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# On Certain Finitely Generated Subgroups of Groups Which Split

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*Abstract.* Define a group $G$ to be in the class $\mathcal{S}$ if for any finitely generated subgroup $K$ of $G$ having the property that there is a positive integer $n$ such that $g^n \in K$ for all $g \in G$, $K$ has finite index in $G$. We show that a free product with amalgamation $A *_C B$ and an HNN group $A*_C$ belong to $\mathcal{S}$, if $C$ is in $\mathcal{S}$ and every subgroup of $C$ is finitely generated.

## Introduction

An easy consequence of $K$ being a finite index subgroup of a group $G$ is that there is a non-zero integer $n$ such that $g^n \in K$ for all $g \in G$. The converse is obviously not true in general. In fact, it is not true even in the case where $G$ is a free group of finite rank, if we do not require that $K$ be finitely generated (see Burnside's problem in [8]). But the converse turns out to be true for certain classes of groups with the additional assumption that $K$ be finitely generated. For convenience, define a group $G$ to be in the class $\mathcal{S}$ if for any finitely generated subgroup $K$ of $G$ having the property that there is a positive integer $n_K$ such that $g^{n_K} \in K$ for all $g \in G$, $K$ has finite index in $G$.

Free groups are in the class $\mathcal{S}$ by the result obtained by Karrass and Solitar in [7], which states that any finitely generated subgroup $K$ of a free group $G$ is of finite index in $G$, if $K$ contains a non-trivial normal subgroup of $G$. Karrass and Solitar's result is an extension of a result that O. Schreier obtained in [9], according to which any non-trivial finitely generated normal subgroup $N$ of $G$ is of finite index. In [5], H. B. Griffiths showed that the fundamental group of any surface is in the class $\mathcal{S}$ (see Theorem 6.2 in [5]). Since every free group is the fundamental group of a surface, Griffiths' result is an extension of the particular case of Karrass and Solitar's result where the normal subgroup is $G^n$, which denotes the subgroup of $G$ generated by all $n$-th powers of elements of $G$.

In this paper, we will show that a free product with amalgamation $A *_C B$ and an HNN extension $A*_C$ are also in the class $\mathcal{S}$, if $C$ is in $\mathcal{S}$ and every subgroup of $C$ is finitely generated. Note that the infinite cyclic group is in the class $\mathcal{S}$ and every subgroup of it is finitely generated. This provides another proof of Theorem 6.2 in [5], since the fundamental group of a surface is either a free product with amalgamation or an HNN extension of free groups along an infinite cyclic group. Using the fact that the fundamental group of a surface is in the class $\mathcal{S}$, and the short exact sequence

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associated to a Seifert fibered space, it can then be proved that the fundamental group of a Seifert fibered space is in the class $\mathcal{S}$ (see Corollary 3.2).

Kleinian 3-manifolds give rise to other examples of groups in the class $\mathcal{S}$. Recall that a 3-manifold whose interior has hyperbolic structure is called a Kleinian 3-manifold. It has been proved that for the fundamental group $G$ of a compact Kleinian manifold $M$ with a non-toroidal boundary component, a finitely generated subgroup $K$ of $G$ has finite index in $G$, if for each $g \in G$, there is a non-zero integer $n_g$ such that $g^{n_g} \in K$ (see Proposition 8.2 in [2]). This result immediately implies that the fundamental group of a compact Kleinian 3-manifold with a non-toroidal boundary component is in the class $\mathcal{S}$. Many compact 3-manifolds can be obtained by taking a torus sum of Seifert fibered spaces and compact Kleinian 3-manifolds. Thus the main result of this paper implies that the fundamental groups of many compact 3-manifolds are in the class $\mathcal{S}$.

