that SSOL, the set of validities of symmetric second-order logic, can be reduced to it. We proceed as follows: First, we define a formula $\mathrm{P}(i)$ of $\mathcal{L}^\square$ with a free propositional variable i. Then, we show that if $\mathrm{P}(i)$ is true in a world w of a relational frame, relative to an assignment function a, we can simulate first- and (symmetric) second-order quantification over $D = R[w] \cap a(i)$ using $\mathcal{L}^\square$. First-order quantification is easily simulated, since we can represent an element $d \in D$ using any proposition which is true in d and no other element of D. To simulate symmetric second-order quantification, $\mathrm{P}(i)$ will be formulated to guarantee that for any $d, e \in D$, there is a world v which serves to represent the unordered pair $\{d, e\}$. This is easily effected by requiring that any $f \in D$ can access v if and only if f is one of d and e. With this, it is easy to simulate quantification over symmetric relations on D, since any such relation corresponds to a set of representations of pairs, and so at least one proposition. This allows us to reduce second-order logic to any logic $\Lambda\pi+$ such that for each cardinal κ, there is a world w of a frame validating Λ in which $\mathrm{P}(i)$ is true relative to some assignment function a such that $|R[w] \cap a(i)| = \kappa$.

We start making this outline precise by introducing abbreviations for formulas of $\mathcal{L}^\square$ expressing the relevant concepts. First, as Fine [12] observed, propositional quantifiers allow us to state that p is true in just one of the accessible worlds, as follows:

$$\mathrm{Q}(p) := \Diamond p \wedge \forall q(\square(p \to q) \vee \square(p \to \neg q)).$$

We use $\mathrm{Ind}(p)$ to state that p represents an individual (an element of the domain D); we use $\mathrm{PotP}(p)$ to state that p is true in a single world of $R[R[w]]$, and so potentially serves to represent an unordered pair of individuals; and we use $\mathrm{Pair}(p, q, r)$ to state that, assuming p and q represent individuals, r represents their unordered pair:

$$\mathrm{Ind}(p) := \mathrm{Q}(p) \wedge \Diamond(p \wedge i),$$
$$\mathrm{PotP}(p) := \Diamond\Diamond p \wedge \forall q(\square\square(p \to q) \vee \square\square(p \to \neg q)),$$
$$\mathrm{Pair}(p, q, r) := \mathrm{PotP}(r) \wedge \forall s(\mathrm{Ind}(s) \to (\Diamond(s \wedge \Diamond r) \leftrightarrow \Diamond(s \wedge (p \vee q)))).$$

With these, we can define the formula $P(i)$, stating that for any p and q representing individuals, there is some r representing their unordered pair:

$$P(i) := \forall p \forall q (\operatorname{Ind}(p) \wedge \operatorname{Ind}(q) \to \exists r \operatorname{Pair}(p, q, r)).$$

Before turning to the simulation of second-order logic, it will be useful to record what it takes for $P(i)$ to be true.

DEFINITION 4.1. Let $\mathfrak{F} = \langle W, R \rangle$ be a relational frame, $w \in W$, and $I \subseteq W$. w and I *provide pairing* if for all $d, e \in R[w] \cap I$, there is some $v \in R[R[w]]$ such that for all $f \in R[w] \cap I$, $f R v$ iff $f \in \{d, e\}$.

LEMMA 4.2. *Let $\mathfrak{F} = \langle W, R \rangle$ be a relational frame, $w \in W$, and a an assignment function. Then $\mathfrak{F}, w, a \Vdash P(i)$ iff w and $a(i)$ provide pairing.*

PROOF. Note that $\mathfrak{F}, w, a \Vdash \operatorname{Ind}(p)$ iff $R[w] \cap a(p)$ is a singleton subset of $a(i)$, and $\mathfrak{F}, w, a \Vdash \operatorname{PotP}(p)$ iff $R[R[w]] \cap a(p)$ is a singleton. Further, if $R[w] \cap a(p) = \{d\} \subseteq a(i)$ and $R[w] \cap a(q) = \{e\} \subseteq a(i)$, then $\mathfrak{F}, w, a \Vdash \operatorname{Pair}(p, q, r)$ iff $R[R[w]] \cap a(r)$ is a singleton $\{v\}$ such that for all $f \in R[w] \cap a(i)$, $f R v$ iff $f \in \{d, e\}$. With these observations, it is routine to establish the claim.   ⊣

The next step requires us to simulate $\mathcal{S}$ using $\mathcal{L}^\square$. We assume that for every variable ξ of $\mathcal{S}$ (first- or second-order), there is a distinct propositional variable p_ξ. With this, we define the recursive mapping $\cdot^*$ from $\mathcal{S}$ to $\mathcal{L}^\square$ whose only non-trivial clauses are the following:

$$\begin{aligned}
(Xyz)^* &:= \exists r (\operatorname{Pair}(p_y, p_z, r) \wedge \diamond\diamond(r \wedge p_X)), \\
(\forall x \varphi)^* &:= \forall p_x (\operatorname{Ind}(p_x) \to \varphi^*), \\
(\forall X \varphi)^* &:= \forall p_X (\varphi^*).
\end{aligned}$$

The next lemma observes that this mapping works as intended.

LEMMA 4.3. *Let $\varphi \in \mathcal{S}$. If $\mathfrak{F} = \langle W, R \rangle$ is a relational frame, $w \in W$, $I \subseteq W$, a an assignment function for $\mathcal{S}$ on $D = R[w] \cap I$, and w and I provide pairing, then*

$$D, a \Vdash \varphi \text{ iff } \mathfrak{F}, w, b \Vdash \varphi^*,$$

where b is the assignment function such that $b(i) = I$, $b(p_x) = \{a(x)\}$, and $b(p_X)$ is the set of $v \in R[R[w]]$ for which there are d and e related by $a(X)$ such that for every $f \in D$, $f R v$ iff $f \in \{d, e\}$.

