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# TEST MAPS AND DISCRETE GROUPS IN $SL(2, \mathbb{C})$ II

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**Abstract.** In this paper we present a new discreteness criterion for a non-elementary subgroup $G$ of $SL(2, \mathbb{C})$ containing elliptic elements by using a loxodromic (resp. an elliptic) transformation as a test map that need not be in $G$.

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**1. Introduction.** The discreteness of Möbius groups is a fundamental problem, which has been discussed by many authors. In 1976, Jørgensen [**4**] established the following discreteness criterion: *A non-elementary subgroup $G$ of Möbius transformations acting on $\overline{\mathbb{R}^2}$ is discrete if and only if for each $f$ and $g$ in $G$, the group $\langle f, g\rangle$ is discrete*. This important result has become standard in literature and shows that the discreteness of a non-elementary Möbius group depends on the information of all its subgroups of rank two. There are many discussions in this direction. Among them, Chen [**2**] proposed to use a fixed Möbius transformation as a test map to test the discreteness of a given Möbius group. More precisely, let $G$ be a non-elementary group and let $f$ be a non-trivial Möbius map. If each group generated by $f$ and an element in $G$ are discrete, then $G$ is discrete. A novelty of this discreteness criterion is that the test map $f$ need not be in $G$, which suggests that the discreteness is not a totally interior affair of the involved group. Following Chen's idea, Yang in [**7**] becomes the first author to generalise the results of [**6**] by using test maps. There are altogether nine cases and the only case left to be solved is the following problem (Conjecture 2.8 in [**7**]).

CONJECTURE 1.1. *Let $G$ be a non-elementary subgroup of $SL(2, \mathbb{C})$ containing elliptic elements and $f$ a loxodromic (resp. an elliptic) transformation. If for each elliptic element $g \in G$ the group $\langle f, g\rangle$ is discrete, then $G$ is discrete.*

**2. Main results.** We begin with some elementary notations about Möbius groups. The reader is referred to [**1**] for more information.

Denote by Möb(2) the group of all (orientation-preserving) Möbius transformations of the extended complex plane $\overline{\mathbb{C}} = \mathbb{R}^2 \cup \infty$. Recall that any matrix $A \in SL(2, \mathbb{C})$ as the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ induces a Möbius transformation $f_A(z) = (az +$

$b)/(cz+d)$. Then Möb(2) is isomorphic to $SL(2,\mathbb{C})/\{\pm I\}$, where $I$ is the identity matrix. Let $\mathrm{tr}^2(f_A) = \mathrm{tr}^2(A)$, where tr denotes the trace of $A$. It is easy to see $\mathrm{tr}^2(f_n) \to \mathrm{tr}^2(f)$ when $f_n$ converges to $f$ in $SL(2,\mathbb{C})$. Non-trivial elements of $SL(2,\mathbb{C})$, or equivalently of Möb(2), can be classified into three types considering the Jordan normal forms.

($i$) Elliptic elements are diagonalizable and have two distinct eigenvalues with absolute value 1, that is, these are conjugated to $\left(\begin{smallmatrix} r & 0 \\ 0 & 1/r \end{smallmatrix}\right)$ with $|r| = 1$. In this case $\mathrm{tr}^2(f)$ is real and $0 \leq \mathrm{tr}^2(f) < 4$.

($ii$) Loxodromic elements are diagonalizable and the eigenvalues do not have absolute value 1, that is, these are conjugated to $\left(\begin{smallmatrix} r & 0 \\ 0 & 1/r \end{smallmatrix}\right)$ with $|r| > 1$. If $\mathrm{tr}^2(f)$ is real and $\mathrm{tr}^2(f) > 4$, then $f$ is called *hyperbolic*, and if $\mathrm{tr}^2(f)$ is not in the interval $[0, +\infty)$, then $f$ is termed as *strictly loxodromic*. We use the term loxodromic to include both hyperbolic and strictly loxodromic elements.

($iii$) Parabolic elements are not diagonalizable. They are conjugated to $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Then $\mathrm{tr}^2(f) = 4$ if $f$ is parabolic.

Recall that Möbius transformations are a finite composition of inversions in spheres and planes of the extended complex plane. Through Poincaré's extension, the action of $f = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ can be extended to an action on the hyperbolic 3-space $\mathbb{H}^3 = \{\omega = z + tj | z \in \mathbb{C}, t > 0\}$ by the formula $f(\omega) = (a\omega + b)/(c\omega + d)$.

A subgroup $G$ of Möb(2) is called elementary if there exists a finite $G$-orbit in the closure of $\mathbb{H}^3$ in Euclidean 3-space. In particular, $G$ is referred to be an elementary group of elliptic type if $G$ contains only elliptic elements and the identity. It is well known that the elements of an elementary group of elliptic type have a common fixed point in $\mathbb{H}^3$ (cf. Theorem 4.3.7 in [**1**]).

For each $f$ and $g$ in Möb(2), let $[f, g]$ denote the commutator $fgf^{-1}g^{-1}$. In their series of important papers, Gehring and Martin [**3**] introduced the following three parameters for the two generator subgroup $\langle f, g\rangle$:

$$\beta(f) = \mathrm{tr}^2(f) - 4, \quad \beta(g) = \mathrm{tr}^2(g) - 4,$$
$$\gamma(f, g) = \mathrm{tr}(fgf^{-1}g^{-1}) - 2.$$

In terms of these parameters, the well-known Jørgensen's inequality gives a sharp lower bound for $|\gamma(f, g)|$ when $|\beta(f)| < 1$ or $|\beta(g)| < 1$. In [**3**], Gehring and Martin sharpen Jørgensen's inequality to the following form.

