BULL. AUSTRAL. MATH. SOC. 47B47
VOL. 23 (1981), 471-475. (47A10, 47A66)

A NOTE ON d-SYMMETRIC OPERATORS

B.C. GUPTA AND P.B. RAMANUJAN

An operator T on a complex Hilbert space is d-symmetric if $\overline{R(\delta_T)} = \overline{R(\delta_{T^*})}$, where $\overline{R(\delta_T)}$ is the uniform closure of the range of the derivation operator $\delta_T(X) = TX - XT$. It is shown that if the commutator ideal of the inclusion algebra $I(T) = \{A : R(\delta_A) \subset \overline{R(\delta_T)}\}$ for a d-symmetric operator is the ideal of all compact operators then T has countable spectrum and T is a quasidiagonal operator. It is also shown that if for a d-symmetric operator $I(T)$ is the double commutant of T then T is diagonal.

Let T be an element of the Banach algebra $B(H)$ of all (bounded linear) operators on a complex Hilbert space H and δ_T the corresponding inner derivation defined by $\delta_T(X) = TX - XT$ on $B(H)$. Let $R(\delta_T)$ denote the range of δ_T and $\overline{R(\delta_T)}$ its uniform closure. An operator T in $B(H)$ is called d-symmetric if $\overline{R(\delta_T)} = \overline{R(\delta_{T^*})}$. For a d-symmetric operator T the inclusion algebra

$$I(T) = \{A \in B(H) : R(\delta_A) \subset \overline{R(\delta_T)}\} ,$$

and the multiplier algebra

$$M(T) = \{A \in B(H) : AR(\delta_T)+R(\delta_T)A \subset \overline{R(\delta_T)}\}$$

are C^*-algebras. Then $I(T)$ contains the C^*-algebra $C^*(T)$ generated

Received 16 February 1981. The authors gratefully acknowledge the support of the University Grants Commission of India.

by T and the identity operator I and is contained in $M(T)$. Further,

$$C(T) = \{A \in B(H) : AB(H)+B(H)A \subset \overline{R(\delta_T)}\}$$

is the commutator ideal of $I(T)$.

In [1, Remark (c) of Corollary 5.5], it is observed that if T is an essentially normal d-symmetric operator with countable spectrum then $C(T) \subset K$, the ideal of all compact operators on H . We show, in this note, that the spectrum of a d-symmetric operator for which $C(T) = K$ is necessarily countable and we deduce certain corollaries. It is also proved that if for a d-symmetric operator $I(T) = \{T\}''$, the double commutant of T then T is a diagonal operator.

In what follows the spectrum, the essential spectrum (the spectrum in the Calkin algebra), and the left essential spectrum of T are designated by $\sigma(T)$, $\sigma_e(T)$ and $\sigma_{1e}(T)$ respectively. The point spectrum and the set of all isolated eigenvalues of finite multiplicity of T are denoted by $\sigma_p(T)$ and $\pi_{00}(T)$ respectively.

For the basic theory of d-symmetric operators we refer to [1], [5]. Furthermore, d-symmetric operators T for which $C(T) = K$ enjoy the following properties which are immediate consequences of the results in [1], and we omit their proofs.

THEOREM 1. *Let* T *be a d-symmetric operator with* $C(T) = K$. *Then*

(a) T *is essentially normal,*

(b) T *has no reducing eigenvalues,*

(c) each projection in $\overline{R(\delta_T)}$ *is finite dimensional,*

(d) $M(T) = \{A \in B(H) : AT-TA \text{ is compact}\}$,

(e) $I(T)/C(T)$ *is a commutative* C^**-subalgebra of the Calkin algebra* $B(H)/K$,

(f) $I(T) = C^*(T) + K$.

We now prove our main result.

THEOREM 2. *Let* T *be a d-symmetric operator for which* $C(T) = K$. *Then* $\sigma(T)$ *is countable.*

Proof. In view of the inclusion relation [3, Theorem 3.3] $\partial\sigma(T) \subset \sigma_{1e}(T) \cup \pi_{00}(T)$, where $\partial\sigma(T)$ is the boundary of $\sigma(T)$, it suffices to show that $\sigma_e(T)$ is countable.

Suppose that $\sigma_e(T)$ is uncountable. Then there exists a perfect subset F of $\sigma_e(T)$ and a continuous positive Borel measure μ with support F [4, p. 176]. Let M_z be the multiplication operator on $H_\mu = L^2(\mu)$ defined by $M_z f(z) = zf(z)$. Let E denote the resolution of identity of M_z . Since $\mu(\lambda) = 0$, we have $E(\lambda) = 0$ for every λ . Therefore $\sigma_p(M_z) = \emptyset$. Let M denote the direct sum of countable copies of M_z . Then M is a normal operator and $\sigma_p(M) = \emptyset$. Therefore $\sigma_e(M) = \sigma(M) = F \subset \sigma_e(T)$. Set $A = T \oplus M$. Since T is essentially normal, Corollary 2.3 of [2] yields a unitary operator U and a compact operator K such that $UAU^{-1} + K = T$. Therefore, for any operator X on $H \oplus \Sigma \oplus H_\mu$,

$$U(AX-XA)U^{-1} = T\left(UXU^{-1}\right) - \left(UXU^{-1}\right)T + K\left(UXU^{-1}\right) - \left(UXU^{-1}\right)K .$$