PROOF. By induction on the complexity of φ. (The proof requires showing that formulas in the image of φ^* are insensitive to the truth of p_x outside of $R[w]$, but this is straightforward from the syntactic positions in which p_x occurs in any such formula. A similar observation is required for p_X.)   ⊣

With this lemma, we can simulate $\mathcal{S}$ on $R[w] \cap I$ whenever w and I provide pairing. Thus, we can reduce SSOL to $\mathcal{L}(C)$ whenever C is a class of frames in which for every cardinality, we can find corresponding w and I which provide pairing. Formally, this condition on classes of frames is defined as follows:

DEFINITION 4.4. Let a class of relational frames C *fully represent* if for every cardinality κ, there is a frame $\mathfrak{F} = \langle W, R \rangle$ in C, $w \in W$ and $I \subseteq W$ such that w and I provide pairing, and $|R[w] \cap I| = \kappa$.

Due to the parameter I, a single frame may be a witness of full representation for many cardinalities. Nonetheless, since there is a proper class of cardinals, any class of relational frames which fully represents must be proper as well. We can now show that whenever C fully represents, ssol can be reduced to $\mathcal{L}(\mathsf{C})$, from which it follows that if $\mathcal{L}(\mathsf{C})$ is finitely definable, then it is recursively isomorphic to ssol (and so second-order logic as usually defined). In fact, the reduction generalizes to any set $\Pi \subseteq \mathcal{L}(\mathsf{C})$, as long as Π includes $\mathrm{K}\pi+$.

THEOREM 4.5. *Let C be a class of relational frames which fully represents, and let* $\Pi \subseteq \mathcal{L}^{\square}$ *such that* $\mathrm{K}\pi+ \subseteq \Pi \subseteq \mathcal{L}(\mathsf{C})$. *Then second-order logic is reducible to* Π. *If C is also defined by a finite set of formulas, then* $\mathcal{L}(\mathsf{C})$ *is recursively isomorphic to second-order logic.*

PROOF. As discussed in Section 2.4, it suffices to reduce second-order logic in the guise of ssol to Π. (The conclusion of recursive isomorphism for a class of frames defined by a finite set from such a reduction is routine; for further details, see the corresponding discussion by Kremer [30].) We may focus just on sentences (closed formulas) of $\mathcal{S}$. The required reduction is then provided by the function mapping any sentence φ of $\mathcal{S}$ to

$$\varphi^{\dagger} := \forall i(\mathrm{P}(i) \to \varphi^*).$$

We show that $\varphi \in$ ssol iff $\varphi^{\dagger} \in \Pi$.

Assume first that $\varphi^{\dagger} \notin \Pi$. Then $\varphi^{\dagger} \notin \mathrm{K}\pi+$. So there is a relational frame $\mathfrak{F} = \langle W, R \rangle$, $w \in W$ and assignment function a such that $\mathfrak{F}, w, a \Vdash \mathrm{P}(i) \wedge \neg\varphi^*$. By Lemma 4.2, w and $a(i)$ provide pairing. Since φ is closed, it follows by Lemma 4.3 that φ is falsified by $R[w] \cap a(i)$. So $\varphi \notin$ ssol.

Assume now that $\varphi^{\dagger} \in \Pi$. Then $\varphi^{\dagger} \in \mathcal{L}(\mathsf{C})$. Assume for contradiction that $\varphi \notin$ ssol. Then for some cardinality κ, the sentence φ is falsified by any set of size κ. Since C fully represents, there is a frame $\mathfrak{F} = \langle W, R \rangle$ in C, $w \in W$ and assignment function a such that w and $a(i)$ provide pairing, and $|R[w] \cap a(i)| = \kappa$. By Lemma 4.2, $\mathfrak{F}, w, a \Vdash \mathrm{P}(i)$. Since φ is closed and falsified by $R[w] \cap a(i)$, it follows with Lemma 4.3 that $\mathfrak{F}, w, a \nVdash \varphi^*$. Thus $\mathfrak{F}, w \nVdash \varphi^{\dagger}$, contradicting $\varphi^{\dagger} \in \mathcal{L}(\mathsf{C})$. So $\varphi \in$ ssol. $\dashv$

4.2. Examples. From Theorem 4.5, we can conclude for any finitely axiomatized normal modal logic Λ that $\Lambda\pi+$ is recursively isomorphic to second-order logic by showing that $\mathrm{R}(\Lambda)$ fully represents. The details depend on Λ, but in many cases, the frames witnessing full representation can be derived from a relatively simple class of frames, which we now introduce. In the following, we assume that every cardinal κ is a set of size κ, and write $U(\kappa)$ for the set of unordered pairs of elements of κ, i.e., $\{\{d, e\} : d, e \in \kappa\}$. We assume that κ, $U(\kappa)$, and $\{\kappa^+\}$ are pairwise disjoint. This is not guaranteed by the usual von Neumann definition of ordinals, as $\{0, 1\} = 2 \in 3$, but the problem is easily overcome by considering the disjoint union of the relevant sets—we omit the details for simplicity.

DEFINITION 4.6. If κ is a cardinal, $\mathfrak{F}_{\kappa}$ is the relational frame $\langle W, R \rangle$ such that:

$W = \{\kappa^+\} \cup \kappa \cup U(\kappa)$,
$R = \{\langle \kappa^+, w \rangle : w \in \kappa\} \cup \{\langle w, v \rangle : w \in \kappa, v \in U(\kappa), \text{ and } w \in v\}$.

