



COMPOSITIO MATHEMATICA

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Compositio Math. **154** (2018), 1–35.

[doi:10.1112/S0010437X17007400](https://doi.org/10.1112/S0010437X17007400)



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ABSTRACT

Let K be the field of fractions of a local Henselian discrete valuation ring \mathcal{O}_K of characteristic zero with perfect residue field k . Assuming potential semi-stable reduction, we show that an unramified Galois action on the second ℓ -adic cohomology group of a K3 surface over K implies that the surface has good reduction after a finite and *unramified* extension. We give examples where this unramified extension is really needed. Moreover, we give applications to good reduction after tame extensions and Kuga–Satake Abelian varieties. On our way, we settle existence and termination of certain flops in mixed characteristic, and study group actions and their quotients on models of varieties.

1. Introduction

Let \mathcal{O}_K be a local Henselian DVR (discrete valuation ring) of characteristic zero with field of fractions K and perfect residue field k , whose characteristic is $p \geq 0$. For example, \mathcal{O}_K could be $\mathbb{C}[[t]]$ or the ring of integers in a p -adic field. Given a variety X that is smooth and proper over K , one can ask whether X has *good reduction*, that is, whether there exists an algebraic space

$$\mathcal{X} \rightarrow \mathrm{Spec} \mathcal{O}_K$$

with generic fiber X that is smooth and proper over \mathcal{O}_K .

1.1 Good reduction and Galois representations

Let ℓ be a prime different from p , let $G_K := \mathrm{Gal}(\overline{K}/K)$ be the absolute Galois group of K , and let I_K be its inertia subgroup. Then the natural ℓ -adic Galois representation

$$\rho_{m,\ell} : G_K \rightarrow \mathrm{Aut}(H_{\mathrm{\acute{e}t}}^m(X_{\overline{K}}, \mathbb{Q}_{\ell}))$$

is called *unramified* if it satisfies $\rho_{m,\ell}(I_K) = \{\mathrm{id}\}$. A necessary condition for X to have good reduction is that for all $m \geq 1$ and all primes $\ell \neq p$, the representation $\rho_{m,\ell}$ is unramified.

Received 24 February 2015, accepted in final form 3 March 2017, published online 18 September 2017.

2010 Mathematics Subject Classification 14J28, 11G25, 11F80, 14E30 (primary).

Keywords: K3 surfaces, good reduction, Galois representations.

The first named author is supported by the ERC Consolidator Grant 681838 ‘K3CRYSTAL’ and the second named author is supported by JSPS KAKENHI Grant Number 26247002.

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1.2 Curves and Abelian varieties

By a famous theorem of Serre and Tate [ST68], which generalizes results of Néron, Ogg, and Shafarevich for elliptic curves to Abelian varieties, the G_K -representation $\rho_{1,\ell}$ detects the reduction type of Abelian varieties.

THEOREM 1.1 (Serre–Tate). *An Abelian variety X over K has good reduction if and only if the G_K -representation on $H_{\text{ét}}^1(X_{\overline{K}}, \mathbb{Q}_\ell)$ is unramified.*

On the other hand, it is not too difficult to give counterexamples to such a result for curves of genus at least 2. Nevertheless, Oda [Oda95] showed that good reduction can be detected by the outer G_K -representation on the étale fundamental group. We refer the interested reader to § 2.4 for references, examples, and details.

1.3 Kulikov–Nakkajima–Persson–Pinkham models

Before coming to the results of this article, we have to make one crucial assumption.

ASSUMPTION (\star). A K3 surface X over K satisfies Assumption (\star) if there exists a finite field extension L/K such that X_L admits a model $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_L$ that is a regular algebraic space with trivial canonical sheaf $\omega_{\mathcal{X}/\mathcal{O}_L}$, and whose geometric special fiber is a normal crossing divisor.

In equal characteristic zero, Assumption (\star) always holds, and the special fibers of the corresponding models have been classified by Kulikov [Kul77], Persson [Per77], and Persson and Pinkham [PP81]. In mixed characteristic, the corresponding classification (assuming the existence of such models) is due to Nakkajima [Nakk00]. If the expected results on resolution of singularities and toroidalization of morphisms were known to hold in mixed characteristic, then Assumption (\star) would follow from Kawamata’s semi-stable minimal model program (MMP) in mixed characteristic [Kaw94] and Artin’s results [Art74] on simultaneous resolutions of families of surface singularities. We refer to Proposition 3.1 for details. Using work of Maulik [Mau14] and some strengthenings due to the second named author [Mat15], we have at least the following.

THEOREM 1.2. *Let X be a K3 surface over K and assume that $p = 0$ or that X admits an ample invertible sheaf \mathcal{L} with $p > \mathcal{L}^2 + 4$. Then X satisfies Assumption (\star).*

1.4 K3 surfaces

In this article, we establish a Néron–Ogg–Shafarevich–Serre–Tate type result for K3 surfaces. Important steps were already taken by the second named author in [Mat15]. Over the complex numbers, similar results are classically known; see, for example, [KK98, ch. 5].

Before coming to the main result of this article, we define a *K3 surface with at worst RDP (rational double point) singularities* to be a proper surface over a field, which, after base change to an algebraically closed field, has at worst rational double point singularities, and whose minimal resolution of singularities is a K3 surface.

THEOREM 1.3. *Let X be a K3 surface over K that satisfies Assumption (\star). If the G_K -representation on $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_\ell)$ is unramified for some $\ell \neq p$, then the following hold.*

- (i) *There exists a model of X that is a projective scheme over \mathcal{O}_K , whose special fiber is a K3 surface with at worst RDP singularities.*
- (ii) *Moreover, there exists an integer N , independent of X and K , and a finite unramified extension L/K of degree at most N , such that X_L has good reduction over L .*

In [HT17, Theorem 35], a similar result is obtained for K3 surfaces over $\mathbb{C}((t))$, but their proof uses methods different from ours. As in the case of Abelian varieties in [ST68], we obtain the following independence of ℓ .

COROLLARY 1.4. *Let X be a K3 surface over K that satisfies Assumption (\star) . Then the G_K -representation on $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_\ell)$ is unramified for one $\ell \neq p$ if and only if it is unramified for all $\ell \neq p$.*

In [ST68], Serre and Tate showed that if an Abelian variety of dimension g over K with $p > 2g + 1$ has potential good reduction, then good reduction can be achieved after a *tame* extension. Here, we establish the following analog for K3 surfaces.

COROLLARY 1.5. *Let X be a K3 surface over K with $p \geq 23$ and potential good reduction. Then X has good reduction after a tame extension of K .*

It is important to note that in part (2) of Theorem 1.3, we cannot avoid field extensions in general. More precisely, we construct the following explicit examples.

THEOREM 1.6. *For every prime $p \geq 5$, there exists a K3 surface $X = X(p)$ over \mathbb{Q}_p , such that:*

- (i) *the $G_{\mathbb{Q}_p}$ -representation on $H_{\text{ét}}^2(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_\ell)$ is unramified for all $\ell \neq p$;*
- (ii) *X has good reduction over the unramified extension \mathbb{Q}_{p^2} ; but*
- (iii) *X does not have good reduction over \mathbb{Q}_p .*

1.5 Kuga–Satake Abelian varieties

Let us recall that Kuga and Satake [KS67] associated to a polarized K3 surface (X, \mathcal{L}) over \mathbb{C} a polarized Abelian variety $\text{KS}(X, \mathcal{L})$ of dimension 2^{19} over \mathbb{C} . Moreover, if (X, \mathcal{L}) is defined over an arbitrary field k , then Rizov [Riz10] and Madapusi Pera [Mad15], building on work of Deligne [Del72] and André [And96], established the existence of $\text{KS}(X, \mathcal{L})$ over some finite extension of k . As an application of Theorem 1.3, we can compare the reduction behavior of a polarized K3 surface to that of its associated Kuga–Satake Abelian variety.

THEOREM 1.7. *Assume $p \neq 2$. Let (X, \mathcal{L}) be a polarized K3 surface over K .*

- (i) *If X has good reduction, then $\text{KS}(X, \mathcal{L})$ can be defined over an unramified extension L/K , and it has good reduction over L .*
- (ii) *Assume that X satisfies Assumption (\star) . Let L/K be a field extension such that both $\text{KS}(X, \mathcal{L})$ and the Kuga–Satake correspondence can be defined over L . If $\text{KS}(X, \mathcal{L})$ has good reduction over L , then X has good reduction over an unramified extension of L .*

1.6 Organization

This article is organized as follows.

In §2, we recall a couple of general facts on models and unramified Galois representations on ℓ -adic cohomology. We also recall the classical Serre–Tate theorem for Abelian varieties and give explicit examples of curves of genus at least 2, where the Galois representation does not detect bad reduction.

In §3, we review potential semi-stable reduction of K3 surfaces, Kawamata’s semi-stable MMP, the Kulikov–Nakkajima–Pinkham–Persson classification list, and the second named

author’s results on potential good reduction of K3 surfaces. We also briefly discuss potential good and semi-stable reduction of Enriques surfaces.

In §4, we establish existence and termination of certain flops, which we need later on to equip our models with suitable invertible sheaves. Moreover, we show that any two smooth models of a K3 surface X over K are related by a finite sequence of flopping contractions and their inverses.

Section 5 is the technical heart of this article: given a K3 surface X over K , a finite Galois extension L/K with group G , and a model of X_L over \mathcal{O}_L , we study extensions of the G -action X_L to this model. Then we study quotients of such models by G -actions, where the most difficult case arises when p divides the order of G (wild action).

In §6, we establish the main results of this article: a Néron–Ogg–Shafarevich type theorem for K3 surfaces, good reduction over tame extensions, as well as the connection to Kuga–Satake Abelian varieties.

Finally, in §7, we give explicit examples of K3 surfaces over \mathbb{Q}_p with unramified Galois representations on their ℓ -adic cohomology groups that do not have good reduction over \mathbb{Q}_p .

Notation and conventions

Throughout the whole article, we fix the following notation:

\mathcal{O}_K	a local Henselian DVR of characteristic zero;
K	its field of fractions;
k	the residue field, which we assume to be perfect;
$p \geq 0$	the characteristic of k ;
ℓ	a prime different from p ;
G_K, G_k	the absolute Galois groups $\text{Gal}(\overline{K}/K), \text{Gal}(\overline{k}/k)$.

If L/K is a field extension, and X is a scheme over K , we abbreviate the base-change $X \times_{\text{Spec } K} \text{Spec } L$ by X_L .

2. Generalities

In this section, we recall a couple of general facts on models of varieties, unramified Galois representations on ℓ -adic cohomology groups, and Néron–Ogg–Shafarevich type theorems.

2.1 Models

We start with the definition of various types of models.

DEFINITION 2.1. Let X be a smooth and proper variety over K .

- (i) A *model of X over \mathcal{O}_K* is an algebraic space that is flat and proper over $\text{Spec } \mathcal{O}_K$ and whose generic fiber is isomorphic to X .
- (ii) We say that X has *good reduction* if there exists a model of X that is smooth over \mathcal{O}_K .
- (iii) We say that X has *semi-stable reduction* if there exists a regular model of X , whose geometric special fiber is a reduced normal crossing divisor with smooth components. (Sometimes, this notion is also called *strictly semi-stable reduction*.)
- (iv) We say that X has *potential good* (respectively *semi-stable*) reduction if there exists a finite field extension L/K such that X_L has good (respectively semi-stable) reduction.

Remark 2.2. Models of curves and Abelian varieties can be treated entirely within the category of schemes; see, for example, [Liu02, ch. 10] and [BLR90]. However, if X is a K3 surface over K with good reduction, then it may not be possible to find a smooth model in the category of schemes, and we refer to [Mat15, § 5.2] for explicit examples. In particular, we are forced to work with algebraic spaces when studying models of K3 surfaces.

2.2 Inertia and monodromy

The G_K -action on \overline{K} induces an action on $\mathcal{O}_{\overline{K}}$ and by reduction, an action on \overline{k} . This gives rise to a continuous and surjective homomorphism $G_K \rightarrow G_k$ of profinite groups. Thus we obtain a short exact sequence

$$1 \rightarrow I_K \rightarrow G_K \rightarrow G_k \rightarrow 1,$$

whose kernel I_K is called the *inertia group*. In fact, I_K is the absolute Galois group of the maximal unramified extension of K . If $p \neq 0$, then the *wild inertia group* P_K is the normal subgroup of G_K that is the absolute Galois group of the maximal tame extension of K . We note that P_K is the unique p -Sylow subgroup of I_K .

DEFINITION 2.3. Let X be a smooth and proper variety over K . Then the G_K -representation on $H_{\text{ét}}^m(X_{\overline{K}}, \mathbb{Q}_\ell)$ is called *unramified* if I_K acts trivially. It is called *tame* if P_K acts trivially.

For an Abelian variety X , it follows from results of Serre and Tate [ST68] that the G_K -representation on $H_{\text{ét}}^m(X_{\overline{K}}, \mathbb{Q}_\ell)$ is unramified for *one* $\ell \neq p$, if and only if it is so for *all* $\ell \neq p$. In Corollary 6.4, we will show a similar result for K3 surfaces. In general, it is not known whether being unramified depends on the choice of ℓ , but it is expected not to.

A relation between good reduction and unramified Galois representations on ℓ -adic cohomology groups is given by the following well-known result, which follows from the proper smooth base change theorem. For schemes, it is stated in [SGA4, Théorème XII.5.1], and in case the model is an algebraic space, we refer to [LZ14, Theorem 0.1.1] or [Art73, ch. VII].

THEOREM 2.4. *If X has good reduction, then the G_K -representation on $H_{\text{ét}}^m(X_{\overline{K}}, \mathbb{Q}_\ell)$ is unramified for all m and for all $\ell \neq p$.*

In view of this theorem, it is natural to ask for the converse direction. Whenever such a converse holds for some class of varieties over K , we obtain a purely representation-theoretic criterion to determine whether such a variety admits a model over \mathcal{O}_K with good reduction.

2.3 Abelian varieties

A classical converse to Theorem 2.4 is the Néron–Ogg–Shafarevich criterion for elliptic curves. Later, Serre and Tate generalized it to Abelian varieties of arbitrary dimension.

THEOREM 2.5 (Serre–Tate [ST68]). *An Abelian variety A over K has good reduction if and only if the G_K -representation on $H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_\ell)$ is unramified for one (respectively all) $\ell \neq p$.*

2.4 Higher genus curves, part 1

Now, the converse to Theorem 2.4 already fails for curves of higher genus. Let X be a smooth and proper curve of genus $g \geq 2$ over K . Let $\text{Jac}(X)$ be its Jacobian, which is an Abelian variety of dimension g over K . Then the exact sequence of étale sheaves on X

$$1 \rightarrow \mu_n \rightarrow \mathbb{G}_m \xrightarrow{\times n} \mathbb{G}_m \rightarrow 1$$

gives rise to G_K -equivariant isomorphisms $H_{\text{ét}}^1(X_{\overline{K}}, \mu_n) \cong \text{Pic}(X_{\overline{K}})[n] \cong H_{\text{ét}}^1(\text{Jac}(X)_{\overline{K}}, \mu_n)$, from which we obtain $H_{\text{ét}}^1(X_{\overline{K}}, \mathbb{Q}_\ell) \cong H_{\text{ét}}^1(\text{Jac}(X)_{\overline{K}}, \mathbb{Q}_\ell)$ by passing to the limit. Moreover, if X has a K -rational point, then there is a natural embedding

$$j : X \rightarrow \text{Jac}(X),$$

and the above isomorphism coincides with j^* (which is independent of the choice of a rational point). By the Serre–Tate theorem (Theorem 2.5), an unramified G_K -representation on $H_{\text{ét}}^1(X_{\overline{K}}, \mathbb{Q}_\ell)$ is equivalent to good reduction of $\text{Jac}(X)$. The following lemma gives a criterion that ensures the latter.

LEMMA 2.6. *Let X be a smooth and proper curve over K that admits a semi-stable scheme model $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_K$ such that the dual graph associated to the components of its special fiber \mathcal{X}_0 is a tree. Then $\text{Jac}(X)$ has good reduction and the G_K -representation on $H_{\text{ét}}^1(X_{\overline{K}}, \mathbb{Q}_\ell)$ is unramified.*

Proof. By [BLR90, §9.2, Example 8], $\text{Pic}_{\mathcal{X}_0/k}^0$ is an Abelian variety, which implies that $\text{Jac}(X)$ has good reduction, and thus the G_K -representation on $H_{\text{ét}}^1(X_{\overline{K}}, \mathbb{Q}_\ell)$ is unramified. \square

Using this lemma, it is easy to produce counterexamples to Néron–Ogg–Shafarevich type results for curves of higher genus.

