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Character sums associated to prehomogeneous vector spaces

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Abstract. Let G be a complex linear algebraic group and $\rho: G \to \mathrm{GL}(V)$ a finite dimensional rational representation. Assume that G is connected and reductive, and that V has an open G-orbit. Let f in $\mathbb{C}[V]$ be a non-zero relative invariant with character $\phi \in \mathrm{Hom}\,(G, \mathbb{C}^\times)$, meaning that $f \circ \rho(g) = \phi(g) f$ for all g in G. Choose a non-zero relative invariant $f^\vee$ in $\mathbb{C}[V^\vee]$, with character ϕ^{-1}, for the dual representation $\rho^\vee: G \to \mathrm{GL}(V^\vee)$. Roughly, the fundamental theorem of the theory of prehomogeneous vector spaces due to M. Sato says that the Fourier transform of $|f|^s$ equals $|f^\vee|^{-s}$ up to some factors. The purpose of the present paper is to study a finite field analogue of Sato's theorem and to give a completely explicit description of the Fourier transform assuming that the characteristic of the base field $\mathbb{F}_q$ is large enough. Now $|f|^s$ is replaced by $\chi(f)$, with χ in Hom $(\mathbb{F}_q^\times, \mathbb{C}^\times)$, and the factors involve Gauss sums, the Bernstein–Sato polynomial $b(s)$ of f, and the parity of the split rank of the isotropy group at $v^\vee \in V^\vee(\mathbb{F}_q)$. We also express this parity in terms of the quadratic residue of the discriminant of the Hessian of log $f^\vee(v^\vee)$. Moreover we prove a conjecture of N. Kawanaka on the number of integer roots of $b(s)$.

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1. Introduction

1.1. Let G be a connected complex linear algebraic group, and $\rho: G \to \mathrm{GL}(V)$ a finite dimensional rational representation. A triple (G, ρ, V) is called a *prehomogeneous vector space* if V has an open G-orbit, say O_0. From now on, we always assume G reductive, except in (1.10). Let $0 \neq f \in \mathbb{C}[V]$ be a relative invariant with the character $\phi \in \mathrm{Hom}(G, \mathbb{C}^\times)$; $f(gv) = \phi(g) f(v)$ for all $g \in G$ and $v \in V$. Let $\rho^\vee : G \to \mathrm{GL}(V^\vee)$ be the dual of ρ. Then it is known that $(G, \rho^\vee, V^\vee)$ also has an open G-orbit, say $O_0^\vee$, and that there exists a relative invariant $0 \neq f^\vee \in \mathbb{C}[V^\vee]$ whose character is ϕ^{-1} (cf. [Gyo1, 1.5]).

Roughly, the fundamental theorem of the theory of prehomogeneous vector spaces due to M. Sato says that

$$\text{Fourier transform of } |f|^s = |f^\vee|^{-s} \times (\text{some factors}) \tag{1}$$

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for $s \in \mathbb{C}$. The purpose of this paper is to study a finite field analogue of (1) and to give an explicit description of the Fourier transform assuming that the characteristic of the base field is large enough.

1.2. In order to state our main result more precisely, let us briefly review the theory of prehomogeneous vector spaces.

(1) There exists a unique G-orbit O_1 (resp. $O_1^\vee$) which is closed in $\Omega := V \backslash f^{-1}(0)$ (resp. $\Omega^\vee := V^\vee \backslash f^{\vee -1}(0)$) [Gyo1, 1.4, (1)]. Put $F := \operatorname{grad} \log f$ and $F^\vee := \operatorname{grad} \log f^\vee$. Then $F(\Omega) = O_1^\vee$ and $F^\vee(\Omega^\vee) = O_1$ [Gyo1, 1.18, (2)].
(2) $\dim V = \dim V^\vee =: n$, $\dim O_1 = \dim O_1^\vee =: m$ [Gyo1, 1.18, (3)], $\deg f = \deg f^\vee =: d$ [SatKim, pp. 71–72], [Gyo1, 1.5, (2)].
(3) $f^\vee(\operatorname{grad}_x) f(x)^{s+1} = b(s) f(x)^s$ and $f(\operatorname{grad}_y) f^\vee(y)^{s+1} = b(s) f^\vee(y)^s$ with some $b(s) \in \mathbb{C}[s]$ [SatKim, p. 72], [Gyo1, 1.6].
(4) $b(s) = b_0 \prod_{j=1}^d (s + \alpha_j)$ with some $b_0 \in \mathbb{C}^\times$ and $\alpha_j \in \mathbb{Q}_{>0}$ [Kas1], (cf. [Gyo1, 2.5.12]).
(5) $b^{\exp}(t) := \prod_{j=1}^d (t \Leftrightarrow \exp(2\pi\sqrt{\Leftrightarrow 1}\alpha_j)) = \prod_{j \geqslant 1} (t^j \Leftrightarrow 1)^{e(j)}$ with some $e(j) \in \mathbb{Z}$. (For example, see [Gyo4].)