$\mathfrak{F}_\kappa$ is constructed just so that κ^+ and κ provide pairing, and $|R[\kappa^+] \cap \kappa| = \kappa$, which can easily be verified. Therefore the class of relational frames fully represents, from which we can draw the following conclusion:

EXAMPLE 4.7.   Kπ+ is recursively isomorphic to second-order logic.

PROOF.   Since the class of relational frames fully represents, the claim follows by Theorem 4.5.   ⊣

More interestingly, $\mathfrak{F}_\kappa$ is easily adapted to meet a number of additional requirements, while κ^+ and κ still provide pairing. For example, we can take the transitive closure of R, and so obtain frames validating the **4** axiom. Adding also a "final" world, seen by every world, we obtain a simple proof of the result of Kaminski and Tiomkin [26] that S4.2π+ is recursively isomorphic to second-order logic. In fact, we note that the result can be strengthened in two ways: First, S4.2 can be strengthened to K2.1. Second, the result extends to $\Lambda\pi$+ for every finitely axiomatized normal modal logic $\Lambda \subseteq$ K2.1, and the resulting unaxiomatizability extends to every subset of formulas which includes Kπ+.

EXAMPLE 4.8.   Second-order logic is reducible to every set of formulas Π such that Kπ+ $\subseteq \Pi \subseteq$ K2.1π+. For every finitely axiomatized normal modal logic $\Lambda \subseteq$ K2.1, $\Lambda\pi$+ is recursively isomorphic to second-order logic.

PROOF.   Consider a frame $\mathfrak{F}_\kappa$, and add a new "final" world f, which every other world can access. Take the reflexive and transitive closure of the accessibility relation of this extended frame. In the resulting frame, κ^+ and κ still provide pairing, while K2.1 is valid. So C(K2.1) fully represents, whence the claim follows by Theorem 4.5.   ⊣

Analogously, we can cover the case of B, which is mentioned by Fine [12] and Kremer [30], but not discussed by Kaminski and Tiomkin [26].

EXAMPLE 4.9.   Second-order logic is reducible to every set of formulas Π such that Kπ+ $\subseteq \Pi \subseteq$ Bπ+. For every finitely axiomatized normal modal logic $\Lambda \subseteq$ B, $\Lambda\pi$+ is recursively isomorphic to second-order logic.

PROOF.   Consider a frame $\mathfrak{F}_\kappa$, and take the reflexive and symmetric closure of the accessibility relation. In the resulting frame, κ^+ and κ still provide pairing, while B is valid. So C(B) fully represents, whence the claim follows by Theorem 4.5.   ⊣

We can also cover the case of K4.2W, which appears not to have been considered in the literature so far. This subsumes the case of KW, which is better known in the context of provability logic as GL; see [5].

EXAMPLE 4.10.   Second-order logic is reducible to every set of formulas Π such that Kπ+ $\subseteq \Pi \subseteq$ K4.2Wπ+. For every finitely axiomatized normal modal logic $\Lambda \subseteq$ K4.2W, $\Lambda\pi$+ is recursively isomorphic to second-order logic.

PROOF.   Analogous to Example 4.8, using the transitive closure instead of the transitive and reflexive closure of the accessibility relation.   ⊣

Inspecting the structure of the frames used in these proofs, it is easy to see that these results can be strengthened further in various ways to some less commonly

considered modal logics. For example, the frames used in the proof of Example 4.10 all do not admit any path along the accessibility relation with more than three steps. Thus, K4.2W can be strengthened by adding $\square\square\square\square\bot$ as an axiom. These strengthened logics also need not be finitely axiomatizable for the reduction of second-order logic to be possible. An example is the logic BSeg, which Hughes and Cresswell [24] show not to be finitely axiomatizable. This normal modal logic is obtained by adding to B every instance of the following axiom schema for $n \geq 1$:

$$\mathbf{Seg}_n := \bigwedge_{i=1}^{n} \Diamond\Diamond p_i \rightarrow \Diamond \bigwedge_{i=1}^{n} \Diamond p_i.$$

The reduction of second-order logic in Example 4.9 can be extended to BSeg by enriching the accessibility relation of any frame $\mathfrak{F}_\kappa$, letting κ^+ access every world and every world access κ^+.

§5. Extensions of K4.3. The extensions of the unimodal logic K4.3 constitute a special case, requiring special treatment for both decidability and unaxiomatizability results. We therefore treat them separately in this section. The classes of frames they define are somewhat more complicated to describe, so we start by introducing the relevant notions. We then consider decidable and unaxiomatizable logics in turn.

5.1. Linear orders and balloons. We first recall the standard order-theoretic notion of a linear order.

DEFINITION 5.1. Let W be a set and R a binary relation on W.
- R is *connected* if for all w and v in W, wRv, $w = v$, or vRw.
- R is a *linear order* if R is a connected partial order (i.e., if R is connected, reflexive, transitive, and antisymmetric).

When no confusion is likely to arise, we identify an ordered set with its ordering relation. Note that a binary relation on a finite set is a linear order just in case it is isomorphic to a finite initial segment $\{0, \ldots, n\}$ of the natural numbers under their natural order $\leq$.

Next, we introduce the notion of a balloon, which we take from [9, pp. 103–104]. Note that the sets H and T in this definitions may be empty.

DEFINITION 5.2. Let W be a set and R a binary relation on W. R is a *reflexive (irreflexive) balloon* if W is the disjoint union of two sets $T = \{t_1, \ldots, t_n\}$ and H such that wRv iff either $w = t_i$ and $v = t_j$, for $i \leq j$ ($i < j$), or $v \in H$.