PROPOSITION 2.7. *If $p \neq 2$, then there exists for infinitely many $g \geq 2$ a smooth and proper curve X of genus g over K such that:*

- (i) *the G_K -representation on $H_{\text{ét}}^m(X_{\overline{K}}, \mathbb{Q}_\ell)$ is unramified for all m and all $\ell \neq p$; and*
- (ii) *X does not have good reduction over K nor over any finite extension.*

Proof. We give examples for $g \not\equiv 1 \pmod{p}$. Let X be a hyperelliptic curve of genus g over K that is one of the examples of [Liu02, Example 10.1.30] with the extra assumptions of [Liu02, Example 10.3.46] (here, we need the assumption $g \not\equiv 1 \pmod{p}$). Then X has stable reduction over K , as well as over every finite extension field L/K . In this example, the special fiber of the stable model is the union of a curve of genus 1 and a curve of genus $(g - 1)$ meeting transversally in one point. In particular, neither X nor any base-change X_L have good reduction, but since the assumptions of Lemma 2.6 are fulfilled, the G_K -representation on $H_{\text{ét}}^m(X_{\overline{K}}, \mathbb{Q}_\ell)$ is unramified for all m and all $\ell \neq p$. \square

We stress that these results are well known to the experts, but since we were not able to find explicit references and explicit examples, we decided to include them here.

2.5 Higher genus curves, part 2

If X is a smooth and proper curve of genus at least 2 over K , then one can also study the outer G_K -representation on its étale fundamental group, which turns out to detect good reduction. More precisely, there exists a short exact sequence of étale fundamental groups

$$1 \rightarrow \pi_1^{\text{ét}}(X_{\overline{K}}) \rightarrow \pi_1^{\text{ét}}(X) \rightarrow G_K \rightarrow 1.$$

For every prime ℓ , this exact sequence gives rise to a well-defined homomorphism from G_K to the outer automorphism group of the pro- ℓ -completion $\pi_1^{\text{ét}}(X_{\overline{K}})_\ell$ of the geometric étale fundamental group

$$\rho_\ell : G_K \longrightarrow \text{Out}(\pi_1^{\text{ét}}(X_{\overline{K}})_\ell).$$

In analogy to Definition 2.3, we will say that this representation is *unramified* if $\rho_\ell(I_K) = \{1\}$. We note that the G_K -representation on $H_{\text{ét}}^1(X_{\overline{K}}, \mathbb{Q}_\ell)$ arises from the residual action of ρ_ℓ on the Abelianization of $\pi_1^{\text{ét}}(X_{\overline{K}})_\ell$. After these preparations, we have the following Néron–Ogg–Shafarevich type theorem for curves of higher genus, which is in terms of fundamental groups rather than cohomology groups.

THEOREM 2.8 (Oda [Oda95, Theorem 3.2]). *Let X be a smooth and proper curve of genus at least 2 over K . Then X has good reduction if and only if the outer Galois action ρ_ℓ is unramified for one (respectively all) $\ell \neq p$.*

3. K3 surfaces and their models

In this section, we first introduce the crucial Assumption (\star) , which ensures the existence of suitable models for K3 surfaces. These models have been studied by Kulikov, Nakajima, Persson, and Pinkham. Following ideas of Maulik, we show how Assumption (\star) would follow from a combination of potential semi-stable reduction (which is not known in mixed characteristic, but expected) and the semi-stable minimal model program (MMP) in mixed characteristic. Then we give some conditions under which Assumption (\star) does hold. After that, we shortly review the second named author’s results on potential good reduction of K3 surfaces. Finally, we show by example that these results do not carry over to Enriques surfaces. Most of the results of this section are probably known to the experts.

3.1 Kulikov–Nakajima–Persson–Pinkham models

We first introduce the crucial assumption that we shall make from now on.

ASSUMPTION (\star) . A K3 surface X over K satisfies Assumption (\star) if there exists a finite field extension L/K such that X_L admits a semi-stable model $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_L$ (in the sense of Definition 2.1) such that $\omega_{\mathcal{X}/\mathcal{O}_L}$ is trivial.

Here, we equip \mathcal{X} with its standard log structure \mathcal{X}^{log} and define the relative canonical sheaf $\omega_{\mathcal{X}/\mathcal{O}_L}$ to be $\bigwedge^2 \Omega_{\mathcal{X}^{\text{log}}/\mathcal{O}_L^{\text{log}}}^1$ using log differentials. Since \mathcal{X}^{log} is log smooth over \mathcal{O}_L , the sheaf $\omega_{\mathcal{X}/\mathcal{O}_L}$ is invertible; see also the discussion in [Mat15, § 3].

The main reason why Assumption (\star) is not known to hold is that potential semi-stable reduction is not known: using resolution of singularities in mixed characteristic (recently announced by Cossart and Piltant [CP14]) and embedded resolution of singularities (Cossart *et al.* [CJS13]), we obtain a model \mathcal{X} , whose special fiber \mathcal{X}_0 has simple normal crossing support, but whose components may have multiplicities. At the moment, it is not clear how to get rid of these multiplicities after base change, unless all of them are prime to p . In case the residue characteristic is zero, these results are classically known to hold; see the discussion in [KM98, § 7.2] for details.

The following result, which is inspired by Maulik’s approach and ideas from [Mau14, § 4], shows that Assumption (\star) essentially holds once we assume potential semi-stable reduction. More precisely, we have the following.

PROPOSITION 3.1. *Assume $p \neq 2, 3$. Let X be a K3 surface over K and assume that there exists:*

- (i) *a finite field extension L'/K ; and*
- (ii) *a smooth surface Y over L' that is birationally equivalent to $X_{L'}$; and*
- (iii) *a scheme model $\mathcal{Y} \rightarrow \text{Spec } \mathcal{O}_{L'}$ of Y with semi-stable reduction.*

Then X satisfies Assumption (\star) .

Proof. Let $\mathcal{Y} \rightarrow \text{Spec } \mathcal{O}_{L'}$ be as in the statement. Since $p \neq 2, 3$, Kawamata’s semi-stable MMP [Kaw94] (see also [KM98, § 7.1] for $p = 0$) produces a scheme $\mathcal{Z} \rightarrow \text{Spec } \mathcal{O}_{L'}$ with nef relative canonical divisor $K_{\mathcal{Z}/\mathcal{O}_{L'}}$ that is a model of a smooth proper surface birationally equivalent to $X_{L'}$, and such that \mathcal{Z} is regular outside a finite set Σ of terminal singularities. We refer to [Kaw94, § 1] for details and the definition of $K_{\mathcal{Z}/\mathcal{O}_{L'}}$, which is a Weil divisor. We also note that it coincides with the Weil divisor class associated to the relative canonical divisor $\omega_{\mathcal{Z}/\mathcal{O}_{L'}}$, see, for example, [Mat15, § 3].

Since $X_{L'}$ is a minimal surface and $K_{\mathcal{Z}/\mathcal{O}_{L'}}$ is nef, the generic fiber of \mathcal{Z} is actually isomorphic to $X_{L'}$, and it follows that $K_{\mathcal{Z}/\mathcal{O}_{L'}}$ is trivial. Outside Σ , this model is already a semi-stable model. From the classification of terminal singularities in [Kaw94, Theorem 4.4] and the fact that $K_{\mathcal{Z}/\mathcal{O}_{L'}}$ is Cartier at points of Σ (since it is trivial), it follows that the geometric special fiber $(\mathcal{Z}_0)_{\bar{k}}$ is irreducible around points of Σ , and that it acquires RDP singularities in these points. Thus, after some finite field extension L/L' , there exists a simultaneous resolution $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_L$ of these singularities by [Art74, Theorem 2]. This \mathcal{X} may exist only as an algebraic space, and it satisfies Assumption (\star) . \square

As already mentioned above, the assumptions are fulfilled if $p = 0$; see [KKMS73, ch. 2] or the discussion in [KM98, § 7.2]. If $p \neq 0$, then they are fulfilled for K3 surfaces that admit a very ample invertible sheaf \mathcal{L} with $p > \mathcal{L}^2 + 4$ by a result of Maulik [Mau14, § 4]. With some extra work, the condition ‘very ample’ can be weakened to ‘ample’ (see [Mat15, argument following Lemma 3.1]) and Theorem 1.2 follows. Thus we have the following result.

THEOREM 3.2 (= Theorem 1.2). *Let X be a K3 surface over K and assume that $p = 0$ or that X admits an ample invertible sheaf \mathcal{L} with $p > \mathcal{L}^2 + 4$. Then X satisfies Assumption (\star) .*

Over \mathbb{C} , Kulikov [Kul77], Persson [Per77], and Pinkham and Persson [PP81] classified the special fibers of the models asserted by Assumption (\star) . We refer to [Mor81, § 1] and [KK98, ch. 5] for overview, and to Nakajima’s extension [Nak00] of these results to mixed characteristic.

3.2 Potential good reduction of K3 surfaces

Now, if X is a K3 surface over K that satisfies Assumption (\star) , then there exists a finite field extension L/K and a model $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_L$ of X_L as asserted by Assumption (\star) . If the G_K -representation on $H_{\text{ét}}^2(X_{\bar{K}}, \mathbb{Q}_\ell)$ is unramified, then the weight filtration on $H_{\text{ét}}^2(X_{\bar{K}}, \mathbb{Q}_\ell)$ that arises from the Steenbrink–Rapoport–Zink spectral sequence (see [Ste76], [RZ82, Satz 2.10], and [Naka00, Proposition 1.9] for details) is trivial. Together with a result of Persson [Per77, Proposition 3.3.6], this implies that the special fiber of \mathcal{X} is smooth, that is, X_L has good reduction. Thus we obtain the following result of the second named author and we refer to [Mat15] for details and a detailed proof.

THEOREM 3.3 (Matsumoto). *Let X be a K3 surface over K that satisfies Assumption (\star) . If the G_K -representation on $H_{\text{ét}}^2(X_{\bar{K}}, \mathbb{Q}_\ell)$ is unramified for one $\ell \neq p$, then X has potential good reduction.*

3.3 Enriques surfaces

The previous theorem does not generalize to other classes of surfaces with numerically trivial canonical sheaves. For example, the G_K -representation on ℓ -adic cohomology of an Enriques surface can neither exclude nor confirm any type in the Kulikov–Nakajima–Persson–Pinkham list for these surfaces. More precisely, we have the following.

LEMMA 3.4. *Let Y be an Enriques surface over K . Then there exists a finite extension L/K such that the G_L -representation on $H_{\text{ét}}^m(Y_{\overline{K}}, \mathbb{Q}_\ell)$ is unramified for all m and all $\ell \neq p$.*

Proof. We only have to show something for $m = 2$. But then the first Chern class induces a G_K -equivariant isomorphism

$$\text{NS}(Y_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \xrightarrow{c_1} H_{\text{ét}}^2(Y_{\overline{K}}, \mathbb{Q}_\ell)(1).$$

After passing to a finite extension L/K , we may assume that $\text{NS}(Y_L) = \text{NS}(Y_{\overline{K}})$. But then the G_L -representation on $\text{NS}(Y_L)$ is trivial; hence it is also trivial on $H_{\text{ét}}^2$, and, in particular, unramified. \square

Moreover, the next example shows that also the G_K -representation on the ℓ -adic cohomology of the K3 double cover X of an Enriques surface Y does not detect potential good reduction of Y . This phenomenon is related to flower pot degenerations of Enriques surfaces, see [Per77, §3.3] and [Per77, Appendix 2].

Example 3.5. Fix a prime $p \geq 5$. Consider $\mathbb{P}_{\mathbb{Z}_p}^5$ with coordinates $x_i, y_i, i = 0, 1, 2$, and inside it the complete intersection of three quadrics

$$\mathcal{X} := \begin{cases} x_1^2 - x_2^2 + y_0^2 & -y_2^2 = 0 \\ x_0^2 & -x_2^2 & +y_1^2 & -y_2^2 = 0 \\ x_0^2 - e^2 x_1^2 + x_2^2 & -p^2 y_2^2 = 0, \end{cases}$$

where $e \in \mathbb{Z}_p^\times$ satisfies $e^2 \not\equiv 0, 1, 2 \pmod p$ (for example, we could take $e = 2$). Then $\iota : x_i \mapsto x_i, y_i \mapsto -y_i$ defines an involution on $\mathbb{P}_{\mathbb{Z}_p}^5$, which induces an involution on \mathcal{X} . We denote by X the generic fiber of \mathcal{X} , and by $Y := X/\iota$ the quotient by the involution.

THEOREM 3.6. *Let $p \geq 5$ and let $X \rightarrow Y$ be as in Example 3.5. Then Y is an Enriques surface over \mathbb{Q}_p , such that:*

- (i) *the K3 double cover X of Y has good reduction;*
- (ii) *the $G_{\mathbb{Q}_p}$ -action on $H_{\text{ét}}^2(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_\ell)$ is unramified for all $\ell \neq p$;*
- (iii) *Y has semi-stable reduction of flower pot type; but*
- (iv) *Y does not have potential good reduction.*

Proof. A straightforward computation shows that X is smooth over \mathbb{Q}_p , and that ι acts without fixed points on X . Thus X is a K3 surface and Y is an Enriques surface over \mathbb{Q}_p . The special fiber of \mathcal{X} is a non-smooth K3 surface with four RDP singularities of type A_1 located at $[0 : 0 : 0 : \pm 1 : \pm 1 : 1]$. Then the blow-up $\mathcal{X}'_1 \rightarrow \mathcal{X}$ of the Weil divisor $\{x_0 - ex_1 = x_2 - py_2 = 0\}$ defines a simultaneous resolution of the singularities of $\mathcal{X} \rightarrow \text{Spec } \mathbb{Z}_p$, and we obtain a smooth model of X over \mathbb{Z}_p . In particular, X has good reduction over \mathbb{Q}_p and the $G_{\mathbb{Q}_p}$ -representation on $H_{\text{ét}}^2(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_\ell)$ is unramified for all $\ell \neq p$.

Next, let $\mathcal{X}'_2 \rightarrow \mathcal{X}$ be the blow-up of the four singular points of the special fiber. Then ι extends to \mathcal{X}'_2 , and the special fiber is the union of four divisors E_i with the minimal desingularization X'_p of the special fiber of \mathcal{X} . The fixed locus of ι on X'_p is the union of the four (-2) -curves of the resolution. Moreover, there exist isomorphisms $E_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ such that ι acts by interchanging the two factors. Thus the quotient \mathcal{X}'_2/ι is a model of Y over \mathbb{Z}_p , whose special fiber is a rational

surface X'_p/ι (a so-called Coble surface) meeting transversally four \mathbb{P}^2 's, that is, a semi-stable degeneration of flower pot type (see, [Per77, §3.3]).

Seeking a contradiction, we assume that Y has potential good reduction. Then there exists a finite extension L/\mathbb{Q}_p and a smooth model $\mathcal{Y} \rightarrow \text{Spec } \mathcal{O}_L$ of Y_L . Let $\mathcal{X}_3 \rightarrow \mathcal{Y} \rightarrow \text{Spec } \mathcal{O}_L$ be its K3 double cover, which is a family of smooth K3 surfaces with generic fiber X_L , whose fixed point free involution specializes to a fixed point free involution in the special fiber of \mathcal{X}_3 .

Now, \mathcal{X}_3 and the base-change of \mathcal{X}'_1 to \mathcal{O}_L both are smooth models of X_L . The isomorphism of generic fibers extends to a birational map of special fibers. The involution on generic fibers extends to rational involutions of the two special fibers, compatible with the just established birational map. Since both special fibers are K3 surfaces, the birational maps and rational involutions extend to isomorphisms and involutions. However, in one special fiber the involution acts without fixed points, whereas it has four fixed curves in the other, which contradicts the assumption. □

4. Existence and termination of flops

Let X be a smooth and proper surface over K with numerically trivial canonical sheaf and assume that we have a smooth model $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_K$. Now, if \mathcal{L} is an ample invertible sheaf on X , then its specialization \mathcal{L}_0 to the special fiber may not be ample, and not even be nef. In this section, we show that there exists a finite sequence of birational modifications (flops) of \mathcal{X} , such that we eventually arrive at a smooth model $\mathcal{X}^+ \rightarrow \text{Spec } \mathcal{O}_K$ of X , such that the restriction of \mathcal{L} to the special fiber of \mathcal{X}^+ is big and nef. We end this section by showing that any two smooth models of X over \mathcal{O}_K are related by a finite sequence of flopping contractions and their inverses.

