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BOUNDS FOR THE DEVIATION OF A FUNCTION FROM THE CHORD GENERATED BY ITS EXTREMITIES

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Abstract

Sharp bounds for the deviation of a real-valued function f defined on a compact interval $[a, b]$ to the chord generated by its end points $(a, f(a))$ and $(b, f(b))$ under various assumptions for f and f', including absolute continuity, convexity, bounded variation, and monotonicity, are given. Some applications for weighted means and f-divergence measures in information theory are also provided.

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1. Introduction

Consider a function $f : [a, b] \to \mathbb{R}$ and assume that it is bounded on $[a, b]$. The chord that connects its end points $A = (a, f(a))$ and $B = (b, f(b))$ has the equation

$$d_f : [a, b] \to \mathbb{R}, \quad d_f(t) = \frac{1}{b-a}[f(a)(b-t) + f(b)(t-a)].$$

We introduce the error in approximating the value of the function $f(t)$ by $d_f(t)$ with $t \in [a, b]$ by $\Phi_f(t)$, that is, $\Phi_f(t)$ is defined by:

$$\Phi_f(t) := \frac{b-t}{b-a} \cdot f(a) + \frac{t-a}{b-a} \cdot f(b) - f(t). \tag{1.1}$$

The main aim of this paper is to provide sharp upper bounds for the absolute value of the difference $\Phi_f(t)$ at each point $t \in [a, b]$ and under various assumptions on the function f or its derivative f'.

In Section 2, we recall some results in the case when f is bounded below by m and above by M and when f is convex on $[a, b]$.

In Section 3, the case when f is of bounded variation and in particular Lipschitzian or monotonic nondecreasing is analyzed, while in Section 4 the case of absolutely continuous functions is investigated.

Sections 5, 6 and 7 provide sharp bounds for $|\Phi_f(t)|$, $t \in [a, b]$ when f' is of bounded variation, Lipschitzian or absolutely continuous.

In Section 8 some applications in estimating the weighted mean generated by f, namely

$$M_f(p, x) := \sum_{i=1}^{n} p_i f(x_i), \tag{1.2}$$

where $p_i \geq 1$, $\sum_{i=1}^{n} p_i = 1$ and $m \leq x_i \leq M$, $i \in \{1, \ldots, n\}$, for a function f defined on an interval containing m and M are also given.

Finally, applications for the *f-divergence functional*

$$I_f(\mathbf{p}, \mathbf{q}) := \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right), \tag{1.3}$$

where $\mathbf{p} = (p_1, \ldots, p_n)$, $\mathbf{q} = (q_1, \ldots, q_n)$ are positive sequences, that was introduces by Csiszár in [4], as a generalized measure of information, a 'distance function' on the set of probability distributions $\mathbb{P}^n$ are also provided.

2. Preliminary results

The following simple result, which provides a sharp upper bound for the case of bounded functions, was stated in [7] as an intermediate result needed to obtain a Grüss type inequality.

THEOREM 2.1. *If $f : [a, b] \to \mathbb{R}$ is a bounded function with $-\infty < m \leq f(t) \leq M < \infty$ for any $t \in [a, b]$, then*

$$|\Phi_f(t)| \leq M - m. \tag{2.1}$$

The multiplicative constant 1 in front of $M - m$ cannot be replaced by a smaller quantity.

PROOF. For the sake of completeness, we present a short proof.

Since f is bounded,

$$m(b-t) \leq (b-t)f(a) \leq (b-t)M, \quad m(t-a) \leq (t-a)f(b) \leq (t-a)M \quad \text{and}$$
$$-(b-a)M \leq -(b-a)f(t) \leq -(b-a)m,$$

so that, by addition and division by $b - a$,

$$-(M-m) \leq \frac{(b-t)f(a) + (t-a)f(b)}{b-a} - f(t) \leq M - m,$$

for each $t \in [a, b]$, that is, the desired inequality (2.1) holds.

Now assume that there exists a constant $C > 0$ such that $|\Phi_f(t)| \leq C(M - m)$ for any f as in the statement of the theorem. Then, for $t = (a + b)/2$, the inequality

$$\left|\frac{f(a) + f(b)}{2} - f\left(\frac{a+b}{2}\right)\right| \leq C(M - m) \tag{2.2}$$

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should hold. If $f : [a, b] \to \mathbb{R}$, $f(t) = |t - (a + b)/2|$, then $f(a) = f(b) = (b - a)/2$, $f((a + b)/2) = 0$, $M = (b - a)/2$ and $m = 0$ and (2.2) becomes $(b - a)/2 \leq C \cdot (b - a)/2$, which implies that $C \geq 1$. □

The case of convex functions was considered in [8] in order to prove another Grüss type inequality. The sharpness of the constant was not addressed in that paper.

THEOREM 2.2. *If $f : [a, b] \to \mathbb{R}$ is a convex function on $[a, b]$, then*

$$0 \leq \Phi_f(t) \leq \frac{(b-t)(t-a)}{b-a}[f'_-(b) - f'_+(a)] \leq \frac{1}{4}(b-a)[f'_-(b) - f'_+(a)] \qquad (2.3)$$

for any $t \in [a, b]$.

If the lateral derivatives $f'_-(b)$ and $f'_+(a)$ are finite, then the second inequality and the constant $\frac{1}{4}$ are sharp.

PROOF. For the sake of completeness, we present a complete proof of (2.3).

Since f is convex, then

$$\frac{t-a}{b-a} \cdot f(b) + \frac{b-t}{b-a} \cdot f(a) \geq f\left[\frac{(b-t)a + (t-a)b}{b-a}\right] = f(t)$$

for any $t \in [a, b]$, that is, $\Phi(t) \geq 0$ for any $t \in [a, b]$.

If either $f'_-(b)$ or $f'_+(a)$ is infinite, then the last part of (2.3) is obvious.

Suppose that $f'_-(b)$ and $f'_+(a)$ are finite. Then, by the convexity of f,

$$f(t) - f(b) \geq f'_-(b)(t - b) \quad \text{for any } t \in (a, b).$$

If we multiply this inequality by $t - a \geq 0$, we deduce that

$$(t-a)f(t) - (t-a)f(b) \geq f'_-(b)(t-b)(t-a), \quad t \in (a, b). \qquad (2.4)$$

Similarly,

$$(b-t)f(t) - (b-t)f(a) \geq f'_+(a)(t-a)(b-t), \quad t \in (a, b). \qquad (2.5)$$

Adding (2.4) to (2.5) and dividing by $b - a$, we deduce that

$$f(t) - \frac{(t-a)f(b) + (b-t)f(a)}{b-a} \geq \frac{(b-t)(t-a)}{b-a}[f'_-(b) - f'_+(a)],$$

for any $t \in (a, b)$, which proves the second inequality for $t \in (a, b)$.

If $t = a$ or $t = b$, the inequality also holds.

Now, assume that (2.3) holds with D and E greater than zero, that is,

$$\Phi_f(t) \leq D \cdot \frac{(b-t)(t-a)}{b-a}[f'_-(b) - f'_+(a)] \leq E(b-a)[f'_-(b) - f'_+(a)]$$

for any $t \in [a, b]$. If we choose $t = (a + b)/2$, then

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$$\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right) \le \frac{1}{4}D(b-a)[f'_-(b) - f'_+(a)]$$
$$\le E(b-a)[f'_-(b) - f'_+(a)]. \tag{2.6}$$

Consider $f : [a, b] \to \mathbb{R}$, $f(t) = |t - (a+b)/2|$. Then f is convex, $f(a) = f(b) = (b-a)/2$, $f((a+b)/2) = 0$, $f'_-(b) = 1$, $f'_+(a) = -1$ and by (2.6) we deduce that

$$\frac{b-a}{2} \le \frac{1}{2}D(b-a) \le 2E(b-a),$$

which implies that $D \ge 1$ and $E \ge \frac{1}{4}$. □

3. The case when f is of bounded variation

We start with the following representation result.