Since $C(T) = K$, we have $K \subset \overline{R\left(\delta_T\right)}$ and so $U\overline{R\left(\delta_A\right)}U^{-1} \subset \overline{R\left(\delta_T\right)}$. As M_z is a d-symmetric operator with no eigenvalues $\overline{R\left(\delta_{M_z}\right)}$ contains all compact operators on H_μ . If now P_0 is non-zero finite rank projection in $\overline{R\left(\delta_{M_z}\right)}$, then there exists a sequence $\left(X_n\right)$ of operators on H_μ such that $\left\|\delta_{M_z}\left(X_n\right)-P_0\right\| \to 0$. Let $\tilde{X}_n$ denote the direct sum of countably many copies of X_n and $Y_n = I_H \oplus \tilde{X}_n$ and let P denote the direct sum of 0 and countably many copies of P_0 . Then P is an infinite dimensional projection and

$$\left\|\delta_A\left(Y_n\right)-P\right\| = \left\|\Sigma \oplus \left(\delta_{M_z}\left(X_n\right)-P_0\right)\right\| = \left\|\delta_{M_z}\left(X_n\right)-P_0\right\| \to 0 .$$

Thus $P \in \overline{R\left(\delta_A\right)}$. So UPU^{-1} is an infinite dimensional projection in

https://doi.org/10.1017/S0004972700007334 Published online by Cambridge University Press

$\overline{R(\delta_T)}$ contradicting Theorem 1 *(c)*. This completes the proof. //

COROLLARY 1. *If T is d-symmetric and $C(T) = K$ then* $\sigma(T) = \sigma_e(T)$.

Proof. Since $\sigma(T)$ is countable,

$$\sigma(T) = \partial\sigma(T) \subset \sigma_{1e}(T) \cup \pi_{00}(T) .$$

If $\lambda \in \pi_{00}(T)$ and if $(T-\lambda I)$ is Fredholm then by [5, Lemma 8] λ is a reducing eigenvalue of T . By Theorem 1 *(b)* no such λ exists. Therefore $\pi_{00}(T) \subset \sigma_e(T)$ and we have $\sigma(T) \subset \sigma_e(T)$. The result follows.

COROLLARY 2. *If T is d-symmetric and $C(T) = K$ then T is biquasitriangular.*

Proof. Since $\sigma(T) = \sigma_e(T)$, the index of $(T-\lambda I)$ is 0 for any λ for which $(T-\lambda I)$ is Fredholm. Therefore the result follows from Theorem 11.10 of [2].

COROLLARY 3. *If T is d-symmetric and $C(T) = K$ then T is quasidiagonal.*

Proof. The proof follows from Corollary 2 and Theorem 11.11 of [2].

COROLLARY 4. *If T is d-symmetric and $C(T) = K$ then T is compact if and only if T is quasinilpotent.*

Proof. Suppose T is quasinilpotent. Then by Corollary 1, $\sigma_e(T) = \sigma(T) = \{0\}$. Therefore T is compact. Conversely, if T is compact then $\sigma(T) = \sigma_e(T) = \{0\}$. Thus T is quasinilpotent.

THEOREM 3. *If T is d-symmetric and $I(T) = \{T\}''$, the double commutant of T , then T is a diagonal operator.*

Proof. Suppose $I(T) = \{T\}''$. Since $I(T)$ is a C^*-algebra, we have $T^* \in \{T\}''$. Therefore T is normal and so $I(T)$ is the von Neumann algebra generated by T and the identity I . Thus $I(T)$ is commutative and it follows from Proposition 5.1 of [1] that T is diagonal.

REMARK. If T is a diagonal operator then $I(T)$ need not be equal to $\{T\}''$. For example, if $T = \sum_{n=1}^{\infty} \frac{1}{n} E_n$, where E_n is a sequence of one dimensional orthogonal projections such that $\sum E_n = I$ then $I(T) = C^*(T)$ [1, Remark (b) of Corollary 5.5] but $C^*(T) \neq \{T\}''$ as the characteristic function $\chi_{\{0\}} \in \{T\}''$ and is not continuous at zero.

References

[1] Joel Anderson, John W. Bunce, James A. Deddens, and J.P. Williams, "C^*-algebras and derivation ranges", *Acta Sci. Math. (Szeged)* 40 (1978), 211-227.

[2] L.G. Brown, R.G. Douglas and P.A. Fillmore, "Unitary equivalence modulo the compact operators and extensions of C^*-algebras", *Proceedings of Conference on Operator Theory*, Halifax, Nova Scotia, 1973, 58-128 (Lecture Notes in Mathematics, 345. Springer-Verlag, Berlin, Heidelberg, New York, 1973).

[3] P.A. Fillmore, J.G. Stampfli, and J.P. Williams, "On the essential numerical range, the essential spectrum, and a problem of Halmos", *Acta Sci. Math. (Szeged)* 33 (1972), 179-192.

[4] Walter Rudin, *Real and complex analysis* (McGraw Hill, New York, London, Sydney, 1966).

[5] Joseph G. Stampfli, "On self-adjoint derivation ranges", *Pacific J. Math.* 82 (1979), 257-277.

Department of Mathematics,
Sardar Patel University,
Vallabh Vidyanagar 388120,
Gujarat,
India;

Department of Mathematics,
Saurashtra University,
Rajkot 360 005,
Gujarat,
India.