With these notions, many classes of frames defined by extensions of K4.3 can be described. For example, K4.3 itself is valid on just those relational frames whose point-generated subframes have an accessibility relation which is transitive and connected. With this, it is easy to see that K3 is valid on every linear order with an endpoint (a world which is accessible from every world). Aside from this, we will also need an account of the classes of relational frames defined by S4.3.1 and K4.3Z. This is provided by the following lemma:

LEMMA 5.3. Let $\mathfrak{F} = \langle W, R \rangle$ be a point-generated relational frame.
(a) S4.3.1 *is valid on* $\mathfrak{F}$ *just in case one of the following two conditions obtains*:
    (i) *R is a reflexive balloon.*
    (ii) $\mathfrak{F}$ *is isomorphic to* $\langle \mathbb{N}, \leq \rangle$.

(b) K4.3Z *is valid on* $\mathfrak{F}$ *just in case one of the following two conditions obtains*:
  (i) *R is an irreflexive balloon*.
  (ii) $\mathfrak{F}$ *is isomorphic to* $\langle \mathbb{N}, < \rangle$.

PROOF. The arguments are routine. A statement equivalent to (a) is also given by Reynolds and Zakharyaschev [49, p. 912].   ⊣

5.2. Decidable logics. In this section, we will show that the two logics just considered, S4.3.1 and K4.3Z, define classes of relational frames whose propositionally quantified modal logic is decidable. We do so by showing that the propositionally quantified modal logics of the various kinds of frames mentioned in Lemma 5.3 are decidable.

The case of the natural numbers under their natural order is already discussed by Fine [12, p. 344]. As Fine notes, the decidability of the propositionally quantified modal logic of the single frame consisting of the natural numbers and their natural ordering follows from the decidability of monadic second-order logic over this frame, and the decidability of this monadic second-order theory was shown by Büchi [6]. This monadic second-order theory is known in the literature as S1S. Fine does not specify whether the intended natural order is $\leq$ or $<$, but the difference is immaterial: it is easy to see that the monadic second-order theories of the two orders are recursively isomorphic. Thus:

LEMMA 5.4. *If* C *is the singleton of* $\langle \mathbb{N}, \leq \rangle$ *or the singleton of* $\langle \mathbb{N}, < \rangle$, *then* $\mathcal{L}(\mathsf{C})$ *is decidable*.

PROOF. Using the standard translation, this is immediate by Proposition 2.9 and the decidability of S1S.   ⊣

The remaining case of balloons requires a bit more thought. The most immediate difficulty is the fact that in a balloon, the cluster of worlds H among which the accessibility relation is universal may be of arbitrary size. It is initially not obvious that it is possible to suitably represent such an arbitrarily large set H using a finite or countably infinite set of natural numbers, in a reduction to S1S. However, using the techniques developed in Section 3, it is easy to see that H may always be assumed to be finite. With this, the reduction is straightforward.

LEMMA 5.5. *If* C *is the class of reflexive balloons or the class of irreflexive balloons, then* $\mathcal{L}(\mathsf{C})$ *is decidable*.

PROOF. We consider the irreflexive case; the reflexive case follows similarly. Along the lines of Lemma 5.4, it suffices to reduce $\mathcal{M}(\mathsf{C})$ to S1S, which we take to be the monadic second-order theory of the single frame $\langle \mathbb{N}, < \rangle$. First, note that if $\varphi \in \mathcal{N}^{\square}$ is satisfiable on some reflexive balloon $\mathfrak{F}$ in C, then by Proposition 3.12, φ is also satisfiable on $\mathfrak{F}^{2^n}_{\mathrm{id}}$, where $n = \mathrm{qd}(\varphi)$, and so on a frame in the class of finite reflexive balloons F. Thus $\mathcal{M}(\mathsf{C})$ is $\mathcal{M}(\mathsf{F})$, so we may reduce $\mathcal{M}(\mathsf{F})$ to S1S.

The required reduction is provided by the recursive mapping with the following non-trivial clauses (with t and h bound by initial existential quantifiers):

$$(Rxy)^{th} := Rxy \lor \neg Ryt,$$
$$(\forall x\psi)^{th} := \forall x(Rxh \to \psi^{th}).$$

This reduction simulates finite balloons using two numbers t and h as parameters: the numbers less than t and h represent T, and the remaining numbers less than h represent H. It is routine to show that φ is valid over F iff $\varphi^* \in$ S1S, as required.   ⊣

With these results, it is straightforward to establish our first example, which further delimits the axiomatizability boundary of Figure 1:

EXAMPLE 5.6. S4.3.1π+ is decidable.

PROOF. Let B be the class of reflexive balloons, and N the singleton of $\langle \mathbb{N}, \leq \rangle$. By Lemma 5.3(a), S4.3.1π+ is $\mathcal{L}(\mathsf{B}) \cap \mathcal{L}(\mathsf{N})$. The decidability of this logic follows from Lemmas 5.4 and 5.5.   ⊣

Moreover, Fine [13] has shown that all normal extensions of S4.3 are finitely axiomatizable. As in the case of KE, we can therefore conclude:

COROLLARY 5.7. *If Λ is a normal extension of* S4.3.1, *then* $\Lambda\pi$+ *is decidable.*

It is worth noting that Segerberg [51] showed in the standard quantifier-free setting that S4.3.1 is the modal logic of the single frame $\langle \mathbb{N}, \leq \rangle$; *a fortiori*, it is also the modal logic of the class of frames it defines. Although these classes of frames have the same quantifier-free modal logic, and both give rise to decidable propositionally quantified modal logics (by Lemma 5.4 and Example 5.6), these propositionally quantified modal logics are not the same: with the additional expressive power of propositional quantifiers, balloons can be distinguished from $\langle \mathbb{N}, \leq \rangle$, for example using the sentence $\Diamond \bar{\forall} \mathbf{5}$.