We start by adjusting [KM98, Definition 3.33] and [KM98, Definition 6.10] to our situation.

DEFINITION 4.1. Let X be a smooth and proper surface over K with numerically trivial $\omega_{X/K}$ that admits a smooth model $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_K$. Then we have the following.

- (i) A proper and birational morphism $f: \mathcal{X} \rightarrow \mathcal{Y}$ over \mathcal{O}_K is called a *flopping contraction* if \mathcal{Y} is normal and if the exceptional locus of f is of codimension at least two.
- (ii) If D is a Cartier divisor on \mathcal{X} , then a birational map $\mathcal{X} \dashrightarrow \mathcal{X}^+$ over \mathcal{O}_K is called a *D-flop* if it decomposes into a flopping contraction $f: \mathcal{X} \rightarrow \mathcal{Y}$ followed by (the inverse of) a flopping contraction $f^+: \mathcal{X}^+ \rightarrow \mathcal{Y}$ such that $-D$ is f -ample and D^+ is f^+ -ample, where D^+ denotes the strict transform of D on \mathcal{X}^+ . If \mathcal{L} is an invertible sheaf on \mathcal{X} , we similarly define an *L-flop*.
- (iii) A morphism f^+ as in (2) is also called a *flop* of f .

In general, one also has to assume that $\omega_{\mathcal{X}/\mathcal{O}_K}$ is numerically f -trivial in the definition of a flopping contraction. However, in our situation this is automatic. Also, a flop of f , if it exists, does not depend on the choice of D by [KM98, Corollary 6.4] and [KM98, Definition 6.10]. This justifies talking about flops without referring to the divisor D .

4.1 Existence of flops

The following is an adaptation of Kollár's proof [Kol89, Proposition 2.2] of the existence of 3-fold flops over \mathbb{C} to our situation, which deals with special flops in mixed characteristic.

PROPOSITION 4.2 (Existence of flops). *Let X be a smooth and proper surface over K with numerically trivial $\omega_{X/K}$ that has a smooth model $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_K$. If \mathcal{L} is an ample invertible*

sheaf on X and C is an integral (but not necessarily geometrically integral) curve on the special fiber \mathcal{X}_0 with $\mathcal{L}_0 \cdot C < 0$, then there exists a flopping contraction $f: \mathcal{X} \rightarrow \mathcal{X}'$ and its \mathcal{L} -flop $f^+: \mathcal{X}^+ \rightarrow \mathcal{X}'$ with the following properties:

- (i) f contracts C and no other curves;
- (ii) $\mathcal{X}^+ \rightarrow \text{Spec } \mathcal{O}_K$ is a smooth model of X ;
- (iii) f and f^+ induce isomorphisms of generic fibers;
- (iv) $\mathcal{L}_0^+ \cdot C^+ > 0$, where \mathcal{L}^+ denotes the extension of \mathcal{L} on \mathcal{X}^+ , and where C^+ denotes the flopped curve (that is, the exceptional locus of f^+).

Proof. Since \mathcal{L} is ample, $\mathcal{L}^{\otimes n}$ is effective for $n \gg 0$, and thus also its specialization $\mathcal{L}_0^{\otimes n}$ to the special fiber \mathcal{X}_0 is effective. In particular, \mathcal{L}_0 has positive intersection with every ample divisor on \mathcal{X}_0 , that is, \mathcal{L}_0 is pseudo-effective. Thus there exists a Zariski–Fujita decomposition on $(\mathcal{X}_0)_{\bar{k}}$

$$(\mathcal{L}_0)_{\bar{k}} = P + N,$$

where P is nef, and where N is a sum of effective divisors, whose intersection matrix is negative definite; see, for example, [Băd01, Theorem 14.14]. Since $\omega_{\mathcal{X}_0/k}$ is numerically trivial, the adjunction formula shows that every reduced and irreducible curve in N is a \mathbb{P}^1 with self-intersection -2 , that is, a (-2) -curve. Moreover, negative definiteness and the classification of Cartan matrices implies that N is a disjoint union of ADE curves. Next, k is perfect and since the Zariski–Fujita decomposition is unique, it is stable under G_k , and thus descends to \mathcal{X}_0 .

After these preparations, let C be as in the statement, that is, $\mathcal{L}_0 \cdot C < 0$. First, we want to show that there exists a morphism $f: \mathcal{X} \rightarrow \mathcal{X}'$ of algebraic spaces that contracts C . Being contained in the support of N , the base-change $C_{\bar{k}} \subset (\mathcal{X}_0)_{\bar{k}}$ is a disjoint union of ADE curves. Since $C^2 < 0$, Artin showed that there exists a morphism of projective surfaces over k

$$f_0: \mathcal{X}_0 \rightarrow \mathcal{X}'_0$$

that contracts C and nothing else (see [Băd01, Theorem 3.9], for example). Since $C_{\bar{k}}$ is a union of ADE-curves, it follows that $(\mathcal{X}'_0)_{\bar{k}}$ has RDP singularities, which are rational and Gorenstein. Thus also \mathcal{X}'_0 has rational Gorenstein singularities.

For all $n \geq 0$, we define

$$\mathcal{X}_n := \mathcal{X} \times_{\text{Spec } \mathcal{O}_K} \text{Spec}(\mathcal{O}_K/\mathfrak{m}^{n+1}).$$

Since f_0 is a contraction with $R^1 f_{0*} \mathcal{O}_{\mathcal{X}_0} = 0$, there exists a blow-down $f_n: \mathcal{X}_n \rightarrow \mathcal{X}'_n$ that extends f_0 , see [CvS09, Theorem 3.1]. Passing to limits, we obtain a contraction of formal schemes

$$\widehat{f}: \widehat{\mathcal{X}} \rightarrow \widehat{\mathcal{X}}'.$$

By [Art70, Theorem 3.1], there exists a contraction of algebraic spaces

$$f: \mathcal{X} \rightarrow \mathcal{X}',$$

whose completion along their special fibers coincides with \widehat{f} . In particular, f is an isomorphism outside C and contracts C to a singular point $w \in \mathcal{X}'$.

Let \widehat{w} be the formal completion of \mathcal{X}' along w , and let

$$\widehat{\mathcal{Z}} \rightarrow \widehat{w}$$

be the formal fiber over \widehat{f} . Then \widehat{w} is a formal affine scheme, say $\mathrm{Spf} R$, and let k' be the residue field, which is a finite extension of k . Let $\mathcal{O}_{K'}$ be the unramified extension of \mathcal{O}_K corresponding to the field extension $k \subseteq k'$. Since $k \subseteq k'$ is separable, k' arises by adjoining a root α of some monic polynomial f with values k . After lifting f to a polynomial with values in \mathcal{O}_K , and using that R is Henselian, we can lift α to R , which shows that $\mathcal{O}_{K'}$ is contained in R . In particular, we can view R as a local $\mathcal{O}_{K'}$ -algebra without residue field extension – we denote by \widetilde{R} the ring R considered as $\mathcal{O}_{K'}$ -algebra.

Then the special fiber of $\mathrm{Spf} \widetilde{R}$ is a rational singularity of multiplicity 2, and thus, by [Lip69, Lemma 23.4], the completion of the local ring of the special fiber is of the form

$$k'[[x, y, z]]/(h'(x, y, z)). \tag{1}$$

Using Hensel’s lemma, we may assume after a change of coordinates that the power series $h'(x, y, z)$ is of the form $z^2 - h_1(x, y)z - h_0(x, y)$ for some polynomials $h_0(x, y), h_1(x, y)$. Using Hensel’s lemma again, the completion of \widetilde{R} is of the form

$$\widehat{\mathcal{O}}_{K'}[[x, y, z]]/(z^2 - H_1(x, y)z - H_0(x, y)) \tag{2}$$

where $H_i(x, y)$ is congruent to $h_i(x, y)$ modulo the maximal ideal of $\widehat{\mathcal{O}}_{K'}$ for $i = 1, 2$; see also [Kaw94, Theorem 4.4]. (If $p \neq 2$, we may even assume $h_1 = 0$ and $H_1 = 0$.) We denote by $t' : \mathrm{Spf} \widetilde{R} \rightarrow \mathrm{Spf} \widetilde{R}$ the involution induced by $z \mapsto H_1(x, y) - z$. It is not difficult to see that t' induces $-\mathrm{id}$ on local Picard groups, see, for example, [Kol89, Example 2.3]. Since R is equal to \widetilde{R} considered as rings, we have established an involution $t : \widehat{w} \rightarrow \widehat{w}$ that induces $-\mathrm{id}$ on local Picard groups. We denote by

$$\widehat{\mathcal{Z}}^+ \rightarrow \widehat{w}$$

the composition $t \circ \widehat{f}$. By [Kol89, Proposition 2.2], this gives the desired flop formally.

By [Art70, Theorem 3.2], there exists a dilatation $f^+ : \mathcal{X}^+ \rightarrow \mathcal{X}'$ of algebraic spaces, such that the formal completion of \mathcal{X}^+ along the exceptional locus of f^+ is given by the just-constructed $\widehat{\mathcal{Z}}^+ \rightarrow \widehat{w}$. Thus there exists a birational and rational map

$$\varphi : \mathcal{X} \dashrightarrow \mathcal{X}^+,$$

which is an isomorphism outside C . From the glueing construction it is clear that \mathcal{X}^+ is a smooth model of X over \mathcal{O}_K . Finally, from the formal picture above, it is clear that the restriction of \mathcal{L}^+ to \mathcal{X}_0^+ has positive intersection with the flopped curve C^+ . \square

4.2 Termination of flops

Having established the existence of certain flops in mixed characteristic, we now show that there is no infinite sequence of them. To do so, one can adjust the proof of termination of flops from [KM98, Theorem 6.17 and Corollary 6.19] over \mathbb{C} to our situation. Instead, we give another argument that was kindly suggested to us by the referee.

We keep the notation and assumptions of Proposition 4.2. Then there are two isomorphisms between the ℓ -adic cohomology groups of the special fibers \mathcal{X}_0 and \mathcal{X}_0^+ .

- (i) The first is by composing the comparison isomorphisms relating the cohomology groups of special and generic fibers of \mathcal{X} and \mathcal{X}^+

$$\alpha : H_{\text{ét}}^2((\mathcal{X}_0^+)_k, \mathbb{Q}_\ell)(1) \cong H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_\ell)(1) \cong H_{\text{ét}}^2((\mathcal{X}_0)_k, \mathbb{Q}_\ell)(1).$$

- (ii) Next, the composition $\varphi := (f^+)^{-1} \circ f : \mathcal{X} \dashrightarrow \mathcal{X}^+$ induces a birational and rational map of special fibers $\varphi_0 : \mathcal{X}_0 \dashrightarrow \mathcal{X}_0^+$, which extends to an isomorphism, since \mathcal{X}_0 and \mathcal{X}_0^+ are minimal surfaces of Kodaira dimension at least 0. Thus we obtain a second isomorphism via pull-back

$$\varphi_0^* : H_{\text{ét}}^2((\mathcal{X}_0^+)_k, \mathbb{Q}_\ell)(1) \cong H_{\text{ét}}^2((\mathcal{X}_0)_k, \mathbb{Q}_\ell)(1).$$

We note that both isomorphisms respect the intersection product coming from Poincaré duality, that is, they are isometries. For a (-2) -curve $C' \subset (\mathcal{X}_0)_k$, we let $[C']$ be the associated cycle class in $H_{\text{ét}}^2((\mathcal{X}_0)_k, \mathbb{Q}_\ell)(1)$ and we define the *reflection in C'* to be the isometry

$$r_{C'} : H_{\text{ét}}^2((\mathcal{X}_0)_k, \mathbb{Q}_\ell)(1) \rightarrow H_{\text{ét}}^2((\mathcal{X}_0)_k, \mathbb{Q}_\ell)(1) \\ x \mapsto x + (x \cdot [C'])[C'].$$

The following lemma compares the isometries α and φ_0^* in terms of reflections in (-2) -curves.

LEMMA 4.3. *We keep the notation and assumptions as in Proposition 4.2 and denote by C_1, \dots, C_m the connected components of C_k . Then*

$$\alpha \circ (\varphi_0^*)^{-1} = r_1 \cdots r_m,$$

where either:

- (i) the connected components C_i are disjoint (-2) -curves and $r_i = r_{C_i}$; or
- (ii) each C_i is the union of two (-2) -curves $C_{i,1}$ and $C_{i,2}$ intersecting in one point and $r_i = r_{C_{i,1}} r_{C_{i,2}} r_{C_{i,1}} = r_{C_{i,2}} r_{C_{i,1}} r_{C_{i,2}}$.

Proof. First, we consider the case, where C is geometrically integral. Let $\mathcal{Z} \subset \mathcal{X} \times_{\mathcal{O}_K} \mathcal{X}^+$ be the closure of the diagonal $\Delta(X) \subset X \times_K X$. Then it is not difficult to see that the isomorphism α is given by $x \mapsto \text{pr}_{1,*}([\mathcal{Z}_0] \cdot \text{pr}_2^*(x))$ (see also Lemma 5.6 below). We set $\mathcal{U} := \mathcal{X} \setminus C$ and $\mathcal{U}^+ := \mathcal{X}^+ \setminus C^+$. Then we have a commutative diagram with exact rows

$$\begin{CD} 0 @>>> H_{C^+}^2((\mathcal{X}_0^+)_k, \mathbb{Q}_\ell)(1) @>>> H_{\text{ét}}^2((\mathcal{X}_0^+)_k, \mathbb{Q}_\ell)(1) @>>> H_{\text{ét}}^2((\mathcal{U}_0^+)_k, \mathbb{Q}_\ell)(1) @>>> 0 \\ @. @V \alpha' VV @V \alpha VV @V \alpha'' VV @. \\ 0 @>>> H_C^2((\mathcal{X}_0)_k, \mathbb{Q}_\ell)(1) @>>> H_{\text{ét}}^2((\mathcal{X}_0)_k, \mathbb{Q}_\ell)(1) @>>> H_{\text{ét}}^2((\mathcal{U}_0)_k, \mathbb{Q}_\ell)(1) @>>> 0 \end{CD}$$

where α'' is also defined by $x \mapsto \text{pr}_{1,*}([\mathcal{Z}_0] \cdot \text{pr}_2^*(x))$ and where α' is the map induced by α . By purity, the left terms are one-dimensional and generated by the classes $[C]$ and $[C^+]$, respectively. Moreover, the right terms are canonically isomorphic to the orthogonal complements of the left terms. Since $\mathcal{Z}|_{\mathcal{U} \times \mathcal{U}^+}$ is the graph of the isomorphism $\varphi|_{\mathcal{U}} : \mathcal{U} \rightarrow \mathcal{U}^+$, it follows that α'' coincides with the pull-back by the isomorphism $\varphi_0|_{\mathcal{U}_0} = \varphi|_{\mathcal{U}_0} : \mathcal{U}_0 \rightarrow \mathcal{U}_0^+$. Since α is an isometry, so is α' , and it maps $[C^+]$ either to $[C]$ or to $-[C]$. From $\alpha(\mathcal{L}_0^+) \cdot \alpha(C^+) = \mathcal{L}_0^+ \cdot C^+ > 0$ and $\alpha(\mathcal{L}_0^+) \cdot C = \mathcal{L}_0^+ \cdot C < 0$, we conclude $\alpha([C^+]) = -[C]$. Putting these observations together, we find $\alpha \circ (\varphi_0^*)^{-1} = r_C$.