When I submitted this paper, the referee informed me of a result of R. G. Burns which is obviously stronger than Griffiths' result (Corollary 3.1). In [1], Burns showed that if $G = A *_C B$ with $A$ free and $C \cong \mathbb{Z}$, and if $C \neq A$ and $C$ is isolated in $A$, then any finitely generated subgroup of $G$ containing a non-trivial subnormal subgroup of $G$ has finite index. Considering this result, it seems appropriate to mention the more general question as to when a non-trivial amalgamated free product can have an infinite index, finitely generated subgroup containing a non-trivial (sub)normal subgroup. I would like to thank the referee for many valuable comments.

In Section 1, we prove a few technical lemmas on graphs of groups. In Section 2, we prove the main theorem of this paper (Theorem 2.1). In Section 3, as an application we give another proof of Theorem 6.2 in [5] and give examples of 3-manifolds whose fundamental groups are in the class $\mathcal{S}$.

## 1 Preliminaries

A graph of groups $(\mathcal{G}, \Gamma)$ is defined to be a connected graph $\Gamma$ together with

(a) a vertex group $G_v$ and an edge group $G_e = G_{\bar{e}}$ corresponding to each vertex $v$ of $\Gamma$ and edge $e$ of $\Gamma$, where $\bar{e}$ is the inverse edge of $e$, and

(b) monomorphisms $\phi_0 \colon G_e \to G_v$ and $\phi_1 \colon G_e \to G_w$ for each edge $e$ of $\Gamma$, where $v$ and $w$ are the vertices of $e$.

Let $(\mathcal{G}, \Gamma)$ be a graph of groups and $T$ be a maximal tree in $\Gamma$. Let $G_T$ be the free product of all vertex groups $G_v$ with the two images of $G_e$ in the corresponding vertex groups amalgamated for each edge $e$ of $T$. If $\Gamma = T$, the fundamental group of $(\mathcal{G}, \Gamma)$ is defined to be $G_T$. If $\Gamma \neq T$, the fundamental group of $(\mathcal{G}, \Gamma)$ relative to $T$ is defined to be the HNN group with base $G_T$, with free part having basis $\{t_e\}$ where $e$ runs over the edges of $\Gamma$ not in $T$, and with the subgroups associated to $t_e$ being the two images of $G_e$. It can be shown that the fundamental group of $(\mathcal{G}, \Gamma)$ is independent of the choice of $T$. We will denote the fundamental group of $(\mathcal{G}, \Gamma)$ by $G_\Gamma$.

For example, the free product with amalgamation $A *_C B$ is the fundamental group of the graph of groups whose underlying graph consists of two vertices, an edge connecting the two vertices and its inverse edge. If $G = A *_C B$ with $A \neq C \neq B$, we say that $G$ has nontrivial amalgamation.

The HNN group $A*_C$ is considered as the fundamental group of the graph of groups whose underlying graph consists of one vertex, an edge from the vertex to itself and its inverse edge.

Let a group $G$ act on a graph $X$. $G$ is said to act without inversions if:

(1) $X$ has no loops;

(2) $ge \neq \bar{e}$ for any edge $e$ of $X$ and element $g$ of $G$.

Let a group $G$ act on a tree $X$ without inversions with quotient graph $\Gamma$. Then there is a graph of groups $(\mathcal{G}, \Gamma)$ whose fundamental group is $G$ and vertex groups and edge groups are $G_v$'s and $G_e$'s, where $G_v$ is the stabilizer of a vertex $v$ and $G_e$ is the stabilizer of an edge $e$. We call this graph of groups the graph of groups associated to the action of $G$ on $X$.

Conversely, if $(\mathcal{G}, \Gamma)$ is a graph of groups with fundamental group $G$ and with maximal tree $T$, then construct a graph $X$ whose vertices are the cosets $gG_v$ of $G_v$ in $G$ and edges are the cosets $gG_e$ of $G_e$ in $G$, where $v$ and $e$ range over all vertices and edges of $\Gamma$, respectively. For $e$ in $T$ with vertices $v$ and $w$, the vertices of $gG_e$ are $gG_v$ and $gG_w$. For $e$ not in $T$ with vertices $v$ and $w$, the vertices of $gG_e$ are $gG_v$ and $gt_eG_w$. It can be shown that $X$ is a tree on which $G$ acts without inversions and the associated graph of groups is $(\mathcal{G}, \Gamma)$ (see [3] or [4]).

