1
Introduction

This book covers the methods by which we can use instantons. What is an instanton? A straightforward definition is the following. Given a quantum system, an instanton is a solution of the equations of motion of the corresponding classical system; however, not for ordinary time, but for the analytically continued classical system in imaginary time. This means that we replace t with $-i\tau$ in the classical equations of motion. Such solutions are alternatively called the solutions of the Euclidean equations of motion.

This type of classical solution can be important in the semi-classical limit $\hbar \to 0$. The Feynman path integral, which we will study in its Euclideanized form in great detail in this book, gives the matrix element corresponding to the amplitude for an initial state at $t = t_i$ to be found in a final state at $t = t_f$ as a "path integral"

$$\begin{aligned}\langle \text{final}, t_f | \text{initial}, t_f \rangle &= \langle \text{final}, t_f | e^{-\frac{i}{\hbar}(t_f - t_i)\hat{h}(\hat{q},\hat{p})} | \text{initial}, t_i \rangle \\ &= \int_{\text{initial},t_i}^{\text{final},t_f} \mathcal{D}p\mathcal{D}q\, e^{\frac{i}{\hbar}\int dt(p\dot{q} - h(p,q))} \qquad (1.1)\end{aligned}$$

where $\hat{h}(\hat{q},\hat{p})$ is the quantum Hamiltonian and $h(q,p)$ is the corresponding classical Hamiltonian of the dynamical system. The "path integral" and integration measure $\mathcal{D}p\mathcal{D}q$ defines an integration over all classical "paths" which satisfy the boundary conditions corresponding to the initial state at t_i and to the final state at t_f. It is intuitively evident, or certainly from the approximation method of stationary phase, that the dominant contribution, as $\hbar \to 0$, should come from the neighbourhood of the classical path which corresponds to a stationary (critical) point of the exponent, since the contributions from non-stationary points of the exponent become suppressed as the regions around them cause wild, self-annihilating variations of the exponential.

However, the situation can occur where the particle (or quantum system in general) is classically forbidden from entering some parts of the configuration

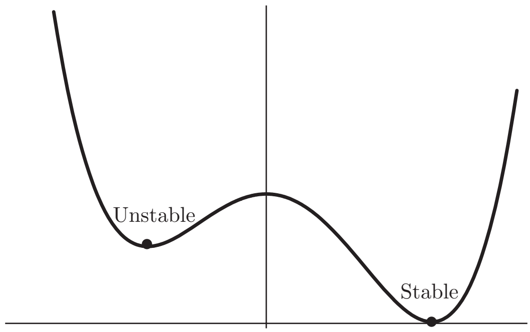

Figure 1.1. A system trapped in the false vacuum will tunnel through the barrier to the state of lower energy

space. In this case we are, generally speaking, considering tunnelling through a barrier, as depicted in Figure 1.1. Classically the particle is not allowed to enter the space where the potential energy is greater than the total energy of the particle. Indeed, if the energy of a particle is given by

$$E = T + V = \frac{\dot{q}^2}{2} + V(q) \tag{1.2}$$

then for a classically fixed energy, regions where $E < V(q)$ require that $T = \frac{\dot{q}^2}{2} < 0$, which means that the kinetic energy has to be negative, and such regions are classically forbidden. Then what takes the role of the dominant contribution in the limit $\hbar \to 0$, since no classical path can interpolate between the initial and final states?

Heuristically such a region is attainable if t becomes imaginary. Indeed, if $t \to -i\tau$ then $\left(\frac{dq}{dt}\right)^2 \to \left(i\frac{dq}{d\tau}\right)^2 = -\left(\frac{dq}{d\tau}\right)^2$, T becomes negative and then perhaps such regions are accessible. Hence it could be interesting to see what happens if we analytically continue to imaginary time, equivalent to continuing from Minkowski spacetime to Euclidean space, which is exactly what we will do. In fact, we will be able to obtain many results of the usual semi-classical WKB (Wentzel, Kramers and Brillouin) approximation [119, 77, 22], using the Euclidean space path integral. The amplitudes that we can calculate, although valid for the small $\hbar$ limit, are not normally attainable in any order in perturbation theory; they behave like $\sim Ke^{-S_0/\hbar}(1 + o(\hbar))$. Such a behaviour actually corresponds to an essential singularity at $\hbar = 0$.

The importance of being able to do this is manifold. Indeed, it is interesting to be able to reproduce the results that can be obtained by the standard WKB method for quantum mechanics using a technique that seems to have absolutely no connection with that method. Additionally, the methods that we will enunciate here can be generalized rather easily to essentially any quantum system, especially to the case of quantum field theory. Tunnelling phenomena

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in quantum field theory are extremely important. The structure of the quantum chromodynamics (QCD) vacuum and its low-energy excitations is intimately connected to tunnelling. Various properties of the phases of quantum field theories are dramatically altered by the existence of tunelling. The decay of the false vacuum and the escape from inflation is also a tunnelling effect that is of paramount importance to cosmology, especially the early universe. The method of instantons lets us study all of these phenomena in one general framework.

1.1 A Note on Notation

We will use the following notation throughout this book:

$$\text{metric} \quad \eta_{\mu\nu} = (1, -1, -1, -1) \tag{1.3}$$

$$\text{Minkowsi time} \quad t \tag{1.4}$$

$$\text{Euclidean time} \quad \tau \tag{1.5}$$

$$\text{Analytic continuation of time} \quad t \to -i\tau \tag{1.6}$$

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