LEMMA 3.1. *If $f : [a, b] \to \mathbb{R}$ is bounded on $[a, b]$ and $Q : [a, b]^2 \to \mathbb{R}$ is defined by*

$$Q(t, s) := \begin{cases} t-b & \text{if } a \le s \le t, \\ t-a & \text{if } t < s \le b, \end{cases} \tag{3.1}$$

then we can write

$$\Phi_f(t) = \frac{1}{b-a}\int_a^b Q(t, s)\, df(s), \qquad t \in [a, b], \tag{3.2}$$

where the integral in (3.2) is in the sense of Riemann–Stieltjes.

PROOF. We need only write

$$\begin{aligned}\int_a^b Q(t, s)\, df(s) &= \int_a^t (t-b)\, df(s) + \int_t^b (t-a)\, df(s) \\ &= (t-b)\int_a^t df(s) + (t-a)\int_t^b df(s) \\ &= (t-b)[f(t) - f(a)] + (t-a)[f(b) - f(t)] \\ &= (b-a)\Phi_f(t)\end{aligned}$$

and the identity is proved. □

The following estimation result holds.

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THEOREM 3.2. *If $f:[a,b]\to\mathbb{R}$ is of bounded variation, then*

$$
|\Phi_f(t)| \le \left(\frac{b-t}{b-a}\right)\cdot\bigvee_a^t(f) + \left(\frac{t-a}{b-a}\right)\cdot\bigvee_t^b(f)
$$
$$
\le \begin{cases} \left[\frac{1}{2}+\left|\frac{t-(a+b)/2}{b-a}\right|\right]\bigvee_a^b(f), \\ \left[\left(\frac{b-t}{b-a}\right)^p+\left(\frac{t-a}{b-a}\right)^p\right]^{1/p}\left[\left(\bigvee_a^t(f)\right)^q+\left(\bigvee_t^b(f)\right)^q\right]^{1/q} \\ \quad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^t(f)-\bigvee_t^b(f)\right|. \end{cases} \tag{3.3}
$$

The first inequality in (3.3) is sharp. The constant $\frac{1}{2}$ is the best possible in the first and third branches.

PROOF. We use the fact that, for $p:[\alpha,\beta]\to\mathbb{R}$ continuous and $v:[\alpha,\beta]\to\mathbb{R}$ of bounded variation, the Riemann–Stieltjes integral $\int_\alpha^\beta p(t)\,dv(t)$ exists and

$$
\left|\int_\alpha^\beta p(t)\,dv(t)\right| \le \sup_{t\in[\alpha,\beta]}|p(t)|\bigvee_\alpha^\beta(v).
$$

Then, by (3.2),

$$
\begin{aligned}
|\Phi_f(t)| &\le \frac{1}{b-a}\left|(t-b)\int_a^t df(s)+(t-a)\int_t^b df(s)\right| \\
&\le \frac{1}{b-a}\left[(b-t)\left|\int_a^t df(s)\right|+(t-a)\left|\int_t^b df(s)\right|\right] \\
&\le \frac{1}{b-a}\left[(b-t)\bigvee_a^t(f)+(t-a)\bigvee_t^b(f)\right],
\end{aligned}
$$

and the first inequality in (3.3) is proved.

Now, by the Hölder inequality,

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$$(b-t)\bigvee_a^t(f)+(t-a)\bigvee_t^b(f)\le\begin{cases}\max\{b-t,t-a\}\left[\bigvee_a^t(f)+\bigvee_t^b(f)\right],\\ [(b-t)^p+(t-a)^p]^{1/p}\\ \quad\times\left[\left(\bigvee_a^t(f)\right)^q+\left(\bigvee_t^b(f)\right)^q\right]^{1/q}\\ \qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ (b-t+t-a)\max\left\{\bigvee_a^t(f),\bigvee_t^b(f)\right\},\end{cases}$$

which produces the last part of (3.3).

For $t=\frac{1}{2}(a+b)$, (3.3) becomes

$$\left|f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{2}\right|\le\frac{1}{2}\bigvee_a^b(f).$$

Assume that there exists a constant $A>0$ such that

$$\left|f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{2}\right|\le A\bigvee_a^b(f). \tag{3.4}$$

If in this inequality we choose $f:[a,b]\to\mathbb{R}$, $f(t)=|t-(a+b)/2|$, then we deduce that $(b-a)/2\le A(b-a)$, which implies that $A\ge\frac{1}{2}$. □

COROLLARY 3.3. *If $f:[a,b]\to\mathbb{R}$ is L_1-Lipschitzian on $[a,t]$ and L_2-Lipschitzian on $[t,b]$, $L_1,L_2>0$, then*

$$\left|\Phi_f(t)\right|\le\frac{(b-t)(t-a)}{b-a}(L_1+L_2)\le\frac{1}{4}(b-a)(L_1+L_2) \tag{3.5}$$

for any $t\in[a,b]$. In particular, if f is L-Lipschitzian on $[a,b]$, then

$$\left|\Phi_f(t)\right|\le\frac{2(b-t)(t-a)}{b-a}L\le\frac{1}{2}(b-a)L. \tag{3.6}$$

The constants $\frac{1}{4}$, 2 and $\frac{1}{2}$ are the best possible.

The proof is obvious by Theorem 3.2 on taking into account that any L-Lipschitzian function is of bounded variation and $\bigvee_a^b(f)\le(b-a)L$. The sharpness of the constants follows by choosing the function $f:[a,b]\to\mathbb{R}$, $f(t)=|t-(a+b)/2|$ which is Lipschitzian with $L=1$.

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COROLLARY 3.4. *If $f:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then*

$$|\Phi_f(t)| \le \left(\frac{b-t}{b-a}\right)[f(t)-f(a)] + \left(\frac{t-a}{b-a}\right)[f(b)-f(t)]$$
$$\le \begin{cases} \left[\dfrac{1}{2}+\left|\dfrac{t-(a+b)/2}{b-a}\right|\right][f(b)-f(a)], \\ \left[\left(\dfrac{b-t}{b-a}\right)^p+\left(\dfrac{t-a}{b-a}\right)^p\right]^{1/p}[[f(t)-f(a)]^q+[f(b)-f(t)]^q]^{1/q} \\ \quad \text{if } p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1, \\ \dfrac{1}{2}[f(b)-f(a)]+\dfrac{1}{2}\left|f(t)-\dfrac{f(a)+f(b)}{2}\right|. \end{cases} \tag{3.7}$$

The first inequality and the constant $\frac{1}{2}$ in the first branch of the second inequality are sharp.

The inequality is obvious from (3.3). For $t=(a+b)/2$, we get in (3.7) that

$$\left|f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{2}\right| \le \frac{1}{2}[f(b)-f(a)]. \tag{3.8}$$

In (3.8), the constant $\frac{1}{2}$ is sharp since for the monotonic nondecreasing function $f:[a,b]\to\mathbb{R}$,

$$f(t)=\begin{cases} 0 & \text{if } t\in\left[a,\dfrac{a+b}{2}\right], \\ 1 & \text{if } t\in\left(\dfrac{a+b}{2},b\right], \end{cases}$$

we obtain in both sides of (3.8) the same quantity $\frac{1}{2}$.