If $H$ is a subgroup of $G$, $H$ acts on $X$ without inversions. Since the $H$-stabilizer of $gG_v$ is $H \cap gG_vg^{-1}$, we obtain the following lemma (see [3], [4] or [11]).

***Lemma 1.1*** *If $G$ is the fundamental group of a graph of groups $(\mathcal{G}, \Gamma)$ and $H < G$, then $H$ is the fundamental group of a graph of groups, where each vertex group is the intersection of $H$ and a conjugate of a vertex group of $(\mathcal{G}, \Gamma)$ and each edge group is the intersection of $H$ and a conjugate of an edge group of $(\mathcal{G}, \Gamma)$.*

The following lemma is well known (see [4]).

***Lemma 1.2*** *Let $(\mathcal{H}, \Gamma_1)$ be a graph of groups with fundamental group $H$ which is finitely generated. Then there is a finite subgraph of groups of $(\mathcal{H}, \Gamma_1)$ whose fundamental group is $H$.*

Recall that a vertex $v$ in a graph is of valence 1 if $v$ has only one edge whose initial vertex is $v$.

***Lemma 1.3*** *Let $G = A *_C B$ with $A \neq C \neq B$ or $G = A*_C$. (Thus the corresponding graph of groups $(\mathcal{G}, \Gamma)$ consists of either two vertices, an edge connecting the two vertices and its inverse edge, or one vertex, an edge from the vertex to itself and its inverse edge.) If $H$ is a finitely generated subgroup of $G$ and its corresponding graph of groups $(\mathcal{H}, \Gamma_1)$ has finite diameter, then it is a finite graph of groups.*

**Proof** Let $T_1$ be a maximal tree of $\Gamma_1$, and let $E$ be the set of edges not in $T_1$. Then $E$ is finite, since the free group on $E$ is a quotient of the finitely generated group $H$. Lemma 1.2 guarantees the existence of a finite subtree $S$ of $T_1$ such that $S$ contains all the vertices of edges in $E$ and the restriction of the graph of groups $(\mathcal{H}, \Gamma_1)$ to $S \cup E$ has the whole of $H$ as its fundamental group, as $H$ is finitely generated. It suffices to show that $T_1 = S$.

Suppose not. Since $T_1$ has finite diameter, it has a vertex $v_0$ at maximal distance from $S$ in $T_1$, and this vertex must have valence 1 in both $\Gamma_1$ and $T_1$, since the vertex is not in $S$. We show this is impossible.

Let $X$ be the tree on which $G$ acts with quotient $\Gamma$ and $H$ acts with quotient $\Gamma_1$. For the HNN case, consider $\Gamma$ as a graph with one vertex and one edge-pair (an edge and its inverse edge), with both edges of the pair starting at the vertex. Hence, at each vertex in $X$, there are two edges starting at that vertex which are in different $G$-orbits, and so in different $H$-orbits. Thus no vertex of $\Gamma_1$ can have valence 1.

Now for the amalgamated free product case, at each vertex of $X$ there are at least two edges, as $A \neq C \neq B$. If the $H$-stabilizer of a vertex of $X$ equals the $H$-stabilizer of some edge at that vertex, then each of the other edges at that vertex will be in a different $H$-orbit from that edge; hence the corresponding vertex of $\Gamma_1$ does not have valence 1. It follows that $H$ is a nontrivial amalgamated free product of the vertex group $H_{v_0}$ and the fundamental group of the remainder of the graph of groups. But, by construction, the latter is the whole of $H$, giving a contradiction. ■

## 2 Main Theorem

A group $G$ is said to *split* over a subgroup $C$ if either $G = A *_C B$ with $A \neq C \neq B$ or $G = A*_C$. If $G$ splits over some subgroup, we say that $G$ is *splittable*. For example, $\mathbb{Z}$ is splittable as $\mathbb{Z} = \{1\}*_{\{1\}}$.