Now, we consider the general case. Since the absolute Galois group G_k acts transitively on the m connected components, they are mutually isomorphic. Since the flops in the disjoint C_i commute, we may assume $m = 1$. As shown in the proof of Proposition 4.2, $C_1 = C_k$ is an ADE configuration of (-2) -curves. Since G_k acts on the irreducible components of C_i transitively, it is not difficult to see from the classification of Dynkin diagrams that only configurations of type A_1 and A_2 can occur. We already treated the A_1 case above and thus we may assume an

A_2 -configuration, that is, $C_{\bar{k}} = C_1 = C_{1,1} \cup C_{1,2}$. Passing to a finite unramified extension K'/K corresponding to an extension k'/k over which C splits, we consider the following diagram of flops and models over $\mathcal{O}_{K'}$:

$$\begin{array}{ccccccc} \mathcal{X} & \xrightarrow{\varphi_1} & \mathcal{X}^1 & \xrightarrow{\varphi_2} & \mathcal{X}^{12} & \xrightarrow{\varphi_1} & \mathcal{X}^{121} \\ & \searrow \varphi_2 & & & & & \\ & & \mathcal{X}^2 & \xrightarrow{\varphi_1} & \mathcal{X}^{21} & \xrightarrow{\varphi_2} & \mathcal{X}^{212} \end{array}$$

where φ_j denotes the flop at $C_{1,j}$ or at the corresponding curve on other models (note that our flops induce isomorphisms between the special fibers). A straightforward computation shows that $r_{C_{1,1}}r_{C_{1,2}}r_{C_{1,1}}$ and $r_{C_{1,2}}r_{C_{1,1}}r_{C_{1,2}}$ both act as $-id$ on the one-dimensional subspace spanned by $[C_{1,1}] + [C_{1,2}]$ inside $H_{\text{ét}}^2((\mathcal{X}_0)_{\bar{k}}, \mathbb{Q}_\ell)(1)$ and as id on its orthogonal complement. Thus $\varphi_1\varphi_2\varphi_1: \mathcal{X} \dashrightarrow \mathcal{X}^{121}$ and $\varphi_2\varphi_1\varphi_2: \mathcal{X} \dashrightarrow \mathcal{X}^{212}$ both satisfy the conditions of the flop of the contraction f . Hence, they coincide by the uniqueness of flops, and we set $\varphi := \varphi_1\varphi_2\varphi_1 = \varphi_2\varphi_1\varphi_2: \mathcal{X} \dashrightarrow \mathcal{X}^+$. Clearly, φ descends to \mathcal{O}_K and coincides with the flop in C established in Proposition 4.2. \square

We define a *generalized (-2) -curve* on a smooth and proper surface X over a perfect field k to be an integral (but not necessarily geometrically integral) curve $C \subset X$ such that $C_{\bar{k}} = C_1 \cup \dots \cup C_m$ is a disjoint union of ADE curves of type A_1 or A_2 . We note that $C^2 = -2m$, that is, such curves are not necessarily of self-intersection -2 . Moreover, we define the reflection r_C in $\text{NS}(X)$ or $H_{\text{ét}}^2(X_{\bar{k}}, \mathbb{Q}_\ell)(1)$ to be equal to $r_1 \cdots r_m$ as in Lemma 4.3, which is equal to the map $x \mapsto x + \sum_{i=1}^m (x \cdot [C_i])[C_i]$. The following lemma is essentially [Huy16, Remark 8.2.13].

LEMMA 4.4. *Let X be a smooth and projective surface over a perfect field with numerically trivial canonical sheaf, and let $x \in \text{NS}(X)$ be a nonzero effective class with $x^2 \geq 0$.*

- (i) *If x is not nef, then there exists a generalized (-2) -curve C with $C \cdot x < 0$ and then $r_C(x)$ is non-zero and effective.*
- (ii) *We define a sequence in $\text{NS}(X)$ by setting $x_0 := x$ and if x_i is not nef, then we choose a generalized (-2) -curve C^i with $C^i \cdot x_i < 0$ and set $x_{i+1} := r_{C^i}(x_i)$. Then $\{x_i\}$ is a finite sequence of non-zero and effective classes in $\text{NS}(X)$ and the last class is nef.*

Proof. Using the Zariski–Fujita decomposition of x (see the proof of Proposition 4.2) and Lemma 4.3, we see that if x is not nef, then a generalized (-2) -curve C with $C \cdot x < 0$ indeed exists. Since Abelian and bielliptic surfaces do not admit smooth rational curves, we may assume that X is a K3 surface or an Enriques surface. Since $r_C(x)^2 = x^2 \geq 0$, it follows from the Riemann–Roch theorem that either $r_C(x)$ or $-r_C(x)$ is effective. Let C_1, \dots, C_m be the connected components of $C_{\bar{k}}$. Then we find $x \cdot r_C(x) = x^2 + \sum_i (x \cdot C_i)^2 > 0$, from which it follows that x and $r_C(x)$ belong to the same component of the cone $\{y \in \text{NS}(X)_{\mathbb{R}} : y^2 > 0\}$, and thus $r_C(x)$ is effective. This establishes assertion (i).

To show assertion (ii), we fix an ample class H of X and let C_1, \dots, C_m be the connected components of $C_{\bar{k}}$. Since G_k acts transitively on these components, we find $x \cdot C_i = (1/m)x \cdot C$ and $H \cdot C_i = (1/m)H \cdot C$, from which we conclude

$$r_C(x) \cdot H = \left(x + \sum_{i=1}^m (x \cdot C_i)C_i \right) \cdot H = x \cdot H + \frac{1}{m}(x \cdot C)(H \cdot C) < x \cdot H,$$

since $x \cdot C < 0$ by assumption and $H \cdot C > 0$ by ampleness of H . Therefore, if $\{x_i\} \in \text{NS}(X)$ is as in assertion (ii), then $\{x_i \cdot H\}$ is a strictly decreasing sequence of positive integers. In particular, it must be of finite length, and its last class must be nef. \square

After these preparations, we obtain the following.

PROPOSITION 4.5 (Termination of flops). *Let (X, \mathcal{L}) and $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_K$ be as in Proposition 4.2. Then every sequence of flops as in Proposition 4.2 is finite. In particular, after a finite sequence*

$$(\mathcal{X}, \mathcal{L}) \dashrightarrow (\mathcal{X}^+, \mathcal{L}^+) \dashrightarrow (\mathcal{X}^{+2}, \mathcal{L}^{+2}) \dashrightarrow \dots \dashrightarrow (\mathcal{X}^{+N}, \mathcal{L}^{+N})$$

of flops we arrive at a smooth model $(\mathcal{X}^{+N}, \mathcal{L}^{+N})$ of X over \mathcal{O}_K such that the specialization \mathcal{L}_0^{+N} is big and nef.

Proof. Let $\dots \dashrightarrow (\mathcal{X}^{+i}, \mathcal{L}^{+i}) \dashrightarrow \dots$ be a sequence of flops in generalized (-2) -curves $C^i \subset (\mathcal{X}^{+i})_0$ as asserted by Proposition 4.2 and Lemma 4.3. Moreover, we have $[\mathcal{L}_0^{+(i+1)}] = [r_{C^i}(\mathcal{L}_0^{+i})]$ by Lemma 4.3. By Lemma 4.4, this sequence is finite and \mathcal{L}_0^{+N} is nef. \square

4.3 Morphism to a projective scheme

In the situation of Proposition 4.5, we obtain a birational morphism to a projective scheme as follows.

PROPOSITION 4.6. *Let X be a smooth and proper surface over K with numerically trivial $\omega_{X/K}$ that admits a smooth model $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_K$. Let \mathcal{L} be an ample invertible sheaf on X and assume that \mathcal{L}_0 is big and nef. Then the natural and a priori rational map*

$$\pi : \mathcal{X} \rightarrow \mathcal{X}' := \text{Proj} \bigoplus_{n \geq 0} H^0(\mathcal{X}, \mathcal{L}^{\otimes n})$$

is a morphism over $\text{Spec } \mathcal{O}_K$ to a projective scheme. More precisely:

- (i) π is a flopping contraction and induces an isomorphism of generic fibers;
- (ii) the induced morphism on special fibers $\pi_0 : \mathcal{X}_0 \rightarrow \mathcal{X}'_0$ is birational and contracts precisely those curves that have zero-intersection with \mathcal{L}_0 . In particular, $(\mathcal{X}'_0)_{\bar{k}}$ is a proper surface with at worst RDP singularities and π_0 is the minimal resolution of singularities.

Proof. Note that also $\omega_{\mathcal{X}_0/k}$ is numerically trivial. Since \mathcal{L}_0 is big and nef, we obtain a proper and birational morphism

$$\varpi : \mathcal{X}_0 \rightarrow W := \text{Proj} \bigoplus_{n \geq 0} H^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes n}).$$

Base-changing to $(\mathcal{X}_0)_{\bar{k}}$, the induced morphism $\varpi_{\bar{k}}$ contracts an integral curve C if and only if it has zero-intersection with \mathcal{L}_0 . Since the intersection matrix formed by contracted curves is negative definite, and since an integral curve with negative self-intersection on a surface with numerically trivial canonical sheaf over an algebraically closed field is a (-2) -curve, it follows from the classification of Cartan matrices, that $W_{\bar{k}}$ has at worst RDP singularities.

Now, $\mathcal{L}_0^{\otimes n}$ is of degree 0 on contracted curves for all n , and over \bar{k} , these curves are ADE curves. Thus we find $R^1 \varpi_* \mathcal{L}_0^{\otimes n} = 0$ for all $n \geq 0$, which implies $H^1(\mathcal{X}_0, \mathcal{L}_0^{\otimes n}) = H^1(W, \mathcal{O}_W(n))$ for all $n \geq 0$, and note that the latter term is zero for $n \gg 0$ by Serre vanishing. Replacing \mathcal{L} by some

sufficiently high tensor power will not change ϖ , and then we may assume that $H^1(\mathcal{X}_0, \mathcal{L}_0^{\otimes n}) = 0$ for all $n \geq 1$. If $f : \mathcal{X} \rightarrow \text{Spec } \mathcal{O}_K$ denotes the structure morphism, then semi-continuity and the previous vanishing result imply $R^1 f_* \mathcal{L}^{\otimes n} = 0$ for all $n \geq 1$. Thus global sections of $\mathcal{L}_0^{\otimes n}$ extend to $\mathcal{L}^{\otimes n}$, and since the former is globally generated for $n \gg 0$ so is the latter. Thus we obtain a morphism of algebraic spaces over $\text{Spec } \mathcal{O}_K$

$$\pi : \mathcal{X} \rightarrow \mathcal{X}' := \text{Proj} \bigoplus_{n \geq 0} H^0(\mathcal{X}, \mathcal{L}^{\otimes n}).$$

Since \mathcal{L} is ample on X , π induces an isomorphism of generic fibers. Moreover, we can identify the induced map π_0 on special fibers with $\varpi : \mathcal{X}_0 \rightarrow W$ from above. □

4.4 Birational relations among smooth models

As an application of existence and termination of flops, Kollár [Kol89, Theorem 4.9] showed that any two birational complex threefolds with \mathbb{Q} -factorial terminal singularities and nef canonical classes are connected by a finite sequence of flops.

We have the following analog in our situation, but since we are dealing with algebraic spaces rather than projective schemes (which is analogous to the case of analytic threefolds in [Kol89]), the flops as defined above do not suffice. We only show that two smooth models are connected by flopping contractions and their inverses.

PROPOSITION 4.7. *Let X be a smooth and proper surface over K with numerically trivial $\omega_{X/K}$ that has good reduction. If $\mathcal{X}_i \rightarrow \text{Spec } \mathcal{O}_K$ are two smooth models of X , then:*

- (i) *the special fibers of \mathcal{X}_1 and \mathcal{X}_2 are isomorphic; and*
- (ii) *\mathcal{X}_1 and \mathcal{X}_2 are connected by a sequence of birational rational maps that are compositions of flopping contractions and their inverses.*

Proof. The special fibers of \mathcal{X}_1 and \mathcal{X}_2 are birational by the Matsusaka–Mumford theorem [MM64, Theorem 2], and since they are minimal surfaces of Kodaira dimension at least 0, they are isomorphic. (Note that this statement also follows from the much more detailed analysis below.)

Now, choose an ample invertible sheaf \mathcal{L} on X . By Proposition 4.5, there exist finite sequences of flops $\mathcal{X}_i \dashrightarrow \cdots \dashrightarrow \mathcal{Y}_i$, $i = 1, 2$, such that \mathcal{L} restricts to big and nef invertible sheaves on the special fibers of \mathcal{Y}_i .

Applying Proposition 4.6 to our models \mathcal{Y}_i , we obtain flopping contractions

$$\mathcal{Y}_i \rightarrow \mathcal{Y}'_i := \text{Proj} \bigoplus_{n \geq 0} H^0(\mathcal{Y}_i, \mathcal{L}^{\otimes n}).$$

Now, \mathcal{L} is ample on \mathcal{Y}'_i . Moreover, the \mathcal{Y}'_i are normal projective schemes and birational outside a finite number of curves in their special fibers. In fact, there exists a birational and rational map between them that is compatible with \mathcal{L} . Thus, by [Kov09, Theorem 5.14], this birational map extends to an isomorphism, and then we obtain a birational map $\mathcal{Y}_1 \dashrightarrow \mathcal{Y}_2$ with decomposition $\mathcal{Y}_1 \rightarrow \mathcal{Y}'_1 \cong \mathcal{Y}'_2 \leftarrow \mathcal{Y}_2$ of the required form.

Putting all these birational modifications together, we have connected \mathcal{X}_1 and \mathcal{X}_2 by a sequence of birational maps of the required form. □

5. Group actions on models

In this section, we study group actions on models. More precisely, we are given a smooth and proper surface X over K with numerically trivial $\omega_{X/K}$, a finite field extension L/K , which is Galois with group G , and a smooth proper model $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_L$ of X_L . Then we study the following questions.

- (i) Does the G -action on X_L extend to \mathcal{X} ?
- (ii) If so, is the special fiber $(\mathcal{X}/G)_0$ of the quotient equal to the quotient \mathcal{X}_0/G of the special fiber?

It turns out, that the answer to question (i) is ‘yes’, when allowing certain birational modifications of the model, and in question (ii), it turns out that the case where $p \neq 0$ and p divides the order of G (wild group actions) is subtle.

5.1 Extending group actions to a possibly singular model

Given a smooth and proper surface X over K with numerically trivial $\omega_{X/K}$ that admits a model \mathcal{X} with good reduction after a finite Galois extension L/K with group G , we first show that the G -action extends to a (mild) birational modification of \mathcal{X} .

PROPOSITION 5.1. *Let X be a smooth and proper surface over K with numerically trivial $\omega_{X/K}$. Assume that there exist:*

- (i) a finite Galois extension L/K with Galois group G ; as well as
- (ii) a smooth model $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_L$ of X_L ; and
- (iii) an ample invertible sheaf \mathcal{L} on X , whose pull-back to X_L restricts to an invertible sheaf on the special fiber \mathcal{X}_0 that is big and nef.

Then there exists a proper birational morphism π

$$\begin{array}{ccc}
 \mathcal{X} & \xrightarrow{\pi} & \mathcal{X}' \\
 \downarrow & \swarrow & \\
 \text{Spec } \mathcal{O}_L & &
 \end{array}$$

of algebraic spaces over \mathcal{O}_L , such that the following hold.

- (i) The natural G -action on X_L extends to \mathcal{X}' and is compatible with the G -action on \mathcal{O}_L .
- (ii) The algebraic space \mathcal{X}' is a projective scheme over $\text{Spec } \mathcal{O}_L$.
- (iii) The generic fibers of \mathcal{X} and \mathcal{X}' are isomorphic via π , whereas the induced morphism on special fibers $\pi_0 : \mathcal{X}_0 \rightarrow \mathcal{X}'_0$ is birational and projective, such that the geometric special fiber $(\mathcal{X}'_0)_{\bar{k}}$ has at worst RDP singularities.

Moreover, if π is not an isomorphism, then \mathcal{X}' is not regular.

Proof. Since \mathcal{X} is regular, the pull-back of \mathcal{L} to X_L extends to an invertible sheaf on \mathcal{X} . By abuse of notation, we shall denote the pull-back to X_L and its extension to \mathcal{X} again by \mathcal{L} . By assumption, the restriction \mathcal{L}_0 of \mathcal{L} to the special fiber \mathcal{X}_0 is big and nef. Note that $\omega_{\mathcal{X}_0/k}$ is numerically trivial. Let

$$\pi : \mathcal{X} \rightarrow \mathcal{X}' := \text{Proj} \bigoplus_{n \geq 0} H^0(\mathcal{X}, \mathcal{L}^{\otimes n})$$

be the morphism of algebraic spaces over \mathcal{O}_L given by Proposition 4.6. Then π has all the properties asserted in claim (iii) of the proposition. Clearly, \mathcal{X}' is a projective scheme over $\text{Spec } \mathcal{O}_L$, and if π is not an isomorphism, then the exceptional locus is non-empty and of codimension 2, which implies that \mathcal{X}' cannot be regular by van der Waerden purity; see, for example, [Liu02, Theorem 7.2.22].