Our second example is established in just the same way as the first:

EXAMPLE 5.8. K4.3Zπ+ is decidable.

PROOF. Analogous to Example 5.6.   ⊣

We can use this example to justify the earlier assertion that the requirement of finiteness in Proposition 2.8 cannot be weakened to recursive enumerability, even among quantifier-free sets of formulas: there is a normal modal logic Λ obtained by strengthening K4.3Z using a recursively enumerable set of axioms such that $\Lambda\pi$+ is undecidable. This follows from the following result:

PROPOSITION 5.9. *Any set $x \subseteq \mathbb{N}$ is reducible to* $(\mathrm{K4.3Z} + \Gamma_x)\pi$+, *where* $\Gamma_x = \{\Box^n \Diamond \top : n \in x\}$.

PROOF. We show that $n \in x$ iff $\Box^n \Diamond \top \in (\mathrm{K4.3Z} + \Gamma_x)\pi$+. The left to right direction is immediate. For the right to left direction, assume $n \notin x$, and let $\mathfrak{F}$ be a finite irreflexive linear order of length $n + 1$. $\mathfrak{F}$ validates K4.3Z + Γ_x but not $\Box^n \Diamond \top$, whence $\Box^n \Diamond \top \notin (\mathrm{K4.3Z} + \Gamma_x)\pi$+.   ⊣

If $x \subseteq \mathbb{N}$ is recursively enumerable but not decidable, it follows that $(\mathrm{K4.3Z} + \Gamma_x)\pi$+ is not decidable either, even though K4.3Z + Γ_x extends K4.3Z using a recursively enumerable set Γ_x, and K4.3Zπ+ is decidable. Similarly, if $x \subseteq \mathbb{N}$ is not recursively enumerable, then neither is $(\mathrm{K4.3Z} + \Gamma_x)\pi$+, so the axiomatizability boundary cannot be extended to all normal modal logics.

5.3. Unaxiomatizable logics. After noting that the decidability of $\mathcal{L}(\{\langle \mathbb{N}, \leq \rangle\})$ follows from Büchi's result (see the previous section), Fine [12, p. 344] asserts that the decidability of S4.3π+ follows from a generalization of Büchi's result by Rabin [48]. This, however, is a mistake: Rabin's result applies to SkS, for any $k \in \mathbb{N} \cup \{\omega\}$, which is the monadic second-order theory of the infinite tree in which every node has exactly k successors. But S4.3 imposes substantially weaker constraints on frames. For example, any linear order is a relational frame validating S4.3. According to Kaminski and Tiomkin [26, p. 41], it follows from this that S4.3π+ is not axiomatizable.

Kaminski and Tiomkin outline the following argument: Using results of Shelah [54] and Gurevich and Shelah [21], second-order arithmetic can be reduced to the monadic second-order theory of linear orders. The monadic second-order theory of linear orders can in turn be reduced to S4.3π+. (Kaminski and Tiomkin note that under a "weak" set-theoretic assumption, results of Gurevich and Shelah [20] show that second-order *logic* can be reduced to the monadic second-order theory of linear orders. Under these assumptions, we therefore obtain that S4.3π+ is not just not axiomatizable, but recursively isomorphic to second-order logic.) In this section, we fill in some details of this observation, and note certain ways in which it can be strengthened.

The reduction of monadic second-order logic over linear orders to S4.3π+ can be effected using the backwards translation, introduced in Section 2.4. The crucial feature of S4.3 which makes this possible is the fact that every linear order is a subframe of a transitive point-generated subframe of a frame in R(S4.3). We therefore present the result in a more general form, applicable to any class of relational frames with this property. Analogous to the results in Section 4, the failure of axiomatizability extends to any smaller set of formulas which includes Kπ+.

PROPOSITION 5.10. *Let* C *be a class of relational frames such that every linear order is a subframe of a transitive point-generated subframe of a frame in* C. *No set of formulas* $\Pi \subseteq \mathcal{L}^{\Box}$ *such that* Kπ+ $\subseteq \Pi \subseteq \mathcal{L}(\mathsf{C})$ *is axiomatizable.*

PROOF. By the results of Shelah and Gurevich mentioned above, it suffices to reduce monadic second-order logic over linear orders to Π. We do so by the function mapping any monadic second-order sentence φ to

$$\varphi^* := \bar{\forall}\mathbf{4} \to (\forall X (\text{Lin} \to \varphi)^X)^{\leftarrow},$$

where Lin is the (first-order) sentence stating that R is a linear order, and ψ^X is the restriction of first-order quantifiers in ψ to X. We show that φ is valid over linear orders if and only if $\varphi^* \in \Pi$.

Assume φ is false in some linear order. By assumption, the order is a subframe $\mathfrak{L}$ of a transitive point-generated subframe $\mathfrak{F}_w$ of a frame $\mathfrak{F}$ in C. Then $\forall X (\text{Lin} \to \varphi)^X$ is false in $\mathfrak{F}_w$. Since $\mathfrak{F}_w$ is transitive, it follows by Proposition 2.10 that $\mathfrak{F}_w, w \nVdash (\forall X (\text{Lin} \to \varphi)^X)^{\leftarrow}$, and so that $\mathfrak{F}_w, w \nVdash \varphi^*$. With Proposition 2.7, $\varphi^* \notin \mathcal{L}(\mathsf{C})$, and so $\varphi^* \notin \Pi$.