Let $G$ be a group which splits over a group $C$, where every subgroup of $C$ is finitely generated, and let $K$ be a finitely generated subgroup of $G$. Suppose there is a non-zero integer $n_K$ such that $g^{n_K} \in K$ for all $g \in G$. Then we will show that $K$ has finite index in $G$, which is the main result of this paper.

Let $N$ be the subgroup of $G$ generated by all the elements of the form $g^{n_K}$ with $g \in G$. It can be easily checked that $N$ is a normal subgroup of $G$ and is contained in $K$. Since $G$ splits over $C$, there is a graph of groups $(\mathcal{G}, \Gamma)$ whose underlying graph $\Gamma$ consists of either two vertices, an edge connecting the two vertices and its inverse edge (amalgamated free product case), or one vertex, an edge from the vertex to itself and its inverse edge (HNN extension case). Let $X$ be the tree described in the previous section so that $G$, $K$ and $N$ act on $X$ with quotients $\Gamma$, $\Gamma_K$ and $\Gamma_N$, respectively. We have the following lemma.

***Lemma 2.1*** *Let $G = A *_C B$ or $G = A*_C$ with a finitely generated subgroup $K$ which contains a non-trivial normal subgroup $N$ of $G$. If the graph $\Gamma_K$ corresponding to $K$ is of infinite diameter, then $N$ is contained in $C$.*

**Proof** Since $K$ is finitely generated, there is a finite subgraph $\Gamma_1$ of $\Gamma_K$ such that the restriction of the graph of groups $(\mathcal{K}, \Gamma_K)$ to $\Gamma_1$ has fundamental group the whole of $K$ by Lemma 1.2. Hence every component of the complement of $\Gamma_1$ in $\Gamma_K$ is a tree and intersects $\Gamma_1$ at only one vertex. Let $S$ be a component of infinite diameter in the complement of $\Gamma_1$ in $\Gamma_K$. Then $S$ is a subtree of infinite diameter which meets $\Gamma_1$ in a single vertex $v_0$ and the fundamental group of $(\mathcal{K}, S)$ is just the vertex group at $v_0$.

Let $p\colon \Gamma_N \to \Gamma_K$ and $q\colon X \to \Gamma_N$ be the projections. Orient $X$ so that every edge is directed towards a fixed vertex $a$ of $(p \circ q)^{-1}(\Gamma_1)$. Let $T_N$ and $T_K$ be fixed maximal trees of $\Gamma_N$ and $\Gamma_K$, respectively. Give an orientation on each of $T_N$ and $T_K$ so that every edge in $T_N$ and $T_K$ is directed towards $q(a)$ and $p\bigl(q(a)\bigr)$, respectively. Let $S'$ be a component of the preimage of $S$ under the projection of $\Gamma_N$ to $\Gamma_K$. Then we will show that $S'$ is a tree.

On the contrary, suppose that there is a circuit in $S'$. Then there is an edge $e'$ in $S'$ which is not in $T_N$. In the orbit above $e'$, there is an edge $e$ such that the initial vertex of $e$ lies in a fixed representative tree $T'_N$ for the action of $N$, *i.e.* a lift of $T_N$ to $X$, but the terminal vertex does not. Since $T_N$ is a maximal tree of $\Gamma_N$, there is an element $\alpha \in N$ which sends the terminal vertex of $e$ to a vertex of $T'_N$, and thus the initial vertex of $e$ to a vertex not in $T'_N$. This implies that both $e$ and $\alpha\bar{e}$ point away from $a$, and so the orbit $Ne$ in $X$ is a reversing $N$-orbit. Since $N \subset K$, the $K$-orbit $Ke$ is also reversing. According to the proof of Lemma 31 on page 217 of the book [4], it can be shown that the image of the reversing $K$-orbit $Ke$ under the projection of $X$ to $\Gamma_K$ is either an edge of $\Gamma_K$ with a maximal tree removed or an edge in $\Gamma_1$. However neither case can happen, as the image $e'$ of $Ke$ is an edge of $S$. Therefore $S'$ must be a tree. Further, $S'$ is of infinite diameter, as $S$ is of infinite diameter.