It remains to establish the G -action on \mathcal{X}' . Since \mathcal{L} is a G -invariant invertible sheaf on X_L , we have an induced G -action on $H^0(X_L, \mathcal{L}^{\otimes n})$ for all $n \geq 0$. We will show that this extends to an action on $H^0(\mathcal{X}, \mathcal{L}^{\otimes n})$. First, we show that there exists a closed subspace $\mathcal{Z} \subset \mathcal{X}$ of codimension at least 2 that is contained in \mathcal{X}_0 , such that the G -action on X_L extends to an action on $\mathcal{U} := \mathcal{X} \setminus \mathcal{Z}$. Since every birational rational map between two normal algebraic spaces is an isomorphism outside a closed subspace of codimension at least 2, there exists for every $g \in G$ a closed subspace $\mathcal{Z}_g \subset \mathcal{X}$ of codimension at least 2 that is contained in \mathcal{X}_0 and such that $g : X_L \rightarrow X_L$ extends to $g : \mathcal{X} \setminus \mathcal{Z}_g \rightarrow \mathcal{X}$. Since \mathcal{X}_0 is a minimal surface, the restriction $g|_{\mathcal{X}_0 \setminus \mathcal{Z}_g} : \mathcal{X}_0 \setminus \mathcal{Z}_g \rightarrow \mathcal{X}_0$ to the special fiber extends to an automorphism $g : \mathcal{X}_0 \rightarrow \mathcal{X}_0$. This defines a G -action on \mathcal{X}_0 . Let $\mathcal{Z}' := \bigcup_{g \in G} \mathcal{Z}_g$ and $\mathcal{Z} := \bigcup_{g \in G} g(\mathcal{Z}')$, where $g(\mathcal{Z}')$ is the image by the action just defined. Since G is a finite group, \mathcal{Z} is closed. This \mathcal{Z} satisfies the above condition, for otherwise there exists a $g \in G$, such that the image of $g : \mathcal{X} \setminus \mathcal{Z} \rightarrow \mathcal{X}$ is not contained in $\mathcal{X} \setminus \mathcal{Z}$, or, equivalently $g^{-1}(\mathcal{Z}) \not\subset \mathcal{Z}$. However, since \mathcal{Z} is G -stable, this cannot happen. Thus we obtain a G -action on $\mathcal{U} = \mathcal{X} \setminus \mathcal{Z}$.

If s is a global section of $\mathcal{L}^{\otimes n}$ over \mathcal{X} and $\sigma \in G$, then $\sigma(s|_{\mathcal{U}})$ is a well-defined global section of $\mathcal{L}^{\otimes n}$ over $\mathcal{U} = \mathcal{X} \setminus \mathcal{Z}$. Since $\mathcal{L}^{\otimes n}$ is a reflexive sheaf on a regular algebraic space, $\sigma(s|_{\mathcal{U}})$ extends uniquely to a global section of $\mathcal{L}^{\otimes n}$ over \mathcal{X} . Thus we obtain a G -action on $H^0(\mathcal{X}, \mathcal{L}^{\otimes n})$, which gives rise to a G -action on \mathcal{X}' that is compatible with the G -action on \mathcal{O}_L , as well as with the natural G -action on X_L . □

Remark 5.2. If all assumptions of Proposition 5.1 except assumption (iii) are satisfied, then Proposition 4.5 shows that there exists another smooth model of X over $\text{Spec } \mathcal{O}_L$ for which all assumptions including assumption (iii) hold, and to which we can apply Proposition 5.1.

5.2 Examples where the action does not extend

In general, it is too much to ask for an extension of the G -action from X_L to \mathcal{X} (notation as in Proposition 5.1). The following example is typical.

Example 5.3 (Arithmetic 3-fold flop). Consider \mathbb{Q}_p with $p \neq 2$ and set $L := \mathbb{Q}_p(\varpi)$, where $\varpi^2 = p$. Then L/\mathbb{Q}_p is Galois with group $G = \mathbb{Z}/2\mathbb{Z}$ and the non-trivial element of G acts as $\varpi \mapsto -\varpi$. We equip

$$\mathcal{X}' := \text{Spec } \mathcal{O}_L[[x, y, z]]/(xy + z^2 - \varpi^2) \rightarrow \text{Spec } \mathcal{O}_L$$

with the G -action that is the Galois action on \mathcal{O}_L , and that is trivial on x, y, z . It is easy to see that the induced G -action on the special fiber \mathcal{X}'_0 is trivial. Next, we consider the two ideal sheaves $\mathcal{I}_{\pm} := (x, y, z \pm \varpi)$ of $\mathcal{O}_{\mathcal{X}'}$ and their blow-ups

$$\pi_{\pm} : \mathcal{X}_{\pm} \longrightarrow \mathcal{X}'.$$

Then \mathcal{X}_{\pm} are regular schemes, \mathcal{X}' is singular at the closed point (x, y, z, ϖ) , π_{\pm} are both resolutions of singularities, and the exceptional locus is a \mathbb{P}^1 in both cases. The ideals \mathcal{I}_{\pm} are not G -invariant and the G -action on \mathcal{X}' does not extend to that on \mathcal{X}_{+} nor on \mathcal{X}_{-} . (Instead, the non-trivial element of G induces an isomorphism $\mathcal{X}_{+} \rightarrow \mathcal{X}_{-}$.) In fact, \mathcal{X}' is an arithmetic version of a 3-fold ordinary double point, and the rational map $\mathcal{X}_{+} \dashrightarrow \mathcal{X}_{-}$ is an arithmetic version of the classical Atiyah flop.

Even worse, the following example (which is a modification of Example 7.1 below, and rests on examples from [Mat15, § 5.3] and [vanL07, § 3]) shows that if we have a G -action on a singular model \mathcal{X}' as in Proposition 5.1, then there may exist resolutions of singularities to which the G -action extends, as well as resolutions to which the G -action does not extend. Moreover, our examples are models of K3 surfaces, that is, such phenomena are highly relevant for our discussion.

Example 5.4. Before giving explicit examples, let us explain the strategy, again, for $K = \mathbb{Q}_p$ and $k = \mathbb{F}_p$ in this example.

Assume $p \neq 2$, let k'/k be the unique extension of degree 2, let K'/K be the corresponding unramified extension, and let $G = \{1, \sigma\}$ be the Galois group of both extensions. Next, let $\mathcal{X}' \rightarrow \text{Spec } \mathcal{O}_K$ be a proper scheme such that the geometric special fiber $(\mathcal{X}'_0)_{\bar{k}}$ has only RDP singularities and at least two of them. Set $S := \text{Sing } \mathcal{X}'_0$, and assume that all points of S are k' -rational but not all k -rational. Then G acts non-trivially on $S(k') = S(\bar{k})$. Let us finally assume that there exist two different resolutions of singularities $\psi_{\pm}: \mathcal{X}_{\pm} \rightarrow \mathcal{X}'$, both of which are isomorphisms outside S , and both of which are obtained by blowing up ideal sheaves \mathcal{I}_{\pm} defined over \mathcal{O}_K . From this setup, we can produce the announced counterexamples.

- (i) The Galois action on $(\mathcal{X}')_{\mathcal{O}_{K'}}$ extends to $(\mathcal{X}_+)_{\mathcal{O}_{K'}}$, as well as to $(\mathcal{X}_-)_{\mathcal{O}_{K'}}$. Thus there do exist resolutions of singularities to which the G -action extends.
- (ii) On the other hand, for each decomposition $S(k') = S_1 \sqcup S_2$, we define $\psi_{S_1, S_2}: \mathcal{X}_{S_1, S_2} \rightarrow (\mathcal{X}')_{\mathcal{O}_{K'}}$ to be the morphism that is equal to ψ_+ (respectively ψ_-) on $(\mathcal{X}')_{\mathcal{O}_{K'}} \setminus S_2$ (respectively on $(\mathcal{X}')_{\mathcal{O}_{K'}} \setminus S_1$). We note that ψ_{S_1, S_2} is also a resolution of singularities. But now, if S_1 and S_2 are not G -stable, then the G -action on $(\mathcal{X}')_{\mathcal{O}_{K'}}$ does not extend to \mathcal{X}_{S_1, S_2} , but induces an isomorphism from \mathcal{X}_{S_1, S_2} to $\mathcal{X}_{\sigma(S_1), \sigma(S_2)}$, where $\sigma \in G$ is the non-trivial element.

We now give explicit examples for $p \geq 5$ and $K = \mathbb{Q}_p$. Fix a prime $p \geq 5$ and choose an integer d such that d is not a quadratic residue modulo p , and such that $d^6 \not\equiv -2^{-4} \cdot 3^{-3} \pmod{p}$ (one easily checks that such d exists). We define the polynomial

$$\begin{aligned} \phi := & x^3 - x^2y - x^2z + x^2w - xy^2 - xyz + 2xyw + xz^2 + 2xzw \\ & + y^3 + y^2z - y^2w + yz^2 + yzw - yw^2 + z^2w + zw^2 + 2w^3. \end{aligned}$$

Then we choose a homogeneous polynomial $f \in \mathbb{Z}[x, y, z, w]$ of degree 3, such that the following congruences hold:

$$\begin{aligned} f &\equiv \phi \pmod{2}, \\ f &\equiv z(x^2 - z^2) + w^3 \pmod{p}. \end{aligned}$$

Next, we choose homogeneous quadratic polynomials $2g, 2h \in \mathbb{Z}[x, y, z, w]$, such that the following congruences hold:

$$\begin{aligned} g^2 - p^2h^2 &\equiv (z^2 + xy + yz)(z^2 + xy) \pmod{2}, \\ g^2 - p^2h^2 &\equiv (y^2 - dx^2)^2 \pmod{p}. \end{aligned}$$

Finally, we define the quartic hypersurface

$$\mathcal{X}' := \mathcal{X}'(p) := \{wf + g^2 - p^2h^2 = 0\} \subset \mathbb{P}_{\mathbb{Z}_p}^3,$$

and denote by $X = X(p)$ its generic fiber. Then X is a smooth K3 surface over \mathbb{Q}_p . The subscheme $S = \text{Sing } \mathcal{X}'_0$ is given by

$$S = \{w = y^2 - dx^2 = z(x^2 - z^2) = 0\} \subset \mathbb{P}^3_{\mathbb{F}_p}.$$

Thus we find six RDP singularities on $(\mathcal{X}'_0)_{\overline{\mathbb{F}_p}}$, all of which are defined over $\mathbb{F}_{p^2} = \mathbb{F}_p[\sqrt{d}]$, and $G = \text{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$ acts non-trivially on $S(k')$, since $\sqrt{d} \notin \mathbb{F}_p$. Finally, the blow-ups $\psi_{\pm}: \mathcal{X}_{\pm} \rightarrow \mathcal{X}'$ of the ideals $\mathcal{I}_{\pm} := (w = g \pm ph = 0)$ are both resolutions of singularities. As explained in the strategy above, this setup yields the desired examples. (We refer to Remark 7.3 for the reason why we use this ϕ .)

5.3 Extending the inertia action to the smooth model

Despite all these discouraging examples, there are situations, in which the G -action on X_L does extend to \mathcal{X} , and not merely to a singular model \mathcal{X}' (notation as in Proposition 5.1). More precisely, we have the following result.

PROPOSITION 5.5. *We keep the notation and assumptions of Proposition 5.1.*

- (i) *If X is an Abelian surface or a hyperelliptic surface, then the G -action on X_L extends to \mathcal{X} .*
- (ii) *Let $H \subseteq G$ be a subgroup, whose action on $H^2_{\text{ét}}(X_{\overline{L}}, \mathbb{Q}_{\ell})$ is trivial (for example, this is the case if $H^2_{\text{ét}}(X_{\overline{L}}, \mathbb{Q}_{\ell})$ is unramified and $H \subseteq I_G$). Then the H -action on X_L extends to \mathcal{X} .*

We first introduce cycle class maps in the context of algebraic spaces.

LEMMA 5.6. *Let \mathcal{X} be a proper and smooth algebraic space over the spectrum $S = \text{Spec } \mathcal{O}_K$ of a strictly Henselian DVR \mathcal{O}_K , and let $\mathcal{Z} \subset \mathcal{X}$ be a closed subspace of codimension c that is flat over S . Then the natural isomorphism*

$$H^{2c}_{\text{ét}}((\mathcal{X}_0)_{\overline{k}}, \mathbb{Z}/n\mathbb{Z}(c)) \rightarrow H^{2c}_{\text{ét}}(\mathcal{X}_{\overline{K}}, \mathbb{Z}/n\mathbb{Z}(c))$$

maps $[\mathcal{Z}_0]$ to $[\mathcal{Z}_K]$.

Proof. This is immediate if we define a cycle class $[\mathcal{Z}]$ in $H^{2c}_{\text{ét}}(\mathcal{X}, \mathbb{Z}/n\mathbb{Z}(c))$ in such a way, that it is compatible with base change $S' \rightarrow S$ of base schemes (and thus, in particular, compatible with restrictions to generic and special fibers). If \mathcal{X} is a scheme, this is defined and shown in [SGA4 $\frac{1}{2}$, Cycle, Numéro 2.3], and thus it remains to treat the case where \mathcal{X} is an algebraic space.

First, let us recall cohomological descent. Let $V \rightarrow \mathcal{Y}$ be an étale covering of schemes. Such a morphism is of cohomological descent by [SGA4, Proposition V^{bis}.4.3.3], and we have a spectral sequence [SGA4, Proposition V^{bis}.2.5.5]

$$E_1^{p,q} := H^q_{\text{ét}}(V_p, a_p^*F) \Rightarrow H^{p+q}_{\text{ét}}(\mathcal{Y}, F),$$

where, for each $p \geq 0$, V_p is the $(p+1)$ -fold fibered product of V over \mathcal{Y} , and where $a_p: V_p \rightarrow \mathcal{Y}$ is the structure map. Next, we consider the more general case, where $V \rightarrow \mathcal{Y}$ is an étale covering of an algebraic space by a scheme. (Note that, then, the V_p are schemes.) We observe that $V \rightarrow \mathcal{Y}$ is still of cohomological descent (proved in the same way) and we obtain the same spectral sequence.

After these preliminary remarks, let $U \rightarrow \mathcal{X}$ be an étale covering by a scheme U . Applying the previous paragraph to the covering $U \times_{\mathcal{X}} \mathcal{Z} \rightarrow \mathcal{Z}$ and the sheaf $R^{2c}i^!\mathbb{Z}/n\mathbb{Z}(c)$ on \mathcal{Z} (where

$i: \mathcal{Z} \rightarrow \mathcal{X}$ is the closed immersion), and using the isomorphism $H_{\mathcal{Z}}^{2c}(\mathcal{X}, \mathbb{Z}/n\mathbb{Z}(c)) \cong H^0(\mathcal{Z}, R^{2c}i^!\mathbb{Z}/n\mathbb{Z}(c))$ (this isomorphism is also proved by reducing to the scheme case), we obtain an isomorphism

$$\text{Ker}(H_{Z_0}^{2c}(U_0, \mathbb{Z}/n\mathbb{Z}(c)) \rightarrow H_{Z_1}^{2c}(U_1, \mathbb{Z}/n\mathbb{Z}(c))) \cong H_{\mathcal{Z}}^{2c}(\mathcal{X}, \mathbb{Z}/n\mathbb{Z}(c)),$$

where $Z_p = U_p \times_{\mathcal{X}} \mathcal{Z}$ (note that this is a scheme). We define $[\mathcal{Z}] \in H_{\mathcal{Z}}^{2c}(\mathcal{X}, \mathbb{Z}/n\mathbb{Z}(c))$ to be $[Z_0] \in H_{Z_0}^{2c}(U_0, \mathbb{Z}/n\mathbb{Z}(c))$ (this class lies indeed in the kernel). Since the cycle map (for schemes) is étale local, this construction does not depend on the choice of the étale covering $U \rightarrow \mathcal{X}$.

Compatibility with change of base schemes reduces to the scheme case. □

Proof of Proposition 5.5. (i) It follows from the assumptions that \mathcal{X}_0 is a smooth and proper surface with numerically trivial $\omega_{\mathcal{X}_0/k}$, and that it has the same ℓ -adic Betti numbers as X . Thus, by the classification of surfaces (see, for example, [BM77, Theorem 6 and the following Proposition]), also \mathcal{X}_0 is Abelian and (quasi-)hyperelliptic, respectively. As seen in the proof of Proposition 5.1, the (geometric) exceptional locus of π is a union of \mathbb{P}^1 's with self-intersection number (-2) . Now, there are no rational curves on Abelian varieties. Also, it follows from the explicit classification and description of (quasi-)hyperelliptic surfaces in [BM77, Proposition 5] that they do not contain any smooth rational curves. In particular, π must be an isomorphism, which implies that the G -action extends to \mathcal{X} .