4. The case when f is absolutely continuous

Now, if $f:[a,b]\to\mathbb{R}$ is absolutely continuous, then f is differentiable almost everywhere and $\int_a^b f'(s)\,ds = f(b)-f(a)$, where the integral is taken in the Lebesgue sense, and we can state the following representation result.

LEMMA 4.1. *If $f:[a,b]\to\mathbb{R}$ is absolutely continuous, then*

$$\Phi_f(t)=\frac{1}{b-a}\int_a^b Q(t,s)f'(s)\,ds, \qquad t\in[a,b], \tag{4.1}$$

where the integral is in the Lebesgue sense and Q is as defined in (3.1).

The proof is similar to that of Lemma 3.1 and the details are omitted.

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We define the Lebesgue p-norms as follows:

$$\|g\|_{[\alpha,\beta],s} := \begin{cases} \operatorname*{ess\ sup}_{t\in[\alpha,\beta]} |g(t)| & \text{if } s=\infty, \\ \left(\int_\alpha^\beta |g(t)|^s dt\right)^{1/s} & \text{if } s\in[1,\infty). \end{cases}$$

The following estimation holds.

THEOREM 4.2. *If f is absolutely continuous, then*

$$\begin{aligned} |\Phi_f(t)| &\le \left(\frac{b-t}{b-a}\right)\cdot\|f'\|_{[a,t],1} + \left(\frac{t-a}{b-a}\right)\cdot\|f'\|_{[t,b],1} \\ &\le \begin{cases} \dfrac{(b-t)(t-a)}{b-a}\|f'\|_{[a,t],\infty} & \textit{if } f'\in L_\infty[a,b], \\ \dfrac{(b-t)(t-a)^{1/q}}{b-a}\|f'\|_{[a,t],p} & \textit{if } f'\in L_p[a,b],\ p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1 \end{cases} \\ &\quad + \begin{cases} \dfrac{(t-a)(b-t)}{b-a}\|f'\|_{[t,b],\infty} & \textit{if } f'\in L_\infty[a,b] \\ \dfrac{(t-a)(b-t)^{1/\beta}}{b-a}\|f'\|_{[t,b],\alpha} & \textit{if } f'\in L_\alpha[a,b],\ \alpha>1,\ \dfrac{1}{\alpha}+\dfrac{1}{\beta}=1 \end{cases} \end{aligned} \tag{4.2}$$

where the second part should be seen as all four possible combinations.

PROOF. The first inequality holds from the representation (4.1) on taking the modulus and applying its properties.

By the integral Hölder inequality,

$$\int_a^t |f'(s)|\,ds \le \begin{cases} (t-a)\operatorname*{ess\ sup}_{s\in[a,t]} |f'(s)| & \text{if } f'\in L_\infty[a,b] \\ (t-a)^{1/q}\left(\int_a^t |f'(s)|^p\,ds\right)^{1/p} & \text{if } f'\in L_p[a,b], \\ & p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1 \end{cases}$$

and

$$\int_t^b |f'(s)|\,ds \le \begin{cases} (b-t)\operatorname*{ess\ sup}_{s\in[t,b]} |f'(s)| & \text{if } f'\in L_\infty[a,b] \\ (b-t)^{1/q}\left(\int_t^b |f'(s)|^p\,ds\right)^{1/p} & \text{if } f'\in L_p[a,b], \\ & p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1 \end{cases}$$

which provides the second part of (4.2). □

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REMARK 1. Some particular inequalities of interest are as follows. If $f' \in L_\infty[a, b]$, then

$$
\begin{aligned}
|\Phi_f(t)| &\le \frac{(b-t)(t-a)}{b-a}[\|f'\|_{[a,t],\infty} + \|f'\|_{[t,b],\infty}] \\
&\le \frac{2(b-t)(t-a)}{b-a}\|f'\|_{[a,b],\infty} \le \frac{1}{2}(b-a)\|f'\|_{[a,b],\infty},
\end{aligned} \tag{4.3}
$$

for any $t \in [a, b]$. The first inequality in (4.3) and the constants 2 and $\frac{1}{2}$ are the best possible.

If $f' \in L_p[a, b]$, $p > 1$, $(1/p) + (1/q) = 1$, then

$$
\begin{aligned}
|\Phi_f(t)| &\le \left[\frac{(b-t)(t-a)}{b-a}\right]^{1/q}\left[\left(\frac{b-t}{b-a}\right)^{1/p}\|f'\|_{[a,t],p} + \left(\frac{t-a}{b-a}\right)^{1/p}\|f'\|_{[t,b],p}\right] \\
&\le \left[\frac{(b-t)(t-a)}{b-a}\right]^{1/q}\left[\left(\frac{b-t}{b-a}\right)^{q/p} + \left(\frac{t-a}{b-a}\right)^{q/p}\right]^{1/q}\|f'\|_{[a,b],p}
\end{aligned} \tag{4.4}
$$

for any $t \in [a, b]$.

In particular, for $p = q = 2$,

$$
\begin{aligned}
|\Phi_f(t)| &\le \sqrt{\frac{(b-t)(t-a)}{b-a}}\left[\sqrt{\frac{b-t}{b-a}} \cdot \|f'\|_{[a,t],2} + \sqrt{\frac{t-a}{b-a}} \cdot \|f'\|_{[t,b],2}\right] \\
&\le \sqrt{\frac{(b-t)(t-a)}{b-a}}\|f'\|_{[a,b],2}
\end{aligned} \tag{4.5}
$$

for any $t \in [a, b]$.

5. The case when f' is of bounded variation

We introduce the following representation of the error Φ_f.

LEMMA 5.1. *If $f : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$ and such that the derivative f' is Riemann integrable on $[a, b]$, then we can state the following representation in terms of the Riemann–Stieltjes integral:*

$$
\Phi_f(t) = \frac{1}{b-a}\int_a^b K(t, s)\, df'(s), \qquad t \in [a, b], \tag{5.1}
$$

where the kernel $K : [a, b]^2 \to \mathbb{R}$ is given by

$$
K(t, s) := \begin{cases} (b-t)(s-a) & \text{if } a \le s \le t, \\ (t-a)(b-s) & \text{if } t < s \le b. \end{cases} \tag{5.2}
$$

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PROOF. Since f' is Riemann integrable on $[a, b]$, it follows that the Riemann–Stieltjes integrals $\int_a^t (s-a)\,df'(s)$ and $\int_t^b (b-s)\,df'(s)$ exist for each $t \in [a, b]$. Now, integrating by parts in the Riemann–Stieltjes integral,

$$\begin{aligned}\int_a^b K(t,s)\,df'(s) &= (b-t)\int_a^t (s-a)\,df'(s) + (t-a)\int_t^b (b-s)\,df'(s)\\ &= (b-t)\left[(s-a)f'(s)|_a^t - \int_a^t f'(s)\,ds\right] + (t-a)\left[(b-s)f'(s)|_t^b - \int_t^b f'(s)\,ds\right]\\ &= (b-t)[(t-a)f'(t) - (f(t)-f(a))] + (t-a)[-(b-t)f'(t) + f(b) - f(t)]\\ &= (t-a)[f(b)-f(t)] - (b-t)[f(t)-f(a)] = (b-a)\Phi_f(t)\end{aligned}$$

for any $t \in [a, b]$, which provides the desired representation (5.1). □

REMARK 2. If we define $\Delta_f : (a, b) \to \mathbb{R}$,

$$\Delta_f(t) = \frac{f(b)-f(t)}{b-t} - \frac{f(t)-f(a)}{t-a},$$

then by (5.1) we can write

$$\Delta_f(t) = \frac{1}{(t-a)(b-t)}\int_a^b K(t,s)\,df'(s) = \int_a^b R(t,s)\,df'(s), \quad t \in (a, b), \tag{5.3}$$

where the new kernel $R : (a, b)^2 \to \mathbb{R}$ is defined by

$$R(t,s) := \begin{cases} \dfrac{s-a}{t-a} & \text{if } a < s \le t,\\ \dfrac{b-s}{b-t} & \text{if } t < s < b.\end{cases}$$

We notice that, for $f(s) := \int_a^s g(z)\,dz$, the last equality in (5.3) produces the identity

$$\frac{1}{b-t}\int_t^b g(z)\,dz - \frac{1}{t-a}\int_a^t g(z)\,dz = \int_a^b R(t,s)\,dg(s), \tag{5.4}$$

as obtained by Cerone in [3, equation 2.12].