1.3. Now assume that every geometric object so far is defined over $\mathbb{Q}$, and let us consider its 'reduction modulo p' assuming $p \gg 0$. (Here $\mathbb{Q}$ may be replaced with any algebraic number field.) Thus we can consider $\Omega(\mathbb{F}_q)$, $O_1^\vee(\mathbb{F}_q)$ etc. for a finite field $\mathbb{F}_q$ if $p = \operatorname{char}(\mathbb{F}_q) \gg 0$. Take a non-trivial additive character $\psi \in \operatorname{Hom}(\mathbb{F}_q, \mathbb{C}^\times)$. For any multiplicative character $\chi \in \operatorname{Hom}(\mathbb{F}_q^\times, \mathbb{C}^\times)$, put $G(\chi, \psi) := \sum_{t \in \mathbb{F}_q^\times} \chi(t)\psi(t)$.

THEOREM A1. *If the characteristic of* $\mathbb{F}_q$ *is sufficiently large, then we have for all* $\chi \in \operatorname{Hom}(\mathbb{F}_q^\times, \overline{\mathbb{Q}_l}^\times)$ *that*

$$q^{-n} \sum_{v \in \Omega(\mathbb{F}_q)} \chi(f(v)) \psi(\langle v^\vee, v \rangle)$$

$$= q^{-m/2} \prod_{j \geqslant 1} \left(\frac{G(\chi^j, \psi)}{\sqrt{q}} \right)^{e(j)} \cdot \chi \left(\frac{b_0 f^\vee(v^\vee)^{-1}}{\prod_{j \geqslant 1} (j^j)^{e(j)}} \right) \cdot \kappa^\vee(v^\vee),$$

for $v^\vee \in O_1^\vee(\mathbb{F}_q)$, *where* $\kappa^\vee(v^\vee) = \pm 1$ *depends on* $v^\vee$ *but not on* χ. *Moreover the above sum vanishes if* $v^\vee \in (\Omega^\vee \backslash O_1^\vee)(\mathbb{F}_q)$.

THEOREM A2. *If the characteristic of* $\mathbb{F}_q$ *is sufficiently large, then we have for all* $\chi \in \operatorname{Hom}(\mathbb{F}_q^\times, \overline{\mathbb{Q}_l}^\times)$ *that*

$$q^{-m} \sum_{v^\vee \in O_1^\vee(\mathbb{F}_q)} \chi(f^\vee(v^\vee)) \psi(\langle v^\vee, v \rangle)$$

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$$= q^{-m/2} \prod_{j \geqslant 1} \left(\frac{G(\chi^j, \psi)}{\sqrt{q}} \right)^{e(j)} \cdot \chi \left(\frac{b_0 f(v)^{-1}}{\prod_{j \geqslant 1} (j^j)^{e(j)}} \right) \cdot \kappa^\vee(F(v)),$$

for $v \in \Omega(\mathbb{F}_q)$, *with the same* $\kappa^\vee$ *as in Theorem* A1.

Moreover for $v^\vee \in (V^\vee \backslash \Omega^\vee)(\mathbb{F}_q)$, resp. $v \in (V \backslash \Omega)(\mathbb{F}_q)$ the character sum in Theorem A1, resp. Theorem A2, vanishes when the order of χ is different from the order in $\mathbb{Q}/\mathbb{Z}$ of each α_j in (1.2, (4)), see (5.2.3.3) and (7.6) below.