(ii) After replacing K by an intermediate extension, we may assume $H = G$. To show that the G -action extends, it suffices to show that the σ -action extends for every $\sigma \in G$. Thus let $\sigma \in G$, and after replacing G by the cyclic subgroup generated by σ , we may assume that G is cyclic, say $G = \text{Gal}(L/K) \cong \mathbb{Z}/n\mathbb{Z}$, and generated by σ .

Let $U \subset \mathcal{X}$ be the maximal open subspace to which the G -action on X_L extends. Then, as in the proof of Proposition 5.1, U contains the generic fiber X_L , as well as an open dense subscheme of the special fiber \mathcal{X}_0 . Let $\Gamma \subset \mathcal{X}^n$ be the closure of the set $\{(x, \sigma(x), \dots, \sigma^{n-1}(x)) \mid x \in U\}$ in \mathcal{X}^n . The group $G = \mathbb{Z}/n\mathbb{Z}$ acts on \mathcal{X}^n by permutation of the factors, and this action restricted to Γ_L coincides with the natural G -action on X_L via $\text{pr}_1: \Gamma_L \xrightarrow{\sim} X_L$.

Consider the diagram of G -representations

$$\begin{array}{ccc} H^2(\mathcal{X}_0^n) & \xleftarrow{\cong} & H^2(X_L^n) \\ \downarrow \cdot[\Gamma_0] & & \downarrow \cdot[\Gamma_L] \\ H^{4n-2}(\mathcal{X}_0^n) & \xleftarrow{\cong} & H^{4n-2}(X_L^n) \end{array} \quad \begin{array}{c} \searrow \\ H^2(\Gamma_L) \cong H^2(X_L) \\ \swarrow \end{array} \quad (3)$$

where we omit the coefficients (Tate twists of \mathbb{Q}_ℓ) of the ℓ -adic cohomology groups from the notation, and we write $\Gamma_{\bar{0}} := (\Gamma_0)_{\bar{k}}$ and so on. The triangle on the right is clearly commutative. The commutativity of the square on the left follows from the fact that the classes $[\Gamma_0]$ and $[\Gamma_L]$ correspond via the isomorphism $H^{4n-4}(\mathcal{X}_0^n) \cong H^{4n-4}(X_L^n)$, as proved in Lemma 5.6. Note that all irreducible components of $\Gamma_{\bar{0}}$ are of dimension 2 (by, for example, [Liu02, Proposition 4.4.16]).

Let $\pi: \mathcal{X} \rightarrow \mathcal{X}'$ be as in Proposition 5.1. Let $E \subset \mathcal{X}_0$ be the exceptional locus of $\pi_0: \mathcal{X}_0 \rightarrow \mathcal{X}'_0$ and E^α be the irreducible components of $E_{\bar{k}}$. Each irreducible component E^α is isomorphic to \mathbb{P}^1 . Since π_0 is a resolution of singularities, the intersection matrix $(E^\alpha \cdot E^\beta)_{\alpha, \beta}$ is negative

definite (by Hodge index theorem), hence invertible. In particular, if we are given $c_\alpha \in \mathbb{Q}_\ell$ for $\alpha = 1, \dots, m$, such that $\sum_{\alpha=1}^m c_\alpha E^\alpha \cdot E^\beta = 0$ for all β , then $c_\alpha = 0$ for all $\alpha = 1, \dots, m$.

Consider the irreducible components of $\Gamma_{\bar{0}}$. First, there is the ‘diagonal’ component, that is, the closure of the set $\{(x, \sigma(x), \dots, \sigma^{n-1}(x)) \mid x \in U_0\}$. If Z is a non-diagonal component (assuming there is one), then Z is contained in $E^{\alpha_1} \times \dots \times E^{\alpha_n}$ for some $\alpha_1, \dots, \alpha_n$. From the Künneth formula and the fact that $H^*(\mathbb{P}^1, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell[\mathbb{P}^1] \oplus \mathbb{Q}_\ell[\text{pt}]$, it follows that the cycle class $[Z] \in H^{4n-4}(\mathcal{X}_0^n)$ is a non-zero $\mathbb{Z}_{\geq 0}$ -combination of $[E_i^\gamma \times E_j^\delta \times \text{pt}^{n-2}]$ with $i \neq j$, where

$$E_i^\gamma \times E_j^\delta \times \text{pt}^{n-2} := \dots \times E^\gamma \times \dots \times E^\delta \times \dots$$

(the i th component is equal to E^γ , the j th component is equal to E^δ , and the remaining components are equal to a point). Hence, if we set $[\Gamma_{\bar{0}}]^{\text{nondiag}} := [\Gamma_{\bar{0}}] - [\text{diag}]$, then there exist $c_{i,j,\gamma,\delta} \in \mathbb{Z}_{\geq 0}$ such that

$$[\Gamma_{\bar{0}}]^{\text{nondiag}} = \sum_{i,j,\gamma,\delta} c_{i,j,\gamma,\delta} [E_i^\gamma \times E_j^\delta \times \text{pt}^{n-2}] \in H^{4n-4}(\mathcal{X}_0^n). \tag{4}$$

We have $c_{i,i,\gamma,\delta} = 0$ for all i, γ, δ .

We want to show $[\Gamma_{\bar{0}}]^{\text{nondiag}} = 0$. For this, we will use the assumption that the G -action on $H^2(X_{\bar{L}})$ is trivial. Using the commutative diagram (3), we see that the map $\cdot [\Gamma_{\bar{0}}] : H^2(\mathcal{X}_0^n) \rightarrow H^{4n-2}(\mathcal{X}_0^n)$ factors through $H^2(X_{\bar{L}})$, and thus every element in its image is G -invariant. In particular, for all α and i , the cycle $[E_i^\alpha \times \mathcal{X}_0^{n-1}] \cdot [\Gamma_{\bar{0}}] \in H^{4n-2}(\mathcal{X}_0^n)$ is G -invariant, where

$$E_i^\alpha \times \mathcal{X}_0^{n-1} := \dots \times E^\alpha \times \dots$$

(the i th component is equal to E^α and the remaining components are equal to \mathcal{X}_0). Now, G acts by $\sigma : [E_j^\beta \times \mathcal{X}_0^{n-1}] \mapsto [E_{j+1}^\beta \times \mathcal{X}_0^{n-1}]$, which implies that for all β , the cycle $[E_i^\alpha \times \mathcal{X}_0^{n-1}] \cdot [\Gamma_{\bar{0}}] \cdot [E_j^\beta \times \mathcal{X}_0^{n-1}] \in H^{4n}(\mathcal{X}_0^n) \cong \mathbb{Q}_\ell$ is independent of j . Since $[E_i^\alpha \times \mathcal{X}_0^{n-1}] \cdot [\text{diag}] \cdot [E_j^\beta \times \mathcal{X}_0^{n-1}]$ is equal to $E^\alpha \cdot E^\beta$, it is also independent of j , and thus $[E_i^\alpha \times \mathcal{X}_0^{n-1}] \cdot [\Gamma_{\bar{0}}]^{\text{nondiag}} \cdot [E_j^\beta \times \mathcal{X}_0^{n-1}]$ is independent of j for all β . In order to compute its value, we use (4) and find

$$[E_i^\alpha \times \mathcal{X}_0^{n-1}] \cdot [\Gamma_{\bar{0}}]^{\text{nondiag}} \cdot [E_j^\beta \times \mathcal{X}_0^{n-1}] = \sum_{\gamma,\delta} (c_{i,j,\gamma,\delta} + c_{j,i,\delta,\gamma})(E^\gamma \cdot E^\alpha)(E^\delta \cdot E^\beta).$$

Since $c_{i,i,\gamma,\delta} = 0$ for all i , this sum is zero for $i = j$. Since it is independent of j , this sum is zero for all i, j . Using invertibility of the matrix $(E^\alpha \cdot E^\beta)$ twice, we obtain $c_{i,j,\gamma,\delta} + c_{j,i,\delta,\gamma} = 0$ for all i, j, γ, δ . Thus $[\Gamma_{\bar{0}}]^{\text{nondiag}} = 0$.

Now, $\text{pr}_i : \Gamma \rightarrow \mathcal{X}$ is a proper birational morphism for all i , where \mathcal{X} is regular, and Γ is integral. Thus, by van der Waerden purity (see, [Liu02, Theorem 7.2.22], for example, and note that this result can easily be extended to algebraic spaces), the exceptional locus of pr_i is either empty or a divisor. If it was a divisor, it would give rise to a non-diagonal component of $\Gamma_{\bar{0}}$, which does not exist by the previous computations. Thus pr_i is an isomorphism for all i , and since the $\text{Gal}(L/K)$ -action extends to Γ , this shows that the $\text{Gal}(L/K)$ -action extends to \mathcal{X} , as desired. \square

Remark 5.7. We stress that the reason for the extension of the G -action to \mathcal{X} rather than \mathcal{X}' in the case of Abelian and hyperelliptic surfaces is their ‘simple’ geometry: they contain no smooth rational curves.

5.4 The action on the special fiber

In the situation of Proposition 5.5, we now want to understand whether the induced G -action on the special fiber \mathcal{X}_0 is trivial. Quite generally, if Y is a smooth and proper variety over some field k , then the natural representation

$$\rho_m : \text{Aut}(Y) \longrightarrow \text{Aut}(H_{\text{ét}}^m(Y_{\bar{k}}, \mathbb{Q}_\ell))$$

is usually neither injective nor surjective. We have the following exceptions.

- (i) If Y is an Abelian variety, then ρ_1 is injective. (Here, $\text{Aut}(Y)$ denotes the automorphism group as an Abelian variety – translations may act trivially on cohomology.)
- (ii) If Y is a K3 surface, then ρ_2 is injective.

Using these results (for references, see below), we have the following.

PROPOSITION 5.8. *We keep the notation and assumptions of Proposition 5.1. Moreover, assume that either:*

- (i) X is an Abelian surface and the G -action on $H_{\text{ét}}^1(X_{\bar{L}}, \mathbb{Q}_\ell)$ is unramified; or
- (ii) X is a K3 surface and the G -action on $H_{\text{ét}}^2(X_{\bar{L}}, \mathbb{Q}_\ell)$ is unramified.

Then the I_G -action on X_L extends to \mathcal{X} , and the induced I_G -action on the special fiber \mathcal{X}_0 is trivial.

Proof. We have already shown the extension of the I_G -action to \mathcal{X} in Proposition 5.5. Moreover, the I_G -action on \mathcal{X}_0 is k -linear. By assumption, the I_G -action on $H_{\text{ét}}^m((\mathcal{X}_0)_{\bar{k}}, \mathbb{Q}_\ell)$ is trivial for $m = 1, 2$, respectively.

If X is an Abelian surface, then the I_G -action on $(\mathcal{X}_0)_{\bar{k}}$ is trivial by the injectivity of ρ_1 in arbitrary characteristic; see, for example, [Mum70, Theorem 3 in § 19]. If X is a K3 surface, then the I_G -action on $(\mathcal{X}_0)_{\bar{k}}$ is trivial by the injectivity of ρ_2 in arbitrary characteristic; see [Ogu79, Corollary 2.5] and [Keu16, Theorem 1.4] (in the case of complex and possibly non-algebraic K3 surfaces, see [Bea85, Proposition IX.6] and [Huy16, Proposition 15.2.1]). In both cases, the I_G -action on $(\mathcal{X}_0)_{\bar{k}}$ is trivial, and thus also the original action on \mathcal{X}_0 is trivial. \square

5.5 Tame quotients

Now, in the situation of Proposition 5.5, it is natural to study the quotient \mathcal{X}/H and its special fiber, where H is a subgroup of G . We start with the following easy result.

PROPOSITION 5.9. *Let X be a smooth and proper variety over K . Let L/K be a finite Galois extension with group G , such that X_L admits a smooth model $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_L$. Moreover, assume that the natural G -action on X_L extends to \mathcal{X} . Let H be a subgroup of G such that:*

- (i) H is contained in the inertia subgroup of G ;
- (ii) H is of order prime to p ; and
- (iii) H acts trivially on the special fiber \mathcal{X}_0 .

Then:

- (i) the quotient \mathcal{X}/H is smooth over $\text{Spec } \mathcal{O}_L^H$;
- (ii) the special fiber of \mathcal{X}/H is isomorphic to \mathcal{X}_0 .

Proof. First of all, the quotient \mathcal{X}/H exists in the category of algebraic spaces [Knu71, ch. IV.1]. Next, let $\widehat{\mathcal{X}}_0$ be the formal completion of \mathcal{X} along the special fiber \mathcal{X}_0 , which is a formal scheme. If $x \in \mathcal{X}_0$ is a closed point, then $\mathcal{O}_{\widehat{\mathcal{X}}_0, x}$ is étale over the localization $A := \widehat{\mathcal{O}}_L \langle y_1, y_2 \rangle_{\mathfrak{m}}$ of the restricted power series ring

$$\widehat{\mathcal{O}}_L \langle y_1, y_2 \rangle := \left\{ \sum_{i_1, i_2 \geq 0} a_{i_1, i_2} y_1^{i_1} y_2^{i_2} \in \widehat{\mathcal{O}}_L[[y_1, y_2]] \mid \begin{array}{l} \text{ord}_{\pi}(a_{i_1, i_2}) \rightarrow \infty \\ \text{as } i_1 + i_2 \rightarrow \infty \end{array} \right\}$$

at the maximal ideal $\mathfrak{m} = (\pi, y_1, y_2)$. The induced H -action on the residue ring $A/(\pi) \cong \kappa(\mathcal{O}_L)[y_1, y_2]_{\mathfrak{m}}$ is trivial. Thus replacing y_i by $(1/|H|) \sum_{\sigma \in H} \sigma(y_i)$ for $i = 1, 2$ (here, we use that the order of H is prime to p) is simply a change of coordinates of A . But then the H -action on $A = \widehat{\mathcal{O}}_L \langle y_1, y_2 \rangle_{\mathfrak{m}}$ is trivial on y_1 and y_2 , and hence $\mathcal{O}_{(\widehat{\mathcal{X}/H})_0, x} \cong \mathcal{O}_{\widehat{\mathcal{X}}_0, x}^H$ is étale over $A^H \cong \widehat{\mathcal{O}}_{L^H} \langle y_1, y_2 \rangle_{\mathfrak{m}}$. From this local and formal description, the smoothness of \mathcal{X}/H follows immediately, and we see that the quotient map $\mathcal{X} \rightarrow \mathcal{X}/H$ induces an isomorphism of special fibers. \square

5.6 Wild quotients

Unfortunately, Proposition 5.9 is no longer true if $p \neq 0$ and H is a subgroup of the inertia subgroup, whose order is divisible by p . Let us illustrate this with a very instructive example. We refer the interested reader to Wewers' article [Wew10] for a more thorough treatment of wild actions and their quotients.

Example 5.10. Consider $K := \mathbb{Q}_p[\zeta_p]$, where ζ_p is a primitive p th root of unity. Then $\pi := 1 - \zeta_p$ is a uniformizer in \mathcal{O}_K , and the residue field is \mathbb{F}_p ; see [Was82, Lemma 1.4], for example. Let L be the finite extension $K[\varpi]$, where $\varpi := \sqrt[p]{\pi}$. Then ϖ is a uniformizer in \mathcal{O}_L , and the residue field is \mathbb{F}_p , that is, L/K is totally ramified. By Kummer theory, L/K is Galois with group $H \cong \mathbb{Z}/p\mathbb{Z}$. More precisely, there exists a generator $\sigma \in H$ such that $\sigma(\varpi) = \zeta_p \cdot \varpi$. We set

$$R := \mathcal{O}_L[x]$$

and extend the H -action to R by requiring that $\sigma(x) = \zeta_p^{p-1} \cdot x$. Then we have $R/(\varpi) \cong \mathbb{F}_p[x]$, and the induced H -action on the quotient is trivial. On the other hand, we find that

$$R^H \cong \mathcal{O}_K[x^p, x \cdot \varpi] \cong \mathcal{O}_K[u, z]/(z^p - \pi u)$$

is normal, but not regular – this is an arithmetic version of the RDP singularity of type A_{p-1} . We also find that the special fiber

$$R^H/(\pi) \cong \mathbb{F}_p[u, z]/(z^p)$$

is not reduced. In particular, Proposition 5.9 does not extend to wild actions without extra assumptions. However, let us make two observations, whose significance will become clear in the proof of Proposition 5.11.