Notice that, in (5.4), the function g can be Riemann integrable and not only absolutely continuous as assumed in [3].

THEOREM 5.2. *Assume that $f : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$. If f' is of bounded variation on $[a, b]$, then*

$$|\Phi_f(t)| \le \frac{(t-a)(b-t)}{b-a} \cdot \bigvee_a^b (f') \le \frac{1}{4}(b-a)\bigvee_a^b (f'), \tag{5.5}$$

where $\bigvee_a^b (f')$ denotes the total variation of f' on $[a, b]$. The inequalities are sharp and the constant $\frac{1}{4}$ is the best possible.

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PROOF. It is well known that, if $p:[\alpha,\beta]\to\mathbb{R}$ is continuous and $v:[\alpha,\beta]\to\mathbb{R}$ is of bounded variation, then the Riemann–Stieltjes integral $\int_\alpha^\beta p(s)\,dv(s)$ exists and

$$\left|\int_\alpha^\beta p(s)\,dv(s)\right| \le \sup_{s\in[\alpha,\beta]}|p(s)|\bigvee_\alpha^\beta(v).$$

Now, utilizing the representation (5.1) and the above property,

$$\begin{aligned}|\Phi_f(t)| &= \frac{1}{b-a}\left|(b-t)\int_a^t (s-a)\,df'(s)+(t-a)\int_t^b (t-s)\,df'(s)\right|\\ &\le \frac{1}{b-a}\left[(b-t)\left|\int_a^t (s-a)\,df'(s)\right|+(t-a)\left|\int_t^b (t-s)\,df'(s)\right|\right]\\ &\le \frac{1}{b-a}\left[(b-t)\bigvee_a^t(f')\sup_{s\in[a,t]}(s-a)+(t-a)\bigvee_t^b(f')\sup_{s\in[t,b]}(t-s)\right]\\ &= \frac{(t-a)(b-t)}{b-a}\left[\bigvee_a^t(f')+\bigvee_t^b(f')\right]=\frac{(t-a)(b-t)}{b-a}\bigvee_a^b(f'). \qquad (5.6)\end{aligned}$$

The last part of (5.5) is obvious by the fact that $(t-a)(b-t)\le\frac{1}{4}(b-a)^2$, $t\in[a,b]$.

For the sharpness of the inequalities in (5.5), assume that there exist $F, G>0$ such that

$$|\Phi_f(t)|\le F\cdot\frac{(t-a)(b-t)}{b-a}\bigvee_a^b(f')\le G(b-a)\bigvee_a^b(f'),$$

with f as in the assumption of the theorem. Then, for $t=(a+b)/2$,

$$\left|\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right|\le\frac{1}{4}F(b-a)\bigvee_a^b(f')\le G(b-a)\bigvee_a^b(f'). \qquad (5.7)$$

Consider the function $f:[a,b]\to\mathbb{R}$, $f(t)=|t-(a+b)/2|$. This function is absolutely continuous, $f'(t)=\operatorname{sgn}(t-(a+b)/2)$, $t\in[a,b]\setminus\{(a+b)/2\}$ and $\bigvee_a^b(f')=2$. Thus, (5.7) becomes

$$\frac{b-a}{2}\le\frac{1}{2}F(b-a)\le 2G(b-a),$$

which implies that $F\ge 1$ and $G\ge\frac{1}{4}$. □

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6. The case when f' is Lipschitzian

The case when the derivative is a Lipschitzian function provides better accuracy in approximating the function f by the straight line d_f as follows.

THEOREM 6.1. *Assume that $f : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$. If f' is K_1-Lipschitzian on $[a, t]$ and K_2-Lipschitzian on $[t, b]$ ($t \in [a, b]$), then*

$$\begin{aligned} |\Phi_f(t)| &\le \frac{1}{2} \cdot \frac{(t-a)(b-t)}{b-a}[(K_1 - K_2)t + K_2 b - K_1 a] \\ &\le \frac{1}{8} \cdot (b-a)[(K_1 - K_2)t + K_2 b - K_1 a], \quad t \in [a, b]. \end{aligned} \tag{6.1}$$

In particular, if f' is K-Lipschitzian on $[a, b]$, then

$$|\Phi_f(t)| \le \tfrac{1}{2}(b-t)(t-a)K \le \tfrac{1}{8}(b-a)^2 K, \quad t \in [a, b]. \tag{6.2}$$

The constants $\frac{1}{2}$ and $\frac{1}{8}$ are the best possible.

PROOF. We use the fact that for an L-Lipschitzian function $p : [\alpha, \beta] \to \mathbb{R}$ and a Riemann integrable function $v : [\alpha, \beta] \to \mathbb{R}$, the Riemann–Stieltjes integral $\int_\alpha^\beta p(s)\, dv(s)$ exists and

$$\left| \int_\alpha^\beta p(s)\, dv(s) \right| \le L \int_\alpha^\beta |p(s)|\, ds.$$

Then

$$\left| \int_a^t (s-a)\, df'(s) \right| \le K_1 \cdot \int_a^t (s-a)\, ds = \tfrac{1}{2} K_1 (t-a)^2 \tag{6.3}$$

and

$$\left| \int_t^b (t-s)\, df'(s) \right| \le K_2 \cdot \int_t^b (t-s)\, ds = \tfrac{1}{2} K_2 (b-t)^2. \tag{6.4}$$

Now, on making use of (5.6), inequalities (6.3) and (6.4) lead to

$$\begin{aligned} |\Phi_f(t)| &\le \frac{1}{b-a}\left[\frac{1}{2}(b-t)(t-a)^2 \cdot K_1 + \frac{1}{2}(t-a)(b-t)^2 \cdot K_2\right] \\ &= \frac{1}{2} \cdot \frac{(t-a)(b-t)}{b-a}[L_1(t-a) + L_2(b-t)], \end{aligned}$$

which produces the first inequality in (6.1). The other inequalities are obvious.

To prove the sharpness of the constants in (6.2), let us assume that there exist $H, K > 0$ such that

$$|\Phi_f(t)| \le H(b-t)(t-a)L \le K(b-a)^2 L \tag{6.5}$$

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for any $t \in [a, b]$ and f an L-Lipschitzian function on $[a, b]$. For $t = (a + b)/2$ we get from (6.5) that

$$\left| \frac{f(a) + f(b)}{2} - f\left(\frac{a+b}{2}\right) \right| \le \frac{1}{4} HL(b-a)^2 \le LK(b-a)^2. \tag{6.6}$$

Consider $f : [a, b] \to \mathbb{R}$, $f(t) = \frac{1}{2}(t - (a + b)/2)^2$. Then $f'(t) = t - (a + b)/2$ is Lipschitzian with the constant $L = 1$ and (6.6) becomes

$$\tfrac{1}{8}(b-a)^2 \le \tfrac{1}{4} H(b-a)^2 \le K(b-a)^2,$$

which implies that $H \ge \frac{1}{2}$ and $K \ge \frac{1}{8}$. □

7. The case when f' is absolutely continuous

The following representation result also holds.