1.4. To state our second result, let us introduce the following notation.

$r :=$ card$\{j | \alpha_j \in \mathbb{Z}\} = \sum_{j \geqslant 1} e(j)$.
$r(\ ) :=$ rank $=$ dimension of a maximal torus.
$s(\ ) :=$ split rank $=$ dimension of a maximal split torus, cf. [Bor2, V, 15.14].
$G_{v^\vee} =$ isotropy group at $v^\vee \in V^\vee(\mathbb{F}_q)$.
$r(v^\vee) := r(G) \Leftrightarrow r(G_{v^\vee})$.
$s(v^\vee) := s(G) \Leftrightarrow s(G_{v^\vee})$.

THEOREM B. *Assume that* char$(\mathbb{F}_q) \gg 0$. *Then*

$$\kappa^\vee(v^\vee) = (\Leftrightarrow 1)^{r(v^\vee) - s(v^\vee)},$$

for $v^\vee \in O_1^\vee(\mathbb{F}_q)$.

1.5. Assume that the characteristic of $\mathbb{F}_q$ is not 2, and let $\chi_{1/2}$ be the unique non-trivial character of $\mathbb{F}_q^\times$ of order 2, i.e., the Legendre symbol. For $v^\vee \in O_1^\vee(\mathbb{F}_q)$, let $h^\vee(v^\vee)$ be the discriminant of the quadratic form Q determined by $((\partial^2 \log f^\vee / \partial y_i \partial y_j)(v^\vee))$, i.e., the discriminant of the quadratic form on $V^\vee(\mathbb{F}_q)/(\text{radical of } Q)$ induced by Q. Cf. (9.1.0, (4)). (Here $\{y_1, \ldots, y_n\}$ denotes a linear coordinate system of $V^\vee$.) Since $h^\vee(v^\vee)$ is an element of $\mathbb{F}_q^\times / \mathbb{F}_q^{\times 2}$, $\chi_{1/2}(h^\vee(v^\vee))$ is well-defined. We define $h(v)$ for $v \in O_1(\mathbb{F}_q)$ similarly using f instead of $f^\vee$. It is known that $(m + r)/2$ is an integer. (See [Gyo2, 7.6], where the proof is based on the mixed Hodge theory. See (5.2.1.3) for an alternative proof.)

THEOREM C. *Assume that* char$(\mathbb{F}_q) \gg 0$. *Then*

$$\kappa^\vee(v^\vee) = \chi_{1/2} \left((\Leftrightarrow 1)^{(m+r)/2} \prod_{j \geqslant 1} j^{e(j)} \cdot h^\vee(v^\vee) \right),$$

for $v^\vee \in O_1^\vee(\mathbb{F}_q)$.

1.6. In the course of the proof of Theorem B, we obtain in (6.3) that

$$r(v^\vee) = r (= \text{the number of the integer roots of } b(s) \text{counting multiplicity}),$$

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for $v^\vee \in O_1^\vee$, which was originally conjectured by N. Kawanaka [Kaw2, (3.4.7), (ii)], [GyoKaw, 3, Remark].

1.7. Note that the character sums of the form $\sum_v \chi_1(f_1(v)) \cdots \chi_l(f_l(v))\psi(\langle v^\vee, v\rangle)$ with relative invariants $f_1, \ldots, f_l$, are actually dealt with in the above Theorems. In fact, we can take χ and $a_i \in \mathbb{Z}_{>0}$ so that $\chi_i = \chi^{a_i}$ for all i, and then we have $\prod_i \chi_i(f_i(v)) = \chi(f(v))$ with $f := \prod_i f_i^{a_i}$.

1.8. Outline of the proof

The proof of the Theorems A–C roughly goes as follows.

(1) We start from the D-module theory [Gyo1, Sect. 3], [Gyo3, Sect. 6], especially the regular holonomic D-modules related to the complex powers of relative invariants,
(2) second, proceed to the study of perverse sheaves on $V(\mathbb{C})$ etc. by the Riemann–Hilbert correspondence [Gyo1, Sect. 3], [Gyo3, Sect. 6],
(3) third, proceed to the study of l-adic étale perverse sheaves on $V(\overline{\mathbb{F}_q})$ etc. by the 'reduction modulo p' (cf. the proof of (3.5.3)),
(4) and then, obtain results concerning $\mathbb{C}$-valued functions on $V(\mathbb{F}_q)$ etc. by considering the Frobenius trace.