- (i) The H -action on the special fiber $R/(\varpi)$ only *seems* to be trivial, but in fact, it has become infinitesimal. More precisely, if $r \in R$ and \bar{r} denotes its residue class in $R/(\varpi)$, then the H -action gives rise to a well-defined and non-trivial derivation

$$\begin{aligned} \theta : R/(\varpi) &\rightarrow R/(\varpi) \\ \bar{r} &\mapsto \left(\frac{\sigma(r) - r}{\pi} \right) \pmod{\varpi}. \end{aligned}$$

- (ii) The *augmentation ideal*, that is, the ideal of R generated by all elements of the form $\sigma(r) - r$, is not principal. In fact, it can be generated by the two elements $\varpi^p x$ and ϖ^{p+1} .

Despite this example, we have the following analog of Proposition 5.9 in the wildly ramified case. The main ideas of its proof are due to Király and Lütkebohmert [KL13, Theorem 2] and Wewers [Wew10, Proposition 3.2].

PROPOSITION 5.11. *Let X be a smooth and proper variety over K . Let L/K be a finite Galois extension with group G , such that X_L admits a smooth model $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_L$. Moreover, assume that the natural G -action on X_L extends to \mathcal{X} . Let H be a subgroup of G such that:*

- (i) H is contained in the inertia subgroup of G ;
- (ii) H is cyclic of order $p \geq 2$; and
- (iii) H acts trivially on the special fiber \mathcal{X}_0 .

Then the H -action induces a global and non-trivial derivation on \mathcal{X}_0 or else both of the following two statements hold true:

- (i) the quotient \mathcal{X}/H is smooth over \mathcal{O}_L^H ;
- (ii) the special fiber of \mathcal{X}/H is isomorphic to \mathcal{X}_0 .

Proof. First of all, the quotient \mathcal{X}/H exists in the category of algebraic spaces [Knu71, ch. IV.1]. Next, we fix once and for all a generator $\sigma \in H$ and a uniformizer $\pi \in \mathcal{O}_L$. We use these to define the following:

$$N(\mathcal{O}_L) := \max\{k \mid \pi^k \text{ divides } \sigma(x) - x \text{ for all } x \in \mathcal{O}_L\},$$

$$J_H(\mathcal{O}_L) := \text{ideal of } \mathcal{O}_L \text{ generated by } \sigma(x) - x \text{ for all } x \in \mathcal{O}_L.$$

Since \mathcal{O}_L is a DVR, the ideal $J_H(\mathcal{O}_L)$ is principal. More precisely, this ideal is generated by $y := \sigma(\pi) - \pi$, and it is also generated by $\pi^{N(\mathcal{O}_L)}$. In [KL13], ideals generated by elements of the form $\sigma(x) - x$ are called *augmentation ideals*. Also, it is not difficult to see that they do not depend on the choice of generator σ , which justifies the subscript H rather than σ .

Next, let $\widehat{\mathcal{X}}_0$ be the formal completion of \mathcal{X} along the special fiber \mathcal{X}_0 , which is a formal scheme. For every point $x \in \mathcal{X}_0$, we define

$$N(\mathcal{O}_{\widehat{\mathcal{X}}_0, x}) := \max\{k \mid \pi^k \text{ divides } \sigma(r) - r \text{ for all } r \in \mathcal{O}_{\widehat{\mathcal{X}}_0, x}\},$$

$$J_H(\mathcal{O}_{\widehat{\mathcal{X}}_0, x}) := \text{ideal of } \mathcal{O}_{\widehat{\mathcal{X}}_0, x} \text{ generated by } \sigma(r) - r \text{ for all } r \in \mathcal{O}_{\widehat{\mathcal{X}}_0, x}.$$

If $\eta \in \mathcal{X}_0$ denotes the generic point, then we have the following:

$$1 \leq N(\mathcal{O}_{\widehat{\mathcal{X}}_0, \eta}) \leq N(\mathcal{O}_{\widehat{\mathcal{X}}_0, x}) \leq N(\mathcal{O}_L), \tag{5}$$

where the leftmost inequality follows from the triviality of the H -action on \mathcal{X}_0 . We distinguish two cases.

Case (I): $N(\mathcal{O}_{\widehat{\mathcal{X}}_0, \eta}) < N(\mathcal{O}_L)$.

Let $x \in \mathcal{X}_0$ be an arbitrary point and set $R := \mathcal{O}_{\widehat{\mathcal{X}}_0, x}$ and $N_\eta := N(\mathcal{O}_{\widehat{\mathcal{X}}_0, \eta})$. Then we define a map

$$\theta : R \rightarrow R/\pi R$$

$$x \mapsto \left(\frac{\sigma(x) - x}{\pi^{N_\eta}} \right) \pmod{\pi},$$

which is easily seen to be a derivation. Since we have $N_\eta < N(\mathcal{O}_L)$, we compute $\theta(\pi) = 0$, and thus θ induces a derivation $\bar{\theta} : R/\pi R \rightarrow R/\pi R$. This globalizes and gives rise to a derivation on the special fiber \mathcal{X}_0 . It follows from the definition of N_η that this derivation is non-zero at the generic point $\eta \in \mathcal{X}_0$, whence non-trivial.

Case (II): $N(\mathcal{O}_{\hat{\mathcal{X}}_0, \eta}) = N(\mathcal{O}_L)$.

Let $x \in \mathcal{X}_0$ be an arbitrary point and set $R := \mathcal{O}_{\hat{\mathcal{X}}_0, x}$. Then the two inequalities at the center and the right of (5) are equalities, which implies that all inclusions in

$$\pi^{N(\mathcal{O}_L)} \cdot R = J_H(\mathcal{O}_L) \cdot R \subseteq J_H(R) \subseteq \pi^{N(R)} \cdot R$$

are equalities. In particular, $J_H(R)$ is a principal ideal, generated by $\pi^{N(\mathcal{O}_L)}$. But then [KL13, Proposition 5] implies that there is an isomorphism of R^H - (respectively \mathcal{O}_L^H -) modules

$$\begin{aligned} R &\cong R^H \oplus R^H \pi \oplus \dots \oplus R^H \pi^{p-1}, \\ \mathcal{O}_L &\cong \mathcal{O}_L^H \oplus \mathcal{O}_L^H \pi \oplus \dots \oplus \mathcal{O}_L^H \pi^{p-1}. \end{aligned}$$

From this description, we conclude that the natural map

$$R^H \otimes_{\mathcal{O}_L^H} \mathcal{O}_L \rightarrow R$$

is an isomorphism. Moreover, if π^H is a uniformizer of \mathcal{O}_L^H , then the previous isomorphism induces an isomorphism

$$R^H / \pi^H R^H \cong R / \pi R.$$

This local computation at completions shows that $\mathcal{X}/H \times_{\mathcal{O}_L^H} \mathcal{O}_L$ is isomorphic to \mathcal{X} , and that the special fiber \mathcal{X}_0 of \mathcal{X} is isomorphic to the special fiber of \mathcal{X}/H . Since \mathcal{X} is smooth over \mathcal{O}_L , \mathcal{X}_0 is smooth over the residue field of \mathcal{O}_L , which implies that also the special fiber of \mathcal{X}/H is smooth over the residue field of \mathcal{O}_L^H . But this implies that \mathcal{X}/H is smooth over \mathcal{O}_L^H . \square

Combining Propositions 5.9 and 5.11, we obtain the following result.

COROLLARY 5.12. *Let X be a smooth and proper variety over K . Let L/K be a finite Galois extension with group G , such that X_L admits a smooth model $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_L$. Assume that the natural G -action on X_L extends to \mathcal{X} , and that the inertia subgroup I_G of G acts trivially on the special fiber \mathcal{X}_0 . Assume also that the special fiber \mathcal{X}_0 admits no non-trivial global vector fields. Then:*

- (i) *the quotient \mathcal{X}/I_G is smooth over $\mathcal{O}_L^{I_G}$;*
- (ii) *the special fiber of \mathcal{X}/I_G is isomorphic to \mathcal{X}_0 .*

Proof. We have a short exact sequence

$$1 \rightarrow P \rightarrow I_G \rightarrow T \rightarrow 1,$$

where P is the unique p -Sylow subgroup of I_G , and where T is cyclic of order prime to p . By definition, P is the wild inertia, and T is the tame inertia.

Being a p -group, P can be written as a successive extension of cyclic groups of order p . Thus, applying Proposition 5.11 inductively, we obtain a smooth algebraic space

$$\mathcal{X}/P \rightarrow \text{Spec } \mathcal{O}_L^P$$

with special fiber \mathcal{X}_0 , which is a model of X_{L^P} .

Finally, applying Proposition 5.9 to the residual T -action on \mathcal{X}/P , we obtain a smooth algebraic space

$$\mathcal{X}/I_G \rightarrow \text{Spec } \mathcal{O}_L^{I_G}$$

with special fiber \mathcal{X}_0 , which is a model of $X_{L^{I_G}}$. □

Remark 5.13. If X is a K3 surface, then the special fiber \mathcal{X}_0 is also a K3 surface and thus admits no non-zero vector fields by a theorem of Rudakov and Shafarevich [RS76].

6. The Néron–Ogg–Shafarevich criterion

We now come to the main result of this article, which is a criterion for good reduction of K3 surfaces, similar to the classical Néron–Ogg–Shafarevich criterion for elliptic curves and its generalization to Abelian varieties by Serre and Tate. Then we give a couple of corollaries concerning potential good reduction, and good reduction after a tame extension. Finally, we relate the reduction behavior of a polarized K3 surface to that of its associated Kuga–Satake Abelian variety.

6.1 The criterion

Let us remind the reader of § 3.1, where we introduced Assumption (\star) and established it in several cases.

THEOREM 6.1. *Let X be a K3 surface over K that satisfies Assumption (\star) . If the G_K -representation on $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_\ell)$ is unramified for some $\ell \neq p$, then we have the following.*

- (i) *There exists a model of X that is a projective scheme over \mathcal{O}_K , whose special fiber is a K3 surface with at worst RDP singularities.*
- (ii) *There exists an integer N , independent of X and K , and a finite unramified extension L/K of degree at most N such that X_L has good reduction over L .*

Proof. By Theorem 3.3, there exists a finite Galois extension M/K , say, with group G and possibly ramified, such that there exists a smooth model of X_M

$$\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_M.$$

Choose an ample invertible sheaf \mathcal{L} on X . Then, by Proposition 4.5, we can replace \mathcal{X} by another smooth model of X such that the pull-back of \mathcal{L} to X_M restricts to an invertible sheaf on \mathcal{X}_0 that is big and nef.

Let I_G be the inertia subgroup of G . By Proposition 5.5, the I_G -action extends to \mathcal{X} and by Proposition 5.8, the induced I_G -action on the special fiber \mathcal{X}_0 is trivial. Thus, by Corollary 5.12 and Remark 5.13, the quotient

$$\mathcal{X}/I_G \rightarrow \text{Spec } \mathcal{O}_L,$$

where $L := M^{I_G}$, is a model of X_L . Since L is a finite and unramified extension of K , this establishes claim (ii) except for the universal bound N .

The pull-back of \mathcal{L} to \mathcal{X}/I_G is still ample on the generic fiber and big and nef when restricted to the special fiber. By Proposition 5.1, there exists a birational morphism over $\text{Spec } \mathcal{O}_L$

$$\pi' : \mathcal{X}/I_G \rightarrow \mathcal{Y},$$

that is an isomorphism on generic fibers, such that the geometric special fiber \mathcal{Y}_0 is a K3 surface with at worst RDP singularities, and such that the $H := \text{Gal}(L/K)$ -action on X_L extends to \mathcal{Y} . Since L/K is unramified, the morphism $\text{Spec } \mathcal{O}_L \rightarrow \text{Spec } \mathcal{O}_K$ is étale, from which it follows that the quotient $\mathcal{Y} \rightarrow \mathcal{Y}/H$ is étale. Thus \mathcal{Y}/H is a projective scheme over \mathcal{O}_K , whose generic fiber is X and whose geometric special fiber is a K3 surface with at worst RDP singularities. This establishes claim (i).

It remains to prove the existence of a universal bound N in claim (ii). Since the Picard rank of a K3 surface is bounded above by 22, there is only a finite list L of Dynkin diagrams that is independent of the characteristic and whose associated root lattices can be embedded in the Néron–Severi lattice of a K3 surface. Therefore, a K3 surface over an algebraically closed field has at most 21 RDP singularities, and all of them are from the list L . By [Art77], there exist only finitely many analytic isomorphism types of RDP singularities with fixed dual resolution graph over algebraically closed fields. For every k' -rational singularity over some perfect field k' that becomes analytically isomorphic to a RDP singularity over \bar{k}' , we have a versal deformation space Def over k' or $W(k')$ (if $\text{char}(k') = 0$ or > 0 , respectively) and a simultaneous resolution algebraic space Res , which is finite over Def by [Art74, Theorem 3]. Since deformation and resolution spaces solve universal problems, the degree of $\text{Res} \rightarrow \text{Def}$ depends only on the analytic isomorphism type of the singularity over \bar{k}' . In particular, there exists an integer N' such that every deformation of a k' -rational singularity over k' that becomes a RDP singularity over \bar{k}' from the list L can be resolved after an extension of degree at most N' . For each Dynkin diagram in L and for almost every characteristic (in fact, it suffices to exclude $2 \leq p \leq 19$), there is only one analytic isomorphism type of RDP singularities and the corresponding degree of $\text{Res} \rightarrow \text{Def}$ is independent of the characteristic. Therefore, the bound N' can be taken to be independent of the characteristic.

Now, let \mathcal{Y}_0 be the special fiber of \mathcal{Y} . Since $(\mathcal{Y}_0)_{\bar{k}}$ has at most 21 non-smooth points, all non-smooth points of \mathcal{Y}_0 become k' -rational after some finite extension k' of k of degree $\leq 21!$. Let K'/K be the corresponding unramified extension of K . From the previous discussion, it follows that after a (possibly ramified) extension L/K' of degree at most N'^{21} , the surface X_L has good reduction. By the above arguments, we can descend a smooth model of X_L over \mathcal{O}_L to the maximal unramified subextension M of L/K' . Thus X_M has good reduction, and M/K is an unramified extension of degree at most $N := 21! \cdot N'^{21}$. This establishes the bound claimed in (ii). \square

Remark 6.2. In the statement (ii) of Theorem 6.1, we cannot avoid field extensions in general: in the next section, we will give examples of K3 surfaces X over \mathbb{Q}_p with unramified $G_{\mathbb{Q}_p}$ -representations on $H_{\text{ét}}^2(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_\ell)$ that do *not* admit smooth models over \mathbb{Z}_p .

Remark 6.3. Unlike curves and Abelian varieties, even if a K3 surface has good reduction over L , then a smooth model of X_L over \mathcal{O}_L need *not* be unique. However, by Proposition 4.7, the special fibers of all smooth models are isomorphic and the models are connected by finite sequences of flopping contractions and their inverses (similar to the classical Atiyah flop).

If a smooth variety over K has good reduction over an unramified extension, then the G_K -representations on $H_{\text{ét}}^m(X_{\overline{K}}, \mathbb{Q}_\ell)$ are unramified for all m and for all $\ell \neq p$ by Theorem 2.4. Thus, as in the case of Abelian varieties in [ST68, Corollary 1 of Theorem 1], we obtain the following independence of the auxiliary prime ℓ .

COROLLARY 6.4. *Let X be a K3 surface over K that satisfies Assumption (\star) . Then the G_K -representation on $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_\ell)$ is unramified for one $\ell \neq p$ if and only if it is unramified for all $\ell \neq p$.*

We remark that this independence of ℓ can be also derived from the weaker criterion of [Mat15] (Theorem 3.3), combined with Ochiai's independence of traces [Och99, Theorem B].

We leave the following easy consequence of Theorem 6.1 to the reader (this also can be derived from the criterion of [Mat15]).

COROLLARY 6.5. *Let X be a K3 surface over K such that the image of inertia*

$$\rho_\ell : I_K \rightarrow \text{GL}(H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_\ell))$$

is finite. If X satisfies Assumption (\star) , then X has potential good reduction. □

If a g -dimensional Abelian variety over K with $p > 2g + 1$ has potential good reduction, then good reduction can be achieved over a tame extension of K by [ST68, Corollary 2 of Theorem 2]. We have the following analog for K3 surfaces.