Since the quotient group $G/N$ acts transitively on the set of edge-pairs of $\Gamma_N$, there would be a circuit in the tree $S'$, if there were a circuit in $\Gamma_N$. It follows that $\Gamma_N$ is a tree.

Consider all the edges adjacent to the vertex $q(a)$. Let $L$ be the subtree of $\Gamma_N$ consisting of such edges with their vertices. Since $G/N$ acts transitively on the set of edge-pairs of $\Gamma_N$ and $S'$ is of infinite diameter, there is an element $\bar{g} \in G/N$ such that $\bar{g}L \subset S'$. From the fact that the vertex group at $v_0$ carries the fundamental group of $(\mathcal{K}, S)$, we deduce the following:

(1) For each vertex $v$ in $S$ other than $v_0$, there is only one edge which has $v$ as its initial vertex.

(2) If $e$ is the edge having $v$ as its initial vertex and $w$ is the other vertex of $e$, then $K_w \geq K_e = K_v$.

(3) If $e$ is an edge having $v$ as its terminal vertex and $w$ is the other vertex of $e$, then $K_v \geq K_e = K_w$.

Hence for the only edge $e_a$ having $p\bigl(\bar{g}q(a)\bigr)$ as its initial vertex, $K_w \geq K_{e_a} = K_{p(\bar{g}q(a))}$, where $w$ is the terminal vertex of $e_a$. For each edge $e$ having $p\bigl(\bar{g}q(a)\bigr)$ as its terminal vertex, $K_{p(\bar{g}q(a))} \geq K_e = K_w$, if $w$ is the initial vertex of $e$. We will show that there is only one edge $e'_a$ in $\bar{g}L$ such that $p(e'_a) = e_a$.

Suppose there is another edge $e''_a$ in $\bar{g}L$ such that $p(e''_a) = e_a$. Then there is an element $\bar{k} \in K/N$ such that $\bar{k}e'_a = e''_a$, and then $\bar{k}$ fixes the vertex $\bar{g}q(a)$. It follows that any element $k$ representing $\bar{k}$ belongs to the vertex group $N_{\bar{g}q(a)}$. But, in view of Lemma 1.1, $N_{e'_a} = N_{\bar{g}q(a)}$ and $N_{e''_a} = N_{\bar{g}q(a)}$, as $K_{e_a} = K_{p(\bar{g}q(a))}$. Thus $k \in N_{e'_a}$, and so $\bar{k}$ fixes the edge $e'_a$, giving a contradiction.

In the amalgamated free product case, consider an edge $e'$ which has $\bar{g}q(a)$ as its terminal vertex. If $u'$ is the initial vertex of $e'$ and $w'$ is the terminal vertex of $e'_a$,

$$N_{w'} \geq N_{e'_a} = N_{\bar{g}q(a)} \geq N_{e'} = N_{u'}.$$

Since $G/N$ acts transitively on the set of edge-pairs of $\Gamma_N$, there is an element $\bar{h} \in G/N$ such that $\bar{h}e' = e'_a$. Note that $\bar{h}\bar{g}q(a) = \bar{g}q(a)$ and $\bar{h}u' = w'$, as two adjacent vertices should be in different orbits in the amalgamated free product case. It follows that

$$N_{w'} = N_{e'_a} = N_{\bar{g}\bar{q}(a)} = N_{e'} = N_{u'}.$$

Since $G/N$ acts transitively on the edge-pairs, every edge group equals the vertex group at each of its vertices. Thus $N$ equals any edge group, and therefore $N = N \cap C$.