LEMMA 7.1. *If $f : [a, b] \to \mathbb{R}$ is differentiable and the derivative f' is absolutely continuous, then*

$$\Phi_f(t) = \frac{1}{b-a} \int_a^b K(t, s) f''(s)\, ds \tag{7.1}$$

for any $t \in [a, b]$, where the integral in (7.1) is considered in the Lebesgue sense.

The proof is similar to that in Lemma 5.1 on integrating the Lebesgue integral $\int_a^b K(t, s) f''(s)\, ds$ by parts. The details are omitted.

THEOREM 7.2. *If f is as in Lemma 7.1, then*

$$|\Phi_f(t)| \le \frac{(b-t)(t-a)}{b-a} \cdot K(t), \qquad t \in [a, b], \tag{7.2}$$

where

$$K(t) := \begin{cases} \|f''\|_{[a,t],1}; & \\ \dfrac{(t-a)^{1/q}}{(q+1)^{1/q}} \|f''\|_{[a,t],p} & \text{if } p > 1,\ \dfrac{1}{p} + \dfrac{1}{q} = 1, \\ & f'' \in L_p[a, b]; \\ \dfrac{1}{2}(t-a)\|f''\|_{[a,t],\infty} & \text{if } f'' \in L_\infty[a, b]; \end{cases}$$
$$+ \begin{cases} \|f''\|_{[t,b],1} & \\ \dfrac{(b-t)^{1/\beta}}{(\beta+1)^{1/\beta}} \|f''\|_{[t,b],\alpha} & \text{if } \alpha > 1,\ \dfrac{1}{\alpha} + \dfrac{1}{\beta} = 1, \\ & f'' \in L_\alpha[a, b]; \\ \dfrac{1}{2}(b-t)\|f''\|_{[t,b],\infty} & \text{if } f'' \in L_\infty[a, b]; \end{cases} \tag{7.3}$$

and the definition of K should be seen as all nine possible combinations.

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PROOF. By inequality (5.6),

$$|\Phi_f(t)| \le \frac{1}{b-a}\left[(b-t)\left|\int_a^t (s-a) f''(s)\, ds\right| + (t-a)\left|\int_t^b (t-s) f''(s)\, ds\right|\right] \tag{7.4}$$

for any $t \in [a, b]$.

Using Hölder's inequality,

$$\left|\int_a^t (s-a) f''(s)\, ds\right| \le \begin{cases} \sup_{s\in[a,t]} (s-a) \int_a^t |f''(s)|\, ds; & \\ \left(\int_a^t (s-a)^q\, ds\right)^{1/q} \left(\int_a^t |f''(s)|^p\, ds\right)^{1/p} & \text{if } p > 1,\ \frac{1}{p}+\frac{1}{q}=1,\ f'' \in L_p[a,b]; \\ \operatorname{ess\,sup}_{s\in[a,t]} |f''(s)| \int_a^t (s-a)\, ds & \text{if } f'' \in L_\infty[a,b]; \end{cases}$$

$$= \begin{cases} (t-a)\|f''\|_{[a,t],1}, & \\ \frac{(t-a)^{1+1/q}}{(q+1)^{1/q}}\|f''\|_{[a,t],p} & \text{if } p > 1,\ \frac{1}{p}+\frac{1}{q}=1,\ f'' \in L_p[a,b]; \\ \frac{1}{2}(t-a)^2\|f''\|_{[a,t],\infty} & \text{if } f'' \in L_\infty[a,b]; \end{cases} \tag{7.5}$$

and, similarly,

$$\left|\int_t^b (b-s) f''(s)\, ds\right| \le \begin{cases} (b-t)\|f''\|_{[t,b],1} & \\ \frac{(b-t)^{1+1/\beta}}{(\beta+1)^{1/\beta}}\|f''\|_{[t,b],\alpha} & \text{if } \alpha > 1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1,\ f'' \in L_\alpha[a,b]; \\ \frac{1}{2}(b-t)^2\|f''\|_{[t,b],\infty} & \text{if } f'' \in L_\infty[a,b], \end{cases} \tag{7.6}$$

for any $t \in [a, b]$.

Finally, on making use of (7.4)–(7.6), we deduce the desired inequality (7.2). □

REMARK 3. The inequalities in (7.2) have some instances of interest that are useful in applications. For example, in terms of the supremum norm,

$$\begin{aligned} |\Phi_f(t)| &\le \frac{1}{2} \cdot \frac{(b-t)(t-a)}{b-a}[(t-a)\|f''\|_{[a,t],\infty} + (b-t)\|f''\|_{[t,b],\infty}] \\ &\le \frac{1}{2} \cdot (b-t)(t-a)\|f''\|_{[a,b],\infty}, \quad t \in [a,b], \end{aligned} \tag{7.7}$$

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where $f'' \in L_\infty[a, b]$. The constant $\frac{1}{2}$ is the best possible in both inequalities. The function $f(t) = \frac{1}{2}(t - (a+b)/2)^2$ produces an equality in (7.7) for $t = (a+b)/2$.

If we assume that $\alpha = p$, $\beta = q$ in (7.2), then we can also write

$$\begin{aligned} |\Phi_f(t)| &\leq \frac{(b-t)(t-a)}{(q+1)^{1/q}(b-a)}[(t-a)^{1/q}\|f''\|_{[a,t],p} + (b-t)^{1/q}\|f''\|_{[t,b],p}] \\ &\leq \frac{(b-t)(t-a)}{(q+1)^{1/q}(b-a)^{1/p}}\|f''\|_{[a,b],p}, \quad t \in [a, b], \end{aligned} \tag{7.8}$$

for $p > 1$, $1/p + 1/q = 1$, $f'' \in L_p[a, b]$, since, by Hölder's inequality,

$$\begin{aligned} &(t-a)^{1/q}\|f''\|_{[a,t],p} + (b-t)^{1/q}\|f''\|_{[t,b],p} \\ &\quad \leq [(t-a)^{q/q} + (b-t)^{q/q}]^{1/q}[\|f''\|^p_{[a,t],p} + \|f''\|^p_{[t,b],p}]^{1/p} \\ &\quad = (b-a)^{1/p}\|f''\|_{[a,b],p}. \end{aligned}$$

If $p = q = 2$, we get the following inequality for the Euclidean norm $\|f''\|_{[a,b],2}$:

$$\begin{aligned} |\Phi_f(t)| &\leq \frac{\sqrt{3}}{3} \cdot \frac{(b-t)(t-a)}{b-a}[\sqrt{t-a}\|f''\|_{[a,t],2} + \sqrt{b-t}\|f''\|_{[t,b],2}] \\ &\leq \frac{\sqrt{3}}{3} \cdot \frac{(b-t)(t-a)}{\sqrt{(b-a)}}\|f''\|_{[a,b],2}, \quad t \in [a, b]. \end{aligned} \tag{7.9}$$

It is an open question whether or not the constant $\sqrt{3}/3$ is the best possible in (7.9).

Finally, by (7.2) we can also write

$$|\Phi_f(t)| \leq \frac{(b-t)(t-a)}{b-a}\|f''\|_{[a,b],1} \leq \frac{1}{4}(b-a)\|f''\|_{[a,b],1} \tag{7.10}$$

for any $t \in [a, b]$.