COROLLARY 6.6. *Let X be a K3 surface over K with potential good reduction. If $p \geq 23$, then good reduction can be achieved after a tame extension.*

Proof. The idea of proof is the same as for Abelian varieties in [ST68], we only adjust the arguments to our situation: since X is projective, there exists an ample invertible sheaf \mathcal{L} defined over K , and then its Chern class $c_1(\mathcal{L})$ gives rise to a G_K -invariant class in $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Z}_\ell)(1)$. Let T_ℓ^2 be the orthogonal complement of $c_1(\mathcal{L})$ with respect to the Poincaré duality pairing. For $\ell \neq p$, we let

$$\rho_\ell : G_K \rightarrow \text{GL}(T_\ell^2)$$

be the induced ℓ -adic Galois representation, and denote by

$$\text{red}_\ell : \text{GL}(T_\ell^2) \rightarrow \text{GL}(T_\ell^2 / \ell T_\ell^2)$$

its reduction modulo ℓ . As usual, we denote by I_K (respectively, P_K) the inertia (respectively, wild inertia) subgroup of G_K . Since X has potential good reduction, $\rho_\ell(I_K)$ is a finite group. Moreover, if ℓ is odd, since $\ker \text{red}_\ell$ has no non-trivial element of finite order (as can be seen by taking the logarithm), $\rho_\ell(I_K)$ is isomorphic to $\text{red}_\ell \circ \rho_\ell(I_K)$ via red_ℓ .

Now, suppose that $\rho_\ell(P_K)$ is non-trivial. Then the order of $\text{red}_\ell \circ \rho_\ell(I_K)$ is divisible by p for all odd ℓ . In particular, if we set $n := \text{rank } T_\ell^2 = 21$, then p divides the order

$$|\text{GL}_n(\mathbb{F}_\ell)| = \ell^{n(n-1)/2} \prod_{s=1}^n (\ell^s - 1)$$

for all odd ℓ . By Dirichlet's theorem on arithmetic progressions, there exist infinitely many primes ℓ such that the residue class of ℓ modulo p generates the group \mathbb{F}_p^\times , which is of order $p - 1$. Choosing such an ℓ , we obtain the estimate $p - 1 \leq n = 21$. (When working directly with $H^2(X_{\overline{K}}, \mathbb{Z}_\ell)(1)$ instead of the primitive cohomology group T_ℓ^2 , we only get the estimate $p - 1 \leq 22$, which includes the prime $p = 23$.)

Thus if $p \geq 23$, then $\rho_\ell(P_K)$ is trivial. But then also the P_K -action on $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Z}_\ell)(1)$ is trivial. Thus there exists a tame extension L/K such that the G_L -action on $H_{\text{ét}}^2(X_{\overline{L}}, \mathbb{Q}_\ell)(1)$ is unramified. By Theorem 6.1, there exists an unramified extension of L'/L such that $X_{L'}$ has good reduction. In particular, X has good reduction after a tame extension of K . □

6.2 Kuga–Satake varieties

Given a polarized K3 surface (X, \mathcal{L}) over \mathbb{C} , Kuga and Satake [KS67] associated to it a polarized Abelian variety, the *Kuga–Satake Abelian variety* $A := \text{KS}(X, \mathcal{L})$, which is of dimension 2^{19} . Although their construction is transcendental, it is shown in work of Rizov [Riz10] and Madapusi Pera [Mad15], building on previous results of Deligne [Del72] and André [And96], that the Kuga–Satake construction exists over arbitrary fields: namely, if (X, \mathcal{L}) is a polarized K3 surface over some field k , then $\text{KS}(X, \mathcal{L})$ exists over some finite extension of k . Then we have the following relation between good reduction of (X, \mathcal{L}) and $\text{KS}(X, \mathcal{L})$.

THEOREM 6.7. *Assume $p \neq 2$. Let (X, \mathcal{L}) be a polarized K3 surface over K .*

- (i) *If X has good reduction, then $\text{KS}(X, \mathcal{L})$ can be defined over an unramified extension L/K , and it has good reduction over L .*
- (ii) *Assume that X satisfies Assumption (\star) . Let L/K be a field extension such that both $\text{KS}(X, \mathcal{L})$ and the Kuga–Satake correspondence (described below) can be defined over L . If $\text{KS}(X, \mathcal{L})$ has good reduction over L , then X has good reduction over an unramified extension of L .*

Proof. We will use the notation and definitions of [Mad15].

(i) The pair (X, \mathcal{L}) gives rise to a morphism $\text{Spec } K \rightarrow \text{M}_{2d}^\circ$, where M_{2d}° denotes the moduli space of primitively polarized K3 surfaces of degree $2d := \mathcal{L}^2$. By assumption, there exists a smooth model of X over \mathcal{O}_K , and by Proposition 4.5, there even exists a smooth model \mathcal{X} of X over \mathcal{O}_K , such that the restriction of \mathcal{L} to the special fiber is big and nef. Thus the morphism $\text{Spec } K \rightarrow \text{M}_{2d}^\circ$ extends to a morphism $\text{Spec } \mathcal{O}_K \rightarrow \text{M}_{2d}$, where M_{2d} denotes the moduli space of primitively quasi-polarized K3 surfaces of degree $2d$. Passing to an unramified extension L/K of degree ≤ 2 if necessary, the previous classifying morphism extends to a morphism $\text{Spec } \mathcal{O}_L \rightarrow \tilde{\text{M}}_{2d}$, see [Mad15, § 5]. Composing with the morphism $\tilde{\text{M}}_{2d} \rightarrow \mathcal{S}(\Lambda_d)$ from [Mad15, Proposition 5.7], we obtain a morphism $\text{Spec } \mathcal{O}_L \rightarrow \mathcal{S}(\Lambda_d)$. We recall from [Mad15, § 4] that there exists a finite and étale cover $\tilde{\mathcal{S}}(\Lambda_d) \rightarrow \mathcal{S}(\Lambda_d)$, such that the Kuga–Satake Abelian scheme is a relative Abelian scheme over $\tilde{\mathcal{S}}(\Lambda_d)$. Thus, after replacing L by a finite and unramified extension if necessary, we can lift the latter morphism to a morphism $\text{Spec } \mathcal{O}_L \rightarrow \tilde{\mathcal{S}}(\Lambda_d)$. Thus we obtain a Kuga–Satake Abelian variety $\text{KS}(X, \mathcal{L})$ over L that has good reduction, and where L is an unramified extension of K .

(ii) By assumption, $\text{KS}(X, \mathcal{L})$ is defined over L and has good reduction over L . Thus the G_L -action on $H_{\text{ét}}^1(\text{KS}(X, \mathcal{L})_{\bar{L}}, \mathbb{Q}_\ell)$ is unramified. By the usual properties of the Kuga–Satake construction, there exists an embedding

$$P_{\text{ét}}^2(X_{\bar{K}}, \mathbb{Q}_\ell)(1) \rightarrow \text{End}(H_{\text{ét}}^1(\text{KS}(X, \mathcal{L})_{\bar{L}}, \mathbb{Q}_\ell)),$$

where $P_{\text{ét}}^2(X_{\bar{K}}, \mathbb{Q}_\ell)(1)$ denotes the orthogonal complement of $c_1(\mathcal{L})$ inside $H_{\text{ét}}^2(X_{\bar{K}}, \mathbb{Q}_\ell)(1)$, and this embedding is G_L -equivariant by assumption. This implies that also the G_L -action on $P_{\text{ét}}^2(X_{\bar{K}}, \mathbb{Q}_\ell)(1)$ is unramified. Since \mathcal{L} is defined over K , the G_L -action on the \mathbb{Q}_ℓ -subvector space generated by $c_1(\mathcal{L})$ inside $H_{\text{ét}}^2(X_{\bar{K}}, \mathbb{Q}_\ell)(1)$ is trivial. From this, we conclude that the G_L -action on $H_{\text{ét}}^2(X_{\bar{K}}, \mathbb{Q}_\ell)(1)$ is unramified. By Theorem 6.1, X has good reduction over an unramified extension of L . □

Remark 6.8. Let us make two comments.

- (i) If (X, \mathcal{L}) is a polarized K3 surface with good reduction, then the previous theorem asserts that $\text{KS}(X, \mathcal{L})$ can be defined over an unramified extension L of K . Thus if $\text{KS}(X, \mathcal{L})$ can be

descended to some field K' with $K \subseteq K' \subseteq L$ (so far, not much is known about fields of definition of Kuga–Satake Abelian varieties), then, since L/K' is unramified and since $\text{KS}(X, \mathcal{L})$ has good reduction over L by the previous theorem, the descended Abelian variety will have good reduction over K' by [ST68].

(ii) We can almost remove the $p \neq 2$ hypothesis in Theorem 6.7: by [KM16, Proposition A.12] (see also the proof of [KM16, Theorem A.1]), there exists a Kuga–Satake morphism with the properties needed to make the proof of Theorem 6.7 work also in residue characteristic 2, but so far only outside the locus of superspecial K3 surfaces.

7. Counterexamples

In this final section we give examples of K3 surfaces X over \mathbb{Q}_p for all $p \geq 5$ with unramified $G_{\mathbb{Q}_p}$ -representation on $H_{\text{ét}}^2(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_\ell)$ that do *not* have good reduction over \mathbb{Q}_p . In particular, the unramified extension from Theorem 6.1 needed to obtain good reduction may be non-trivial. The examples in question already appeared in [Mat15, § 5.3] and rest on examples due to van Luijk [vanL07, § 3].

Example 7.1. Fix a prime $p \geq 5$. We choose integers a, c such that $a \not\equiv 0, \frac{27}{16} \pmod p$, such that $c \equiv 1 \pmod 8$, and such that c is not a quadratic residue modulo p . Then we choose a homogeneous polynomial $f \in \mathbb{Z}[x, y, z, w]$ of degree 3, such that the following congruences hold:

$$\begin{aligned} f &\equiv \phi \pmod 2, \\ f &\equiv x^3 + y^3 + z^3 + aw^3 \pmod p, \end{aligned}$$

where ϕ is as in Example 5.4. Finally, we define the quartic hypersurface

$$\mathcal{X} := \mathcal{X}(p) := \left\{ wf + \left(pz^2 + xy + \frac{p}{2}yz \right)^2 - \frac{cp^2}{4}y^2z^2 = 0 \right\} \subset \mathbb{P}_{\mathbb{Z}_p}^3,$$

and denote by $X = X(p)$ its generic fiber.

THEOREM 7.2. *Let $p \geq 5$ and let X and \mathcal{X} be as in Example 7.1. Then X is a smooth K3 surface over \mathbb{Q}_p , such that:*

- (i) *the $G_{\mathbb{Q}_p}$ -representation on $H_{\text{ét}}^2(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_\ell)$ is unramified for all $\ell \neq p$;*
- (ii) *\mathcal{X} is a projective model of X over \mathbb{Z}_p , whose geometric special fiber is a K3 surface with RDP singularities of type A_1 ;*
- (iii) *X has good reduction over the unramified extension $\mathbb{Q}_p[\sqrt{c}]$.*
- (iv) *X does not have good reduction over \mathbb{Q}_p .*

Proof. Smoothness of X follows from considering the equations over \mathbb{Z} , reducing modulo 2 and checking smoothness there. Claims (ii) and (iii) are straightforward computations (for claim (iii), blow up the ideal \mathcal{I}_+ or \mathcal{I}_- defined below), and, since X has good reduction after an unramified extension, also claim (i) follows. We refer to [Mat15, § 5.3] for computations and details.

To show claim (iv), we argue by contradiction, and assume that there exists a smooth and proper algebraic space $\mathcal{Z} \rightarrow \text{Spec } \mathbb{Z}_p$ with generic fiber X . Since the generic fibers of \mathcal{X} and \mathcal{Z} are isomorphic, such an isomorphism extends to a birational, but possibly rational, map

$$\alpha : \mathcal{Z} \dashrightarrow \mathcal{X}.$$

Next, let \mathcal{L} be an ample invertible sheaf on \mathcal{X} , for example, the restriction of $\mathcal{O}(1)$ from the ambient $\mathbb{P}_{\mathbb{Z}_p}^3$. Restricting \mathcal{L} to the generic fiber \mathcal{X}_η , and pulling it back via α , we obtain an ample invertible sheaf $\alpha_\eta^*(\mathcal{L}_\eta)$ on \mathcal{Z}_η . Since \mathcal{Z} is smooth over $\text{Spec } \mathcal{O}_K$, this invertible sheaf on \mathcal{Z}_η extends uniquely to an invertible sheaf on \mathcal{Z} that we denote by \mathcal{M} .

By Proposition 4.5, there exists a rational and birational map

$$\varphi : \mathcal{Z} \dashrightarrow \mathcal{Z}^+,$$

where \mathcal{Z}^+ is another model of X with good reduction, and such that the transform \mathcal{M}^+ of \mathcal{M} on \mathcal{Z}^+ is ample on the generic fiber, and big and nef on the special fiber. We denote by $\alpha^+ : \mathcal{Z}^+ \dashrightarrow \mathcal{X}$ the composition $\alpha \circ \varphi^{-1}$. Then

$$\mathcal{Z}^+ \rightarrow (\mathcal{Z}^+)' := \text{Proj} \bigoplus_{n \geq 0} H^0(\mathcal{Z}^+, (\mathcal{M}^+)^{\otimes n})$$

is a birational morphism that contracts precisely those curves on the special fiber \mathcal{Z}_0^+ that have zero-intersection with \mathcal{M}_0^+ , and nothing else. By construction, we have $(\alpha_\eta^+)^*\mathcal{L} \cong \mathcal{M}^+$ and thus, by [Kov09, Theorem 5.14], there exists an isomorphism $(\mathcal{Z}^+)' \xrightarrow{\sim} \mathcal{X}$ over $\text{Spec } \mathbb{Z}_p$.

Thus we have shown that the model \mathcal{X} admits a simultaneous resolution $\alpha^+ : \mathcal{Z}^+ \rightarrow \mathcal{X}$ of singularities over \mathbb{Z}_p . But then let $x \in \mathcal{X}_0$ be an \mathbb{F}_p -rational singular point: for example, the point $x = w = y + z = 0$. Then let $\mathcal{O}_{\mathcal{X},\bar{x}}$ be the strict local ring, and denote by $\text{Cl}(\mathcal{O}_{\mathcal{X},\bar{x}})$ its Picard group. Then α^+ induces a $G_{\mathbb{Q}_p}$ -equivariant surjection $(R^1\alpha_*^+\mathcal{O}_{\mathcal{Z}^+}^*)_{\bar{x}} \rightarrow \text{Cl}(\mathcal{O}_{\mathcal{X},\bar{x}})$. However, this is impossible for the following reasons.

- (i) The group $(R^1\alpha_*^+\mathcal{O}_{\mathcal{Z}^+}^*)_{\bar{x}}$ is generated by the class of the exceptional curve, which is \mathbb{F}_p -rational, and thus the $G_{\mathbb{Q}_p}$ -action on it is trivial.
- (ii) The $G_{\mathbb{Q}_p}$ -action on $\text{Cl}(\mathcal{O}_{\mathcal{X},\bar{x}})$ is non-trivial. More precisely, if we define the following ideals of $\mathcal{O}_{\mathcal{X},\bar{x}}$

$$\mathcal{I}_\pm := \left(w, \left(pz^2 + xy + \frac{p}{2}yz \right) \pm \frac{\sqrt{c}}{2}pyz \right),$$

then their classes in $\text{Cl}(\mathcal{O}_{\mathcal{X},\bar{x}})$ satisfy $[\mathcal{I}_+] = -[\mathcal{I}_-] \neq [\mathcal{I}_-]$, and since $G_{\mathbb{Q}_p}$ acts on $\mathbb{Q}_p[\sqrt{c}]$ as $\sqrt{c} \mapsto -\sqrt{c}$, the $G_{\mathbb{Q}_p}$ -action on $\text{Cl}(\mathcal{O}_{\mathcal{X},\bar{x}})$ is non-trivial.

This contradiction shows that X does not have good reduction over \mathbb{Q}_p , and establishes claim (iv). □

Remark 7.3. This example is the one the second named author gave in [Mat15, § 5.3]. There, the choice of ϕ ensured that X was a smooth K3 surface, as well as of Picard number one. In the present paper, we only need smoothness, and therefore, could have used a simpler polynomial. The same remark holds for Example 5.4.

ACKNOWLEDGEMENTS

It is a pleasure for us to thank François Charles, Christopher Hacon, Annabelle Hartmann, Brendan Hassett, Tetsushi Ito, Keerthi Madapusi Pera, Davesh Maulik, as well as Yoichi Mieda for discussions and comments. We also thank the referee for careful proofreading and many comments, including an idea to simplify the arguments in § 4.2. The second named author thanks the department of mathematics at the TU München for kind hospitality while visiting the first named author there.

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