Since a D-module is nothing but a system of *linear* partial differential equations, we can intuitively say that we are studying functions via the linear differential equations characterizing them. Therefore the ambiguity of multiplication by a scalar is inevitable, and our main problem is to explicitly determine this scalar.

Our present approach to the determination of the scalar is based on the Laumon's product formula (cf. (3.1.5)) expressing the Frobenius determinant as a product of local constants. We shall give a slightly more detailed explanation of this approach in (1.9) below, in the course of explaining the overall structure of the present paper.

An alternative approach to the determination of the scalar is considered in [Gyo2] (see [Gyo4] for the detail), where the mixed Hodge modules due to M. Saito are considered in place of D-modules in the step (1), and then the Weil estimate is obtained in the step (4) via the calculation of the weight filtrations in the step (3) and via the famous work of P. Deligne on the Weil 'conjecture'. In this way, we can explicitly determine the archimedean absolute value of the scalar. At present, we can not completely eliminate the remaining ambiguity of the argument of the scalar by this approach.

1.9. Content

This paper consists of 9 sections.

Section 1. Introduction.

Section 2 is devoted to the review of the theory of prehomogeneous vector spaces.

Section 3. We shall study the character sums using l-adic étale sheaves via the Grothendieck–Lefschetz trace formula. As we shall see in (3.5), (an essential part of) the sheaf of our concern is smooth and of rank one. Therefore, the trace and the determinant of the Frobenius are in fact the same, and thus the Frobenius determinant becomes of our main interest. In this section, first we review the product formula of G. Laumon (3.1.5) which describes the Frobenius determinant in terms of the local constants. Next, we review a formula (3.2.1), which describes geometrically how the Frobenius determinant varies when the coefficient sheaf is twisted by the Lang torsor. Using this formula together with a generality concerning the Deligne–Fourier transformation (3.3), we obtain an expression for the ratio of the twisted and the untwisted Frobenius determinant in terms of global monodromy etc. (3.4.5). In the final stage, the twisted Frobenius determinant becomes the character sum appearing in Theorem A1 (or rather the one appearing in (5.2.1.0)), and the untwisted one corresponds to the case where χ is the trivial character.

Section 4. What is important as for the expression (3.4.5) is that there is an expression (4.1.4) for the determinant of the difference operator on the Aomoto complex (4.1.2), and this expression highly resembles (3.4.5). Moreover, we can explicitly determine the determinant of the difference operator at least in the situation of our concern (cf. (4.2.4), (4.2.5) and (4.3.3)).

Section 5. Thus, comparing (3.4.5) and (4.1.4), we obtain enough information concerning the ratio of the twisted and the untwisted Frobenius determinant, and this enables us to prove Theorem A1. In the course of the proof, we obtain Theorem A1† in (5.2.3.2), which refines both Theorems A1 and (3.5.3, (2)). In particular, Theorem A1† gives an expression of the values of the character sum appearing in Theorem A1 for all $v^\vee \in V^\vee(\mathbb{F}_q)$ in terms of the trace of the Frobenius, whose explicit determination is still open.

Section 6 is devoted to the proof of Theorem B. The basic idea is (6.2, (9)), which enables us to express the right-hand side of Theorem B using the operation $|_{q\mapsto q^{-1}} =$ (the substitution of q by q^{-1}).

Section 7. Here we show that Theorem A2 follows from Theorem A1. In the same time, we also obtain Theorem A2† whose meaning is similar to Theorem A1†.

Section 8 contains a formula (Proposition (8.2.3)) which yields in (9.3) an expression for the untwisted Frobenius determinant in terms of the Hessian of the restriction of f to $\{v \in O_1 | \langle v^\vee, v\rangle = 1\}$. This formula is an easy consequence of some known facts which are first reviewed.