8. Applications for weighted means

For a function $f : [a, b] \to \mathbb{R}$, $\mathbf{x} = (x_1, \ldots, x_n) \in [a, b]^n$ and $\mathbf{p} = (p_1, \ldots, p_n)$ a probability sequence, that is, $p_i \geq 0$, $i \in \{1, \ldots, n\}$ and $\sum_{i=1}^n p_i = 1$, we define the mean

$$M_f(\mathbf{p}; \mathbf{x}) := \sum_{i=1}^n p_i f(x_i). \tag{8.1}$$

If $f(t) = t$, $t \in [a, b]$, then

$$M_f(\mathbf{p}; \mathbf{x}) = A(\mathbf{p}; \mathbf{x}) = \sum_{i=1}^n p_i x_i,$$

which is the *arithmetic mean* of $\mathbf{x}$ with the weights $\mathbf{p}$.

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The main aim of the present section is to provide sharp bounds for the error in approximating $\mathcal{M}_f(\mathbf{p}; \mathbf{x})$ in terms of the simpler quantity

$$f(a) \cdot \frac{b - A(\mathbf{p}; \mathbf{x})}{b - a} + f(b) \cdot \frac{A(\mathbf{p}; \mathbf{x}) - a}{b - a}. \tag{8.2}$$

The following proposition contains some results of this type.

PROPOSITION 8.1. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function on $[a, b]$, $\mathbf{x} = (x_1, \dots, x_n) \in [a, b]^n$ and $\mathbf{p}$ a probability sequence. Define the error functional $\mathcal{E}_f(\mathbf{p}; \mathbf{x})$ by*

$$\mathcal{E}_f(\mathbf{p}; \mathbf{x}) := f(a) \cdot \frac{b - A(\mathbf{p}; \mathbf{x})}{b - a} + f(b) \cdot \frac{A(\mathbf{p}; \mathbf{x}) - a}{b - a} - \mathcal{M}_f(\mathbf{p}; \mathbf{x}). \tag{8.3}$$

(i) *If $-\infty < m \le f(t) \le M < \infty$ for any $t \in [a, b]$, then*

$$|\mathcal{E}_f(\mathbf{p}; \mathbf{x})| \le M - m. \tag{8.4}$$

The inequality is sharp.

(ii) *If $f : [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$, then*

$$|\mathcal{E}_f(\mathbf{p}; \mathbf{x})| \le \left[\frac{1}{2} + \sum_{i=1}^{n} p_i \left|\frac{x_i - (a + b)/2}{b - a}\right|\right] \bigvee_a^b (f). \tag{8.5}$$

The constant $\frac{1}{2}$ is the best possible in (8.5).

(iii) *If $f : [a, b] \to \mathbb{R}$ is L-Lipschitzian on $[a, b]$, then*

$$\begin{aligned} |\mathcal{E}_f(\mathbf{p}; \mathbf{x})| &\le \frac{2L}{b - a} \sum_{i=1}^{n} p_i (b - x_i)(x_i - a) \\ &\le \frac{2L}{b - a} [b - A(\mathbf{p}; \mathbf{x})][A(\mathbf{p}; \mathbf{x}) - a] \le \frac{1}{2} L(b - a). \end{aligned} \tag{8.6}$$

All the inequalities in (8.6) are sharp.

PROOF. Let us confine ourselves to proving inequality (8.6). The other inequalities follow likewise. Applying inequality (3.6) for $t = x_i$, $i \in \{1, \dots, n\}$,

$$\left| f(x_i) - \frac{f(a)(b - x_i) + f(b)(x_i - a)}{b - a} \right| \le \frac{2L}{b - a} (b - x_i)(x_i - a), \tag{8.7}$$

for any $i \in \{1, \dots, n\}$. Multiplying (8.7) by p_i, summing over i from 1 to n and utilizing the generalized triangle inequality

$$\sum_{i=1}^{n} |\alpha_i| \ge \left| \sum_{i=1}^{n} \alpha_i \right|,$$

we deduce the first inequality in (8.6).

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Further, we use the following Chebyshev inequality:

$$\sum_{i=1}^{n} p_i\alpha_i\beta_i \le \sum_{i=1}^{n} p_i\alpha_i \sum_{i=1}^{n} p_i\beta_i, \tag{8.8}$$

provided that $p_i \ge 0$, $\sum_{i=1}^{n} p_i = 1$ and $(\alpha_i)_{i=\overline{1,n}}$, $(\beta_i)_{i=\overline{1,n}}$ are *asynchronous*, that is,

$$(\alpha_i - \alpha_j)(\beta_i - \beta_j) \le 0 \quad \text{for any } i, j \in \{1, \ldots, n\}.$$

Then, by (8.8),

$$\begin{aligned}\sum_{i=1}^{n} p_i(b - x_i)(x_i - a) &\le \sum_{i=1}^{n} p_i(b - x_i) \sum_{i=1}^{n} p_i(x_i - a) \\ &= [b - A(\mathbf{p}; \mathbf{x})][A(\mathbf{p}; \mathbf{x}) - a]\end{aligned}$$

and the second inequality in (8.6) is proved. The last part is obvious.

The sharpness of the inequality follows from the case $n = 1$. The details are omitted. □

If $f : [a, b] \to \mathbb{R}$ is a convex function, then the following result is known in the literature as the (discrete) *Lah–Ribarić inequality*:

$$\sum_{i=1}^{n} p_i f(x_i) \le \frac{1}{b-a}\{f(a)[b - A(\mathbf{p}; \mathbf{x})] + f(b)[A(\mathbf{p}; \mathbf{x}) - a]\}. \tag{8.9}$$

For a generalization to a positive linear functional that incorporates both the original Lah–Ribarić integral inequality and its discrete version due to Beesack and Pečarić [2], see [12, p. 98].

In terms of the error functional $\mathcal{E}_f(\mathbf{p}; \mathbf{x})$, we can then write $\mathcal{E}_f(\mathbf{p}; \mathbf{x}) \ge 0$, when f is convex and $\mathbf{p}$, $\mathbf{x}$ are as above. Now, on utilizing Theorem 2.2, we can state the following converse of the Lah–Ribarić inequality (8.9).

PROPOSITION 8.2. *If $f : [a, b] \to \mathbb{R}$ is convex on $[a, b]$ and the lateral derivatives $f'_-(b)$, $f'_+(a)$ are finite, then*

$$\begin{aligned}(0 \le)\mathcal{E}_f(\mathbf{p}; \mathbf{x}) &\le \frac{f'_-(b) - f'_+(a)}{b-a} \sum_{i=1}^{n} p_i(b - x_i)(x_i - a) \\ &\le \frac{f'_-(b) - f'_+(a)}{b-a}[b - A(\mathbf{p}; \mathbf{x})][A(\mathbf{p}; \mathbf{x}) - a] \\ &\le \tfrac{1}{4}(b-a)[f'_-(b) - f'_+(a)].\end{aligned} \tag{8.10}$$

The inequalities are sharp and $\frac{1}{4}$ is the best possible.

The following results in terms of the derivative of a function f can be also stated.

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PROPOSITION 8.3. *Assume that $f : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$.*

(i) *If f' is of bounded variation on $[a, b]$, then*

$$|\mathcal{E}_f(\mathbf{p}; \mathbf{x})| \le \frac{1}{b-a} \bigvee_a^b (f') \sum_{i=1}^n p_i(b - x_i)(x_i - a)$$
$$\le \frac{1}{b-a} \bigvee_a^b (f')[A(\mathbf{p}; \mathbf{x}) - a][b - A(\mathbf{p}; \mathbf{x})] \le \frac{1}{4}(b-a) \bigvee_a^b (f'). \tag{8.11}$$

All inequalities in (8.11) are sharp. The constant $\frac{1}{4}$ is the best possible.