In the HNN case, suppose that there is no edge group of $\Gamma_N$ which equals the vertex group at both of its vertices. Note that every vertex group equals the edge group at one of the edges adjacent to the vertex and contains the edge group at each of the other edges adjacent to the vertex, as $G/N$ acts transitively on the set of vertices of $\Gamma_N$. Let $v'$ be an arbitrary vertex of $\Gamma_N$, and let $e'_v$ be the only adjacent edge to $v'$ such that the vertex group at $v'$ equals the edge group at $e'_v$. Since the edge groups at other edges adjacent to $v'$ than $e'_v$ is contained in the vertex group at $v'$, the group $N$ can be described as the union of such edge groups $N_{e'_v}$. Recall that $A*_C$ has a presentation of the form $\langle A, t : t^{-1}ct = \phi(c)\rangle$, where $\phi\colon C \to C$ is a group automorphism. Since $t^n \in N$, $t^n$ should be in one of the edge groups $N_{e'_v}$, which lies in a conjugate of $C$. But this is impossible. Therefore there is an edge group which equals the vertex group at each of its vertices. Since $G/N$ acts transitively on the edges of $\Gamma_N$, every edge group equals the vertex group at each of its vertices, and so $N$ is contained in $C$. ■

***Theorem 2.1*** *Let $G = A *_C B$ or $G = A*_C$, where $C$ is in the class $\mathcal{S}$ and every subgroup of $C$ is finitely generated. Let $K$ be a finitely generated subgroup of $G$. If there is an integer $n$ such that $g^n \in K$ for all $g \in G$, then the index $|G : K|$ is finite.*

**Proof** Let $N$ be the normal subgroup generated by all the elements of the form $g^n$ with $g \in G$ as described earlier. Let $X$ be the tree on which $G$, $K$ and $N$ act with quotients $\Gamma$, $\Gamma_K$, and $\Gamma_N$, respectively. If $\Gamma_K$ is a graph of infinite diameter, $N$ is contained in $C$ according to Lemma 2.1. However since $G$ splits non-trivially, there is no integer $r$ such that $g^r \in C$ for all $g \in G$, and so $N$ can not be contained in $C$. Therefore $\Gamma_K$ is a graph of finite diameter. By Lemma 1.3, $\Gamma_K$ is a finite graph of groups.

Let $g \in G$. Since $\Gamma_K$ is a finite graph, there is a finite number of edges, say $g_1e, \dots, g_me$ in the fundamental domain for the action of $K$ on $X$. There is an element $h \in K$ such that $hge = g_ie$ for some $i = 1, \dots, m$. Since $g_i^{-1}hge = e$, $g_i^{-1}hg \in G_e$. We may assume without loss of generality that $G_e = C$. Since $C$ is in $\mathcal{S}$ and $C \cap N$ is finitely generated, the index $|C : C \cap N|$ is finite. Let $c_1(C \cap N), \dots, c_l(C \cap N)$ be the cosets of $C \cap N$ in $C$. Then $g_i^{-1}hg \in c_j(C \cap N)$ for some $j = 1, \dots, l$. Since $c_j(C \cap N) \subset c_jN$, $g_i^{-1}h^{-1}g \in c_jN$. It follows that

$$g \in hg_ic_jN = Nhg_ic_j \subset Kg_ic_j,$$

as $N$ is normal in $G$ and $h \in K$, $N \subset K$. Therefore there are at most $ml$ right cosets of $K$ in $G$, which implies that $K$ is of finite index in $G$. ■

## 3 3-Manifold Groups

One can easily see that the infinite cyclic group satisfies the hypotheses on $C$ in Theorem 2.1. This gives rise to the following result which Griffiths obtained in [5].