Assume $\varphi^* \notin \Pi$. Then $\varphi^* \notin$ Kπ+, so there is a relational frame $\mathfrak{F}$ and world w in which $\bar{\forall}\mathbf{4}$ is true but $(\forall X (\text{Lin} \to \varphi)^X)^{\leftarrow}$ is false (relative to any assignment function). Then $\mathfrak{F}_w$ has a transitive accessibility relation, and so by Propositions 2.7

and 2.10, $\forall X(\text{Lin} \to \varphi)^X$ is false in $\mathfrak{F}_w$. So there is a (subframe of $\mathfrak{F}_w$ which is a) linear order in which φ is false. Therefore φ is not valid over linear orders.   ⊣

From this result, we can conclude that $\text{S4.3}\pi+$ is not axiomatizable. This extends to every normal modal logic included in K3:

EXAMPLE 5.11.   For every normal modal logic $\Lambda \subseteq \text{K3}$, $\Lambda\pi+$ is not axiomatizable.

PROOF.   Let $\mathfrak{L}$ be a reflexive linear order, and $\mathfrak{F}$ the result of adding to $\mathfrak{L}$ an initial element i and a final element f. Then $\mathfrak{F}$ is a member of $\mathsf{R}(\text{K3})$, while $\mathfrak{L}$ is a subframe of $\mathfrak{F}_i$. So, $\mathsf{R}(\text{K3})$ satisfies the condition of Proposition 5.10, from which the claim follows.   ⊣

§6. Multimodal logics. So far, we have considered only examples of unimodal logics. But, as urged already by Scott [50, p. 161], applications of modal logics very often require multiple modalities. We therefore turn to the multimodal case. This opens up a vast field of new complexity questions, and we will not attempt as systematic an investigation as we aimed to provide for the unimodal case.

6.1. Fusions. Many interesting multimodal logics can be obtained using general methods for combining unimodal logics. There are different ways of combining modal logics; see [35] for an introduction. The simplest way of combining two normal modal logics Λ and Λ' is the formation of their fusion, which is the normal bimodal logic axiomatized by the theorems of Λ for one modality, and those of Λ' for the other modality. This definition straightforwardly generalizes to fusions of arbitrary countable sets of normal modal logics. For the following definition, recall that we use K for the smallest normal modal logic in whatever modal language is contextually salient, which need not be unimodal.

DEFINITION 6.1.   Let I be a countable set, and for each $i \in I$, let $\Lambda_i \subseteq \mathcal{L}^{\Box_i}$ be a normal unimodal logic. Then the *fusion of* Λ_i (for $i \in I$), written $\bigotimes_{i \in I} \Lambda_i$, is $\mathsf{K} + \bigcup_{i \in I} \Lambda_i$, the smallest normal modal logic in $\mathcal{L}^{\{\Box_i : i \in I\}}$ including $\bigcup_{i \in I} \Lambda_i$.

In the case of combining two logics, we write their fusion as $\Lambda_1 \otimes \Lambda_2$. When combining unimodal logics, we assume tacitly that they employ distinct modal operators. For example, we simply write $\text{S5} \otimes \text{S5}$ for the fusion of S5 in a unimodal language using different modal operators. We leave the choice of these modal operators unspecified, and choose them according to the relevant context.

For our purposes, fusions involving unimodal logics below the axiomatizability boundary in Figure 1 are uninteresting, since the resulting propositionally quantified modal logics will only be axiomatizable in the trivial case in which one of the fused logics is inconsistent (i.e., identical to $\mathcal{L}^{\Box}$). This follows from the following observation.

PROPOSITION 6.2.   *Let $\bigotimes_{i \in I} \Lambda_i$ be a fusion of consistent normal unimodal logics. Then for all $i \in I$, $(\bigotimes_{i \in I} \Lambda_i)\pi+$ is a conservative extension of $\Lambda_i\pi+$, in the sense that for all $\varphi \in \mathcal{L}^{\Box_i}$, $\varphi \in (\bigotimes_{i \in I} \Lambda_i)\pi+$ iff $\varphi \in \Lambda_i\pi+$.*

PROOF.   Consider any $i \in I$, and $\mathcal{L}^{\Box_i}$-formula $\varphi \notin \Lambda_i\pi+$. Then there is a relational frame $\langle W, R \rangle$ validating Λ_i but not φ. As shown by Makinson [38], any consistent normal unimodal logic is included in either Triv or Ver. Let

$J = \{j \in I : \Lambda_j \subseteq \text{Triv}\}$. Let $\mathfrak{F}$ be the relational frame $\langle W, R_{\square_j}\rangle_{j \in I}$ such that $R_i = R$, $R_j = \text{id}$ for $j \in J$, and $R_j = \emptyset$ in all other cases. Then $\mathfrak{F}$ validates $\bigotimes_{i \in I} \Lambda_i$ but not φ. ⊣

We therefore consider fusions of normal modal logics above the axiomatizability boundary in Figure 1. One such logic has already been considered in the literature on propositionally quantified modal logics: Antonelli and Thomason [1] show that $(\text{S5} \otimes \text{S5})\pi+$ is recursively isomorphic to second-order logic. Proofs of this result can also be found in Kuhn [34] and Belardinelli et al. [2]. Since S5 is already a highly restrictive logic, this indicates that only very rare fusions of unimodal logics define classes of frames whose propositionally quantified modal logics are axiomatizable.

One example for such a rare fusion is the finite fusion of normal modal logics of the form KAlt_n. For any $n \in \mathbb{N}$, the unimodal logic KAlt_n is valid on just those relational frames in which every world can access at most n worlds. Since the satisfiability of a formula with modal depth m is invariant under taking m-step point-generated subframes, the size of frames which need to be considered to determine satisfiability is finitely bounded in terms of m and the indices n of the fused logics.

EXAMPLE 6.3. Let I be a finite set and, $n_i \in \mathbb{N}$ for all $i \in I$. Then $(\bigotimes_{i \in I} \text{KAlt}_{n_i})\pi+$ is decidable.