(ii) *If f' is K-Lipschitzian on $[a, b]$ $(K > 0)$, then*

$$|\mathcal{E}_f(\mathbf{p}; \mathbf{x})| \le \tfrac{1}{2}K \sum_{i=1}^n p_i(b - x_i)(x_i - a)$$
$$\le \tfrac{1}{2}K[b - A(\mathbf{p}; \mathbf{x})][A(\mathbf{p}; \mathbf{x}) - a] \le \tfrac{1}{8}(b-a)^2 K. \tag{8.12}$$

The constants $\frac{1}{2}$ and $\frac{1}{8}$ are the best possible.

The proof is obvious by Theorems 5.2 and 6.1 and the details are omitted.

The above results can be useful in providing various inequalities between means. For instance, if we denote by $G(\mathbf{p}, \mathbf{x})$ the *geometric mean* $\prod_{i=1}^n x_i^{p_i}$, then for the convex function $f(t) = -\ln t$, we can write, for $0 < m \le x_i \le M < \infty$, $i \in \{1, \ldots, n\}$,

$$\mathcal{E}_f(\mathbf{p}; \mathbf{x}) = \ln G(\mathbf{p}; \mathbf{x}) - \ln[m^{(M-A(\mathbf{p};\mathbf{x}))/(M-m)} \cdot M^{(A(\mathbf{p};\mathbf{x})-m)/(M-m)}]$$
$$= \ln\left[\frac{G(\mathbf{p}; \mathbf{x})}{m^{(M-A(\mathbf{p};\mathbf{x}))/(M-m)} \cdot M^{(A(\mathbf{p};\mathbf{x})-m)/(M-m)}}\right],$$

$$\bigvee_m^M (f) = \ln\left(\frac{M}{m}\right),$$

f is L-Lipschitzian with the constant $L = \|f'\|_{\infty,[m,M]} = 1/m$ and

$$\frac{f'(M) - f'(m)}{M - m} = \frac{1}{mM}, \quad \bigvee_m^M (f') = \frac{M - m}{mM}.$$

Also, f' is K-Lipschitzian with the constant $K = \|f''\|_{\infty,[m,M]} = 1/m^2$.

Applying Proposition 8.1, we get

$$0 \le \ln\left[\frac{G(\mathbf{p}; \mathbf{x})}{m^{(M-A(\mathbf{p};\mathbf{x}))/(M-m)} \cdot M^{(A(\mathbf{p};\mathbf{x})-m)/(M-m)}}\right]$$
$$\le \left[\frac{1}{2} + \sum_{i=1}^n p_i \left|\frac{x_i - (m+M)/2}{M - m}\right|\right] \ln\left(\frac{M}{m}\right),$$

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while Propositions 8.2 and 8.3 give

$$
\begin{aligned}
0 &\le \ln\left[\frac{G(\mathbf{p};\mathbf{x})}{m^{(M-A(\mathbf{p};\mathbf{x}))/(M-m)}\cdot M^{(A(\mathbf{p};\mathbf{x})-m)/(M-m)}}\right] \\
&\le \min\left\{\frac{2}{m(M-m)},\frac{1}{mM},\frac{1}{2m^2}\right\}\cdot\sum_{i=1}^{n}p_i(M-x_i)(x_i-m) \\
&\le \min\left\{\frac{2}{m(M-m)},\frac{1}{mM},\frac{1}{2m^2}\right\}\cdot[M-A(\mathbf{p};\mathbf{x})][A(\mathbf{p};\mathbf{x})-m] \\
&\le \min\left\{\frac{M-m}{2m},\frac{(M-m)^2}{4mM},\frac{(M-m)^2}{8m^2}\right\}.
\end{aligned}
$$

REMARK 4. All the results in this section can be stated for positive linear functionals defined on linear spaces of functions. Applications for Lebesgue integrals in the general setting of measurable spaces can also be provided. However, for the sake of brevity, we do not state them here.

9. Applications for f-divergences

Given a convex function $f:[0,\infty)\to\mathbb{R}$, the *$f$-divergence functional*

$$
I_f(\mathbf{p},\mathbf{q}) := \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right), \tag{9.1}
$$

where $\mathbf{p}=(p_1,\ldots,p_n)$, $\mathbf{q}=(q_1,\ldots,q_n)$ are positive sequences, was introduced by Csiszár in [4] as a generalized measure of information, a 'distance function' on the set of probability distributions $\mathbb{P}^n$. As in [4], we interpret undefined expressions as follows:

$$
\begin{gathered}
f(0)=\lim_{t\to 0+} f(t), \quad 0f\left(\frac{0}{0}\right)=0, \\
0f\left(\frac{a}{0}\right)=\lim_{q\to 0+} f\left(\frac{a}{q}\right)=a\lim_{t\to\infty}\frac{f(t)}{t}, \quad a>0.
\end{gathered} \tag{9.2}
$$

The following results were essentially given by Csiszár and Körner [5].

(i) If f is convex, then $I_f(\mathbf{p},\mathbf{q})$ is jointly convex in p and q.
(ii) For every $p,q\in R_+^n$,

$$
I_f(\mathbf{p},\mathbf{q}) \ge \sum_{j=1}^{n} q_j f\left(\frac{\sum_{j=1}^{n} p_j}{\sum_{j=1}^{n} q_j}\right). \tag{9.3}
$$

If f is strictly convex, equality holds in (9.3) if and only if

$$
\frac{p_1}{q_1}=\frac{p_2}{q_2}=\cdots=\frac{p_n}{q_n}.
$$

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If f is normalized, that is, $f(1) = 0$, then for every $p, q \in R_+^n$ with $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i$, we have the inequality

$$I_f(\mathbf{p}, \mathbf{q}) \geq 0. \tag{9.4}$$

In particular, if $\mathbf{p}, \mathbf{q} \in \mathbb{P}^n$, then (9.4) holds. This is the well-known *positive property* of the f-divergence.

We now give some examples of divergence measures in information theory which are particular cases of f-divergence.

(1) *Kullback–Leibler distance* [10]. The Kullback–Leibler distance $D(\cdot, \cdot)$ is defined by

$$D(\mathbf{p}, \mathbf{q}) := \sum_{i=1}^n p_i \log\left(\frac{p_i}{q_i}\right).$$

If we choose $f(t) = t \ln t$, $t > 0$, then obviously

$$I_f(\mathbf{p}, \mathbf{q}) = D(\mathbf{p}, \mathbf{q}).$$

(2) *Variational distance* (l_1-distance). The variational distance $V(\cdot, \cdot)$ is defined by

$$V(\mathbf{p}, \mathbf{q}) := \sum_{i=1}^n |p_i - q_i|.$$

If we choose $f(t) = |t - 1|$, $t \in [0, \infty)$, then

$$I_f(\mathbf{p}, \mathbf{q}) = V(\mathbf{p}, \mathbf{q}).$$

(3) *Hellinger discrimination* [1]. The Hellinger discrimination is defined by $\sqrt{2h^2(\cdot, \cdot)}$, where $h^2(\cdot, \cdot)$ is given by

$$h^2(\mathbf{p}, \mathbf{q}) := \tfrac{1}{2} \sum_{i=1}^n (\sqrt{p_i} - \sqrt{q_i})^2.$$

It is obvious that if $f(t) = \frac{1}{2}(\sqrt{t} - 1)^2$, then

$$I_f(\mathbf{p}, \mathbf{q}) = h^2(\mathbf{p}, \mathbf{q}).$$

(4) *Triangular discrimination* [14]. We define triangular discrimination between $\mathbf{p}$ and $\mathbf{q}$ by

$$\Delta(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^n \frac{|p_i - q_i|^2}{p_i + q_i}.$$

It is obvious that if $f(t) = (t - 1)^2/(t + 1)$, $t \in (0, \infty)$, then

$$I_f(\mathbf{p}, \mathbf{q}) = \Delta(\mathbf{p}, \mathbf{q}).$$

Note that $\sqrt{\Delta(\mathbf{p}, \mathbf{q})}$ is known in the literature as the Le Cam distance.