LEMMA 2.1. *Let $\langle f, g\rangle$ be a discrete and non-elementary group of $SL(2,\mathbb{C})$ with $\beta(f) = \beta(g)$. Then $|\gamma(f, g)| > 0.193$.*

We also need the following.

LEMMA 2.2. *Let $\langle f, g\rangle$ be an elementary group of elliptic type in $SL(2,\mathbb{C})$. Then $\gamma(f, g) < 0$.*

*Proof.* We may assume, up to conjugation, that $f = \left(\begin{smallmatrix} r & 0 \\ 0 & 1/r \end{smallmatrix}\right)$ and $g$ fixes the point $(0, 0, 1)$ in the upper half-space model of $\mathbb{H}^3$. Hence, $g$ has the matrix form as $\left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix}\right)$ with $|a|^2 + |b|^2 = 1$ (cf. Theorem 2.5.1 in [**1**]).

Recall that $r = e^{i\theta_0}$ for some $\theta_0 \neq 0 \pmod{2\pi}$, it follows that

$$\beta(f) = \left(r + \frac{1}{r}\right)^2 - 4 = e^{2i\theta_0} + e^{-2i\theta_0} - 2 = 2[\cos(2\theta_0) - 1] < 0.$$

Therefore, we have $\gamma(f, g) = \mathrm{tr}(fgf^{-1}g^{-1}) - 2 = |b|^2\beta(f) < 0$. □

THEOREM 2.3. *Let $G$ be a non-elementary subgroup of $SL(2, \mathbb{C})$ containing elliptic elements and $f$ be a loxodromic (resp. an elliptic) transformation. If for each elliptic element $g \in G$ the group $\langle f, g\rangle$ is discrete, then $G$ is discrete.*

*Proof.* Suppose on the contrary that $G$ is not discrete. We only need to consider the case that $G$ is dense in $SL(2, \mathbb{C})$ by Section 1 of **[5]** and Theorem 2.9 of **[7]**.

Let $f$ be represented by the matrix $\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ with $b \neq 0 \neq c$. This is possible since $G$ is non-elementary. Setting $h = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, then we get $hgh^{-1} = \begin{pmatrix} a+ct & -ct^2+(d-a)t+b \\ c & d-ct \end{pmatrix}$. Since $G$ is dense in $SL(2, \mathbb{C})$, there exists a sequence $\{h_n\}$ in $G$ which converges to $h$.

We denote $h_ngh_n^{-1}$ by $\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ and $l_n = h_ngh_n^{-1}fh_ng^{-1}h_n^{-1}$. By a calculation, we explicitly obtain

$$l_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}\begin{pmatrix} d_n & -b_n \\ -c_n & a_n \end{pmatrix}$$

$$= \begin{pmatrix} ra_nd_n - \frac{1}{r}b_nc_n & -a_nb_n\left(r - \frac{1}{r}\right) \\ c_nd_n\left(r - \frac{1}{r}\right) & \frac{1}{r}a_nd_n - rb_nc_n \end{pmatrix}.$$

By the assumption, it follows from $h_ngh_n^{-1} \in G$ being elliptic that the groups $\langle f, l_n\rangle \subset \langle f, h_ngh_n^{-1}\rangle$ are discrete for all $n$.

We complete the proof by dividing into two cases:

- **$f$ is loxodromic.**
  We take $t$ to be the root of the quadratic equation $cx^2 + (a-d)x - b = 0$ satisfying $d - ct \neq 0$. This will lead to $\lim_{n\to\infty} b_n = 0$. It follows that

  $$\lim_{n\to\infty} |\gamma(f, l_n)| = \lim_{n\to\infty} |\mathrm{tr}([f, l_n]) - 2| = \lim_{n\to\infty} |a_nb_nc_nd_n|\left|r - \frac{1}{r}\right|^4 = 0.$$

  Therefore, the groups $\langle f, l_n\rangle$ are discrete and elementary groups for sufficiently large $n$ by Lemma 2.1.

  On the other hand, the loxodromic $l_n$ does not fix infinity since the limit as $n$ approaches infinity of $c_nd_n(r - \frac{1}{r})$ does not equal to 0. This is the desired contradiction.
- **$f$ is elliptic.**
  From the above, we obtain $\gamma(f, l_n) = a_nb_nc_nd_n|r - \frac{1}{r}|^4$, which converges to $|r - \frac{1}{r}|^4c(ct+a)(ct-d)[ct^2 + (a-d)t - b]$ as $n \to \infty$. Thanks to the fundamental theorem of algebra, we can take the value of $t$ such that $|r - \frac{1}{r}|^4c(ct+a)$

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$(ct-d)[ct^2+(a-d)t-b]$ is sufficiently small and positive, say,

$$\left|r-\frac{1}{r}\right|^4 c(ct+a)(ct-d)[ct^2+(a-d)t-b]=0.1.$$

By Lemma 2.1, we see that the discrete groups $\langle f, l_n\rangle$ must be elementary for sufficient large $n$, which is a contradiction with Lemma 2.2.

□

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