PROOF. Let φ be a formula of modal depth m which is satisfiable on a frame validating $(\bigotimes_{i \in I} \Lambda_i)\pi+$. By Proposition 2.7, φ is satisfiable on an m-step-generated subframe, which is bounded in size by $\Sigma_{j=0}^m (\Sigma_{i \in I} n_i)^j$. Decidability follows, since such subframes are recursively enumerable. ⊣

The decidability of $\text{KAlt}_n\pi+$ follows as a special case; this establishes the last outstanding claim presented in Figure 1. This decidability result in the unimodal case was already noted by ten Cate [8, p. 222]. ten Cate also observes that the corresponding monadic second-order theory $\mathcal{M}^\square(\text{KAlt}_n)$ is undecidable as long as $n \geq 2$. Indeed, it is easy to see that accessibility relations of such frames may serve to encode a pairing function, so that these monadic second-order theories are recursively isomorphic to second-order logic. This highlights that despite the existence of the backwards translation on transitive frames, $\mathcal{L}^O$ is in general less expressive than $\mathcal{M}^O$. For more on questions of expressivity, see [8].

6.2. Products and linear fusions. Many applications of multimodal logics require non-trivial interaction axioms between the modalities. For example, consider a combined modal-temporal logic with two modal operators, $\square$ for *necessary* and A for *always*. One might plausibly think that what is necessarily always the case is always necessary, and vice versa. That is, one might include in this modal-temporal logic the principle:

$$\square \text{A} p \leftrightarrow \text{A} \square p.$$

The particular interaction principles to include in a given multimodal logic will of course depend on the application, but there are some general ways of combining modal logics which lead to non-trivial interaction axioms, and which arise naturally in many contexts.

Continuing with the example of a modal-temporal logic, consider the following simple way of constructing relational frames for such a language: We take a set

of alethically possible worlds W as given, as well as a set of times T. The worlds of our frame are pairs of alethically possible worlds and times, i.e., the members of $W \times T$. The accessibility relations for the two modalities operate on the respective components of these pairs: $R_\square$ relates pairs with the same time, and R_A relates pairs with the same alethically possible world. Such a class of frames for a combined modal-temporal logic was influentially developed by Kaplan [28]. It is an instance of a very general way of combining unimodal frames into a multimodal frame, and by extension combining normal unimodal logics to a normal multimodal logic. This is known as taking *products*.

Given the results established above, the most natural question concerning the axiomatizability of propositionally quantified modal logic over product frames is whether $\mathcal{L}(\mathsf{PE})$ is axiomatizable, where PE is the class of products of unimodal relational frames in which the accessibility relation is an equivalence relation. The resulting logic is a strengthening of $(\mathsf{S5} \otimes \mathsf{S5})\pi+$ which adds various interaction axioms, including the one stated above for $\square$ and A. This question is answered by Fritz [16], who shows that this extension is also recursively isomorphic to second-order logic.

In the case of modal-temporal logics, Montague [39] took a slightly different approach: like Kaplan, he started from a Cartesian product $W \times T$, with the same relation for the temporal modality A. However, Montague's accessibility relation for the alethic modality $\square$ is universal. More generally, Dorr and Goodman [11] argue that the correct logic of $\square$ and A includes the principle $\square p \to Ap$: what is necessary is always the case. On relational frames, this principle is valid just in case R_A is a subset of $R_\square$. Abstracting from Montague's particular models, this suggests a natural class of frames for our modal-temporal language $\mathcal{L}^{\square A}$, namely the class of bimodal frames in which both accessibility relations are equivalence relations, with R_A being a refinement of $R_\square$. This class is defined by $\mathsf{S5} \otimes \mathsf{S5} + \square p \to Ap$.

This example suggest a simple general way of combining any two normal unimodal logics Λ_1 and Λ_2 (in this order), the result of which we call their *linear* fusion. It is obtained by adding to the fusion of Λ_1 and Λ_2 the axiom $\square_1 p \to \square_2 p$, where $\square_1$ and $\square_2$ are the modalities of Λ_1 and Λ_2, respectively. This straightforwardly generalizes to any finite sequence of normal modal logics:

DEFINITION 6.4. Let $\Lambda_0, \dots, \Lambda_n$ be normal unimodal logics in languages $\mathcal{L}^{\square_0}, \dots, \mathcal{L}^{\square_n}$, respectively. Then the *linear fusion of* $\Lambda_0, \dots, \Lambda_n$ is defined as follows:

$$\Lambda_0 \vec{\otimes} \cdots \vec{\otimes} \Lambda_n := \Lambda_0 \otimes \cdots \otimes \Lambda_n + \square_0 p \to \square_1 p + \cdots + \square_{n-1} p \to \square_n p.$$

We show that in contrast to Kaplan's product frames, the Montague-inspired approach to modal-temporal logic via linear fusions leads to a decidable propositionally quantified modal logic, as $(\mathsf{S5} \vec{\otimes} \mathsf{S5})\pi+$ is decidable. Indeed, this extends to the linear fusion of any finite number of copies of S5.

EXAMPLE 6.5. $(\mathsf{S5} \vec{\otimes} \cdots \vec{\otimes} \mathsf{S5})\pi+$ is decidable.

PROOF. We discuss the case of $(\mathsf{S5} \vec{\otimes} \mathsf{S5})\pi+$; it is clear that the argument can be iterated for longer linear fusions. Our main task is to show that $\mathsf{R}(\mathsf{S5} \vec{\otimes} \mathsf{S5})$ is computably reducible. We do so by showing that it is r-reducible, for $r : n \mapsto 2^{(2^{n+1}n)}$.