***Corollary 3.1*** *Let $G$ be the fundamental group of a surface, and let $K$ be a finitely generated subgroup of $G$. If there is a positive integer $n$ such that $g^n \in K$ for all $g \in G$, then $K$ has finite index in $G$. In other words, surface groups are in the class $\mathcal{S}$.*

Now we will discuss which 3-manifold groups are in the class $\mathcal{S}$. See [6] or [10] for 3-manifold terminologies and basic facts. Let $M$ be a compact, orientable, irreducible 3-manifold. If the fundamental group of $M$ is finite, then obviously the fundamental group is in the class $\mathcal{S}$. Suppose that the fundamental group of $M$ is infinite. If the boundary is not empty and compressible, $M$ can be split along a disk to obtain a new 3-manifold. Thus the fundamental group of $M$ is of the form $A *_C B$ or $A*_C$, where $C$ is a trivial group. In this case, the fundamental group of $M$ is in the class $\mathcal{S}$, as the trivial group satisfies the hypotheses on $C$ in Theorem 2.1.

Suppose that $M$ is a compact, orientable, irreducible with incompressible boundary (possibly empty boundary). It is known that $M$ is a torus sum of 3-manifolds, or a compact 3-manifold whose interior has hyperbolic structure, or a Seifert fibered space. In the case where $M$ is a torus sum of 3-manifolds, the fundamental group of $M$ is of the type $A *_C B$ or $A*_C$, where $C$ is $\mathbb{Z} \times \mathbb{Z}$, the free abelian group of rank 2. Since $\mathbb{Z} \times \mathbb{Z}$ satisfies the hypotheses on $C$ in Theorem 2.1, the fundamental group of $M$ is in the class $\mathcal{S}$.

Consider the case of a compact Kleinian manifold $M$ with a non-toroidal boundary component. R. Canary's result in [2] implies that the fundamental group of $M$ is in the class $\mathcal{S}$, as pointed out in the introduction. We do not know whether the fundamental group of a Kleinian manifold with only toroidal boundaries is in the class $\mathcal{S}$. However the fundamental group of a Seifert fibered space is in the class $\mathcal{S}$.

***Corollary 3.2*** *Let $G$ be the fundamental group of a Seifert fibered space $M$, and let $K$ be a finitely generated subgroup of $G$. If there is a positive integer $n$ such that $g^n \in K$ for all $g \in G$, then $K$ has finite index in $G$.*

**Proof** Since $G$ is the fundamental group of a Seifert fibered space, there is a short exact sequence

$$1 \longrightarrow Z \longrightarrow G \xrightarrow{p} Q \longrightarrow 1,$$

where $Z$ denotes the cyclic subgroup of $G$ generated by the regular fiber and $Q$ is an 2-dimensional orbifold group. The group $Z$ is infinite except in the case where $M$ is covered by the 3-dimensional sphere $S^3$ (see Lemma 3.2 in [10]). If $M$ is covered by $S^3$, then $G$ is a finite group, in which case the conclusion follows easily. Suppose $M$ is not covered by $S^3$. It is well-known that the orbifold group $Q$ contains a surface group $S$ as its finite index subgroup. For any $g \in S$, $g^n \in p(K) \cap S$. Since surface groups are in the class $\mathcal{S}$, $p(K) \cap S$ is of finite index in $S$. It follows that $p^{-1}\big(p(K)\big)$ has finite index in $G$. It can be easily seen that $p^{-1}\big(p(K)\big) = \langle t, K\rangle$, where $Z$ is generated by $t$ and $\langle t, K\rangle$ is the subgroup of $G$ generated by $t$ and $K$. Each element $x$ in $\langle t, K\rangle$ can be written as $t^m \cdot k$ for some integer $m$ and $k \in K$. Since $t^n \in K$, there are less than $n + 1$ left cosets of $K$ in $\langle t, K\rangle$. Hence $K$ has finite index in $p^{-1}\big(p(K)\big)$, and so $K$ has finite index in $G$. ■

If we combine the results obtained above, we obtain the following theorem.

***Theorem 3.1*** *Let $M$ be a compact, orientable, irreducible 3-manifold which is not a Kleinian manifold having toroidal boundary components only (possibly empty boundary). Let $G$ be the fundamental group of $M$, and $K$ be a finitely generated subgroup of $G$. If there is a positive integer $n$ such that $g^n \in K$ for all $g \in G$, then $K$ has finite index in $G$.*

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