A CHARACTERIZATION OF BLOCK-GRAPHS

Frank Harary

(received January 10, 1962)

The purpose of this note is to characterize "block-graphs", a collection of graphs defined by a construction involving certain subgraphs called "blocks". A related operation on a graph leads to the study of "cut-point-graphs". The precise relationship between these two operations is made explicit. In order that this characterization be self-contained, we include the necessary definitions.

1. Introduction. A graph G is defined as a finite non-empty set V of elements called points together with a given collection X of unordered pairs of distinct points. Each element of X is called a line of the graph. If line x consists of points v_1 and v_2, then v_1 and v_2 are incident with x and are adjacent to each other. Two graphs G and H are isomorphic, written $G = H$, if there is a 1 - 1 correspondence between their sets of points which preserves adjacency. A subgraph of G consists of subsets of V and X which themselves form a graph. The subgraph of G generated by a set S of points contains S and all lines of G joining two points of S. If G is a graph and v is any point of G, then the graph $G - v$ obtained from G by removing point v is the maximal subgraph not containing v. Thus $G - v$ is generated by $V - \{v\}$. A path of G is an alternating sequence of distinct points and lines of the form $v_1, x_1, v_2, x_2, v_3, \ldots, v_n$ such that each line x_i is incident with v_i and v_{i+1}. This path is said to join v_1 and v_n A graph is connected if there is a path joining every pair of points. A component of G is a maximal connected subgraph. A cut point of a connected graph

Canad. Math. Bull. vol. 6, no. 1, January 1963.

G is a point v such that G - v is disconnected. For any graph G, v is called a cut point of G if v is a cut point of its component. A graph is called a block if it has more than one point, is connected, and has no cut points. A block of a graph G is a maximal subgraph of G which is itself a block.

The block-graph B(G) of a given graph G is that graph whose points are the blocks $B_1, B_2, \ldots, B_N$ of G and whose lines are determined by taking two points B_i and B_j as adjacent if and only if they contain a cut point of G in common. A graph is called a block-graph if it is the block-graph of some graph.

In a previous note [1], we have derived a formula for the number N of blocks of a given connected graph G in terms of the number r of cut points and the number k_i of components in the subgraph obtained on removing the i'th cut point of G:

$$(1) \qquad N = 1 - r + \sum_{i=1}^{r} k_i .$$

There is another graph which can be constructed from a given graph, which is related to its block-graph. The cut-point-graph C(G) of a given graph G is that graph whose points are the cut points $v_1, v_2, \ldots, v_r$ of G, in which two points are adjacent if and only if they both lie in a common block. A graph is called a cut-point-graph if it is the cut-point-graph of some graph.

Let $B^2(G) = B(B(G))$; thus $B^2(G)$ is the block-graph of B(G). Similarly, we define $B^n(G)$ for any positive integer n. A complete graph is one in which every pair of distinct points are adjacent. Let K_p be the complete graph of p points. Note that in particular K_1 is the graph with one point and no lines. We define $B(K_1)$ as empty. If G is a block then $B(G) = K_1$ and we define C(G) to be empty.

https://doi.org/10.4153/CMB-1963-001-x Published online by Cambridge University Press

A <u>cycle</u> of a graph is the union of two paths joining two distinct points u and v which intersect only at u and v. The <u>length</u> of a path or cycle is the number of lines in it. An <u>end point</u> of G is incident with exactly one line, called an <u>end line.</u> The following two theorems may be found in the book by König [2] and are based on results of Whitney [3] and Whyburn [4].

THEOREM A. For any graph G with more than two points, the following statements are equivalent:

(1) G is connected and has no cut points (definition of a block).

(2) Every two distinct points of G lie on a cycle.

(3) Every two distinct lines of G lie on a cycle.

(4) For any three distinct points of G, there exists a path joining every pair of them which contains the third.

THEOREM B. The intersection of any two distinct blocks of a graph consists of at most one point.

Hence every line of G is in exactly one block and any point lying in two distinct blocks of G is a cut point.

2. <u>Characterization</u>.

THEOREM 1. If H is a block-graph, then every block of H is complete.

<u>Proof</u>. By hypothesis, there exists a graph G such that H = B(G). Assume H has a block H_1 which is not complete. Then there are two points u_1 and u_2 in H_1 which are not adjacent. By Theorem A, u_1 and u_2 lie on a cycle of H_1 Since u_1 and u_2 are not adjacent, they lie in a cycle z of H_1 of length at least 4. This leads to a contradiction since in G, u_1 and u_2 cannot then correspond to blocks. For the union of the blocks of G corresponding to the points of H_1

https://doi.org/10.4153/CMB-1963-001-x Published online by Cambridge University Press

lying on the cycle z is itself connected and has no cut points, contradicting the maximality property of u_1 (and of u_2) as coming from a block of G.

COROLLARY 1a. To each point v of G, there corresponds a block of $B(G)$ which is complete and whose number of points equals the number of blocks of G containing v.

Proof. Let v be a cut point of G and let $B_1, B_2, \ldots, B_n$ be all the blocks of G which contain v. Then in $B(G)$ the corresponding n points generate a complete subgraph K_n. This complete subgraph is a block of $B(G)$ for it is connected and has no cut points, being complete, and it is maximal by the same reasoning as in the proof of Theorem 1.

COROLLARY 1b. To each block H_i of the block-graph $H = B(G)$, there corresponds a cut point v_i of G.