So let $n \in \mathbb{N}$, and consider any point-generated subframe $\mathfrak{F} = \langle W, R_\square, R_\boxtimes \rangle$ of a frame validating S5 $\vec{\otimes}$ S5. We use $\mathcal{L}^{\square\boxtimes}$ as the bimodal language of S5 $\vec{\otimes}$ S5, assuming that the latter contains $\square p \rightarrow \boxtimes p$. $\mathfrak{F}$ will be reduced to a finite frame in two stages. First, let $\mathfrak{F}' = \langle W', R'_\square, R'_\boxtimes \rangle$ be the frame $\mathfrak{F}^{2^n}_{\mathrm{id}}$. By Proposition 3.12, $\mathfrak{F}$ and $\mathfrak{F}'$ are n-equivalent. Second, let $\mathfrak{F}''$ be the frame $\mathfrak{F}'^{2^{(2^n n)}}_{R'_\boxtimes}$. Since the equivalence classes of $R'_\boxtimes$ are bounded in size by 2^n, it follows with Proposition 3.12 that $\mathfrak{F}'$ and $\mathfrak{F}''$ are n-equivalent as well.

Since equivalence classes of $R'_\boxtimes$ are bounded in size by 2^n, and both $R'_\square$ and $R'_\boxtimes$ are universal within such an equivalence relation, there are only up to 2^n duplicate classes of such equivalence classes in $\mathfrak{F}'$. Therefore, $\mathfrak{F}''$ is bounded in size by $2^n \cdot 2^{(2^n n)} = r(n)$. By construction, it is easy to see that $\mathfrak{F}''$ also validates S5 $\vec{\otimes}$ S5. Thus R(S5 $\vec{\otimes}$ S5) is r-reducible. Since S5 $\vec{\otimes}$ S5 is finitely axiomatizable, it follows with Lemma 3.2 that (S5 $\vec{\otimes}$ S5)$\pi+$ is decidable. $\dashv$

Along the lines of Example 6.3, it is easy to see that for any linear fusion $\Lambda = \Lambda_0 \vec{\otimes} \cdots \vec{\otimes} \Lambda_n$, if Λ_0 is KAlt$_m$ for some $m \in \mathbb{N}$ and every other Λ_i is KΓ_i for some finite set of axioms Γ_i, then $\Lambda\pi+$ is decidable. However, already (S5 $\vec{\otimes}$ KAlt$_2$)$\pi+$ is recursively isomorphic to second-order logic. This follows by a straightforward pairing argument, similar to the case of $\mathcal{M}^\square$(KAlt$_n$) for $n \geq 2$, discussed above.

§7. Conclusion. In this paper, we have developed general methods to answer questions concerning the axiomatizability of propositionally quantified modal logics on classes of relational frames. Central among these methods is the technique of reducing the number of duplicates of equivalence classes of worlds in relational frames. This led directly to several new decidability results, including the decidability of KE$\pi+$ (Example 3.15). The technique can also fruitfully be combined with standard decidability arguments, such as reductions to S1S, as we did in order to show that S4.3.1$\pi+$ is decidable (Example 5.6). Finally, we showed how the reductions of duplicates can be iterated, which allowed us to prove that the multimodal logic (S5 $\vec{\otimes}$ S5)$\pi+$ is decidable (Example 6.5). We also improved an existing construction for showing propositionally quantified modal logics of classes of relational frames to be recursively isomorphic to second-order logic. This allowed us to simplify proofs for known cases, and to cover new cases such as K4.2W$\pi+$ (Example 4.10).

The various examples used to illustrate these new and improved methods amount to a detailed picture of which normal unimodal logics define classes of relational frames whose propositionally quantified modal logics are axiomatizable, which is presented in Figure 1. For many of the commonly considered normal modal logics Λ, the complexity of $\Lambda\pi+$ decreases (weakly) monotonically with the strength of Λ. Among these logics, the question of the axiomatizability of $\Lambda\pi+$ therefore introduces what Ding [10, p. 1148] calls the "axiomatizability boundary," which is indicated in Figure 1. This boundary, as well as a corresponding decidability boundary, gives rise to further problems which are left open here.

First, even though we have developed general methods, our approach to locating the axiomatizability boundary has for the most part still been driven by considering

particular examples. In some cases, we were able to obtain more general results, such as the observation that $\Lambda\pi+$ is decidable for every normal extension Λ of KE (Corollary 3.16), but much of the lattice of normal modal logics is not covered by any such general result established here.

Second, one might also take a more abstract perspective on the issue, and ask whether the property of having an axiomatizable propositionally quantified extension $\Lambda\pi+$, for axiomatizable normal modal logics Λ, is itself decidable. For an overview of similar complexity questions concerning metalogical properties of modal logics, see [9, Chapter 17].

Third, it is not yet clear exactly how much of the lattice of normal modal logics divides neatly along the kind of axiomatizability boundary discussed here. Proposition 2.8 describes a relatively rich space of normal modal logics which does provide such a boundary. However, it follows from Proposition 5.9 that this cannot be extended to all normal modal logics. This limits, but does not pin down exactly, the spaces of normal modal logics to which we may be able to extend the axiomatizability boundary.

Finally, our discussion has been especially unsystematic in the multimodal case. This is partly due to the fact that this case has not been considered in the literature, with the exception of the particular matter of normal bimodal logics in which both unimodal fragments obey the principles of S5: It is known that in the case of S5, the two most common methods of combining modal logics, namely fusions and products, both lead to a propositionally quantified bimodal logic which is recursively isomorphic to second-order logic. To this we have added a decidability result, showing that taking the *linear* fusion of S5 with itself leads to a decidable propositionally quantified bimodal logic (Example 6.5). However, many further complexity questions are left open in this area, including many questions on systems with interaction axioms between the modalities motivated by particular applications.

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