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(5) *Chi-square distance.* We define the χ^2-distance by

$$\chi^2(\mathbf{p}, \mathbf{q}) := \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{q_i}.$$

It is clear that if $f(t) = (t-1)^2$, $t \in [0, \infty)$, then

$$I_f(\mathbf{p}, \mathbf{q}) = \chi^2(\mathbf{p}, \mathbf{q}).$$

(6) *Rényi's divergences* [13]. For $\alpha \in \mathbb{R}\backslash\{0, 1\}$, consider

$$\rho_\alpha(\mathbf{p}, \mathbf{q}) := \sum_{i=1}^{n} p_i^\alpha q_i^{1-\alpha}.$$

It is obvious that if $f(t) = t^\alpha$ $(t \in (0, \infty))$, then

$$I_f(\mathbf{p}, \mathbf{q}) = \rho_\alpha(\mathbf{p}, \mathbf{q}).$$

Rényi's divergences $R_\alpha(\mathbf{p}, \mathbf{q}) := (1/\alpha(\alpha - 1)) \ln[\rho_\alpha(\mathbf{p}, \mathbf{q})]$ were introduced for all real orders $\alpha \neq 0$, $\alpha \neq 1$ (and continuously extended for $\alpha = 0$ and $\alpha = 1$) in [11], where the reader may find many inequalities valid for these divergences, without, as well as with, some restrictions on $\mathbf{p}$ and $\mathbf{q}$.

For other examples of divergence measures, see the paper [9] and the books [11] and [15], where further references are given.

Now, for $0 < r < 1 < R < \infty$, we consider the expression

$$\frac{1}{R-r}[(R-1)f(r) + (1-r)f(R)]$$

and are interested in comparing it with the f-divergence $I_f(\mathbf{p}, \mathbf{q})$ which can be extended for larger classes than convex functions with the same definition (9.1) and the same conventions as those from (9.2).

PROPOSITION 9.1. *Let $f : [r, R] \to \mathbb{R}$ be a bounded function on the interval $[r, R]$ with $0 < r < 1 < R < \infty$. Assume that $\mathbf{p}, \mathbf{q} \in \mathbb{P}^n$ are such that*

$$r \leq \frac{p_i}{q_i} \leq R, \quad \textit{for each } i \in \{1, \ldots, n\}, \tag{9.5}$$

and define the error functional

$$\delta_f(\mathbf{p}, \mathbf{q}; r, R) := \frac{1}{R-r}[(R-1)f(r) + (1-r)f(R)] - I_f(\mathbf{p}, \mathbf{q}).$$

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(i) *If* $-\infty < m \le f(t) \le M < \infty$ *for any* $t \in [r, R]$, *then*

$$|\delta_f(\mathbf{p}, \mathbf{q}; r, R)| \le M - m. \tag{9.6}$$

The inequality is sharp.

(ii) *If* $f : [r, R] \to \mathbb{R}$ *is of bounded variation on* $[r, R]$, *then*

$$|\delta_f(\mathbf{p}, \mathbf{q}; r, R)| \le \left[\frac{1}{2} + \frac{1}{R - r}\sum_{i=1}^{n}\left|p_i - \frac{r + R}{2} \cdot q_i\right|\right] \bigvee_r^R (f). \tag{9.7}$$

The constant $\frac{1}{2}$ *is the best possible in* (8.5).

(iii) *If* $f : [r, R] \to \mathbb{R}$ *is L-Lipschitzian on* $[r, R]$, *then*

$$\begin{aligned} |\delta_f(\mathbf{p}, \mathbf{q}; r, R)| &\le \frac{2L}{R - r}[(R - 1)(1 - r) - \chi^2(p, q)] \\ &\le \frac{2L}{R - r}(R - 1)(1 - r) \le \frac{1}{2}L(R - r), \end{aligned} \tag{9.8}$$

where the Pearson χ^2*-divergence is obtained from* (9.1) *for the convex function* $f(t) = (1 - t)^2$, $t \in \mathbb{R}$ *and given by the equivalent expressions:*

$$\chi^2(p, q) := \sum_{j=1}^{n} q_j\left(\frac{p_j}{q_j} - 1\right)^2 = \sum_{j=1}^{n}\frac{(p_j - q_j)^2}{q_j} = \sum_{j=1}^{n}\frac{p_j^2}{q_j} - 1. \tag{9.9}$$

PROOF. The proof follows in a similar manner to that of Proposition 8.1 on choosing $a = r$, $b = R$, $p_i = q_i$ and $x_i = p_i/q_i$, $i \in \{1, \ldots, n\}$, and the details are omitted. □

In the case of convex functions we can give the following result.

PROPOSITION 9.2. *If* $f : [r, R] \to \mathbb{R}$ *is convex on* $[r, R]$ *and the lateral derivatives* $f'_-(R)$, $f'_+(r)$ *are finite, then*

$$\begin{aligned} (0 \le)\delta_f(\mathbf{p}, \mathbf{q}; r, R) &\le \frac{f'_-(R) - f'_+(r)}{R - r}[(R - 1)(1 - r) - \chi^2(p, q)] \\ &\le \frac{f'_-(R) - f'_+(r)}{R - r}(R - 1)(1 - r) \\ &\le \frac{1}{4}(R - r)[f'_-(R) - f'_+(r)], \end{aligned} \tag{9.10}$$

provided that $\mathbf{p}, \mathbf{q} \in \mathbb{P}^n$ *are such that* (9.5) *holds. The inequalities are sharp and* $\frac{1}{4}$ *is the best possible.*

The result in Proposition 9.2 was first obtained by the author in [6].

Finally, we can state the following result.

PROPOSITION 9.3. *Assume that* $f : [r, R] \to \mathbb{R}$ *is absolutely continuous on* $[r, R]$ *and that* $\mathbf{p}, \mathbf{q} \in \mathbb{P}^n$ *are such that* (9.5) *holds.*

(i) *If* f' *is of bounded variation on* $[r, R]$, *then*

$$|\delta_f(\mathbf{p}, \mathbf{q}; r, R)| \le \frac{1}{R-r} \bigvee_r^R (f')[(R-1)(1-r) - \chi^2(p, q)] \le \frac{1}{R-r}(R-1)(1-r) \bigvee_r^R (f') \le \frac{1}{4}(R-r) \bigvee_r^R (f'). \tag{9.11}$$

All inequalities in (9.11) *are sharp. The constant* $\frac{1}{4}$ *is the best possible.*

(ii) *If* f' *is* K*-Lipschitzian on* $[r, R]$ $(K > 0)$, *then*

$$|\delta_f(\mathbf{p}, \mathbf{q}; r, R)| \le \tfrac{1}{2}K[(R-1)(1-r) - \chi^2(p, q)] \le \tfrac{1}{2}K(R-1)(1-r) \le \tfrac{1}{8}(R-r)^2 K. \tag{9.12}$$

The constants $\frac{1}{2}$ *and* $\frac{1}{8}$ *are the best possible.*

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