Proof. Let H_i be a block of $H = B(G)$. By Theorem 1, H_i is complete. Let H_i have as its points in H the blocks $B_1, B_2, \ldots, B_n$ of G. Since H_i is a complete subgraph of H, every pair of these blocks contains a common cut point of G. Let v_i be the point of G such that $\{v_i\} = B_1 \cap B_2$. Assume that $\{v_j\} = B_2 \cap B_3$, $v_j \neq v_i$. Then in G, $B_1 \cup B_2 \cup B_3$ is connected and has no cut point, contrary to the maximality of each of the distinct blocks B_1, B_2, B_3 of G. By mathematical induction, the same point v_i is contained in all n blocks B_k. Hence v_i is the cut point of G corresponding to block H_i of H.

THEOREM 2. If every block of H is complete, then H is a block-graph.

Proof. Let H be a given graph in which every block is complete. Form its block-graph $B(H)$. Now construct a graph G by starting with the graph $B(H)$ and adding to each point H_i

https://doi.org/10.4153/CMB-1963-001-x Published online by Cambridge University Press

of B(H) a number of end lines equal to the number of points of the block H_i which are not cut points of H. Then it is readily seen that B(G) is isomorphic with H.

The construction of this proof is illustrated in Figure 1, in which the end lines of G are indicated by dashes.

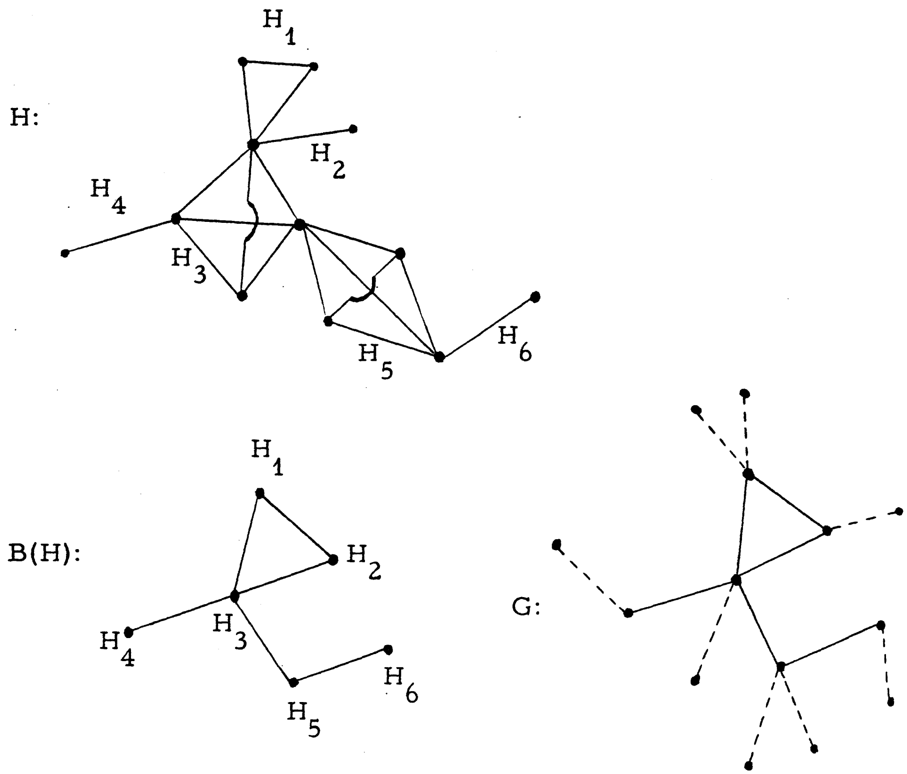

Figure 1

The proof of Theorem 2 has the following consequence.

COROLLARY 2a. For any connected graph G, $B^2(G) = C(G)$.

To prove Corollary 2a, note that there is a one-to-one correspondence between the blocks of B(G) and the cut points of G such that two cut points of G lie on a common block if and only if the corresponding two blocks of B(G) contain a common cut point of B(G).

https://doi.org/10.4153/CMB-1963-001-x Published online by Cambridge University Press

COROLLARY 2b. The operations of forming the block-graph and the cut-point-graph of a given graph commute:

$$B(C(G)) = C(B(G)) = B^3(G).$$

Combining Theorems 1 and 2, we obtain the following.

<u>Characterization</u>. A graph is a block-graph if and only if all its blocks are complete.

THEOREM 3. The set of all cut-point-graphs coincides with the set of all block-graphs. In other words, every cut-point-graph is a block-graph, and conversely.

<u>Proof</u>. It is easy to verify the direct part of this theorem. For by Corollary 2a, the cut-point-graph $C(G)$ is the block-graph of $B(G)$, and hence is itself a block-graph.

To prove the converse, we need to show that every block-graph is a cut-point-graph. Let H be a given block-graph. By definition $H = B(G)$ for some graph G. But the construction of the proof of Theorem 2 shows that the graph G itself has every block complete. Therefore G is also a block-graph by the characterization. Thus H is the block graph of a block-graph. Hence by Corollary 2a, H is a cut-point-graph.

REFERENCES

1. F. Harary, An elementary theorem on graphs. Amer. Math. Monthly 66 (1959) 405-407.

2. D. König, Theorie der endlichen und unendlichen Graphen. Leipzig, 1936; reprinted New York, 1950.

3. H. Whitney, Non-separable and planar graphs. Trans. Amer. Math. Soc. 34(1932) 339-362.

4. G. T. Whyburn, Cyclicly connected continuous curves. Proc. Natl. Acad. Sci. U. S. A. 13(1927) 31-38.