Section 9. In Lemma (9.1.7) we relate the above Hessian with $h^\vee(v^\vee)$. Finally we prove Theorem C in (9.3).

1.10. Historical remark, motivation, and related works

The character sums studied in the present paper were first taken up in 1981 by Z. Chen [Che] in the general setting, although some special cases had already

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been discussed in, e.g., [Sta], [Tsa]. In 1983, N. Kawanaka [Kaw1] has taken up the same character sums, independently of Z. Chen. His motivation lies in the theory of complex linear representations of $G(\mathbb{F}_q)$ with reductive G: he showed that these character sums actually appear in the character tables of the finite reductive groups, i.e., the groups of the rational points of connected reductive groups over $\mathbb{F}_q$ [Kaw2]. Since the explicit determination of the character tables is the main problem in the representation theory of finite reductive groups (an open problem, at least in May 1996), it is more or less inevitable to study these character sums. Later the second named author started to work jointly with N. Kawanaka. The result of this joint work was announced in [GyoKaw]. Using the result of [GyoKaw], F. Sato [SatF1] studied the L-functions obtained by twisting the ζ-functions of prehomogeneous vector spaces by Dirichlet characters. In particular, he obtained the functional equations satisfied by them, where our character sums appear in the same way as the classical Gauss sums appear in the functional equations of the Dirichlet L-functions. Some progress after [GyoKaw] was announced in [Gyo2], where two conjectures were formulated, which are now the main theorems of the present paper.

Although our main interest here is the finite field analogue of (1.1, (1)), let us give a short sketch of what is known when the base field is a local field of characteristic zero.

Archimedean local fields ($\mathbb{R}$, $\mathbb{C}$)

The functional equation of the form (1.1, (1)) was first proved by M. Sato [SatM] under the assumptions

(1) that G is a reductive group,
(2) that every irreducible component of $V\backslash O_0$ is of codimension one,

together with some additional mild assumptions. Indeed, this result is the very origin of the theory of prehomogeneous vector spaces. F. Sato [SatF2] obtained a similar functional equation for $|f_1|^{s_1}, \ldots, |f_l|^{s_l}$, where he did not assume (1). In [Gyo1], another generalization was obtained where only (1) is assumed. These results do not give the explicit form of '(some factors)' in (1.1, (1)). In the $\mathbb{C}$-case, these factors are explicitly determined already in [SatM] up to signature, and in [Igu2] without ambiguity. Both works assume (1), (2) and some mild assumptions. In the $\mathbb{R}$-case, an algorithm based on the microlocal analysis to calculate these factors is given in [Kas2], [KasKimMur]. Actual calculation has been done for some special cases by T. Suzuki and M. Muro using this algorithm. Besides, functional equations of the form (1.1, (1)) for some special cases have been obtained by various methods by many authors including I. M. Gelfand–G.E. Shilov, M. Sato, T. Shintani, F. Sato, B. Datskovski–D. J. Wright, I. Muller, I. Satake–J. Faraut, T. Suzuki, Y. Teranishi, S. Rallis–G. Schiffmann, E. M. Stein, S. S. Gelbart, R. Godement–H. Jacquet.

Non-archimedean local fields (finite extensions of $\mathbb{Q}_p$)

For these fields a functional equation of the form (1.1, (1)) was first proved by J. I. Igusa [Igu1] assuming the 'Finite Orbit Condition' and some other restrictions. Some of these restrictions were removed in [SatF1] and [Kim]. However, very little is known about the explicit form of '(some factors)' in (1.1, (1)), except in some examples (see, e.g., [SatF1]), or in the case that $G(\mathbb{Q}_p)$ acts transitively on $\Omega(\mathbb{Q}_p)$, which has been investigated by J. I. Igusa [Igu2].

2. Prehomogeneous vector spaces

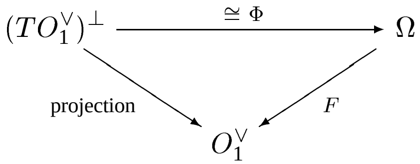
In this section, we review the theory of prehomogeneous vector spaces [Gyo1], [Gyo2], [Gyo3]. We keep the notation of (1.1) and (1.2).

2.1. LEMMA [Gyo1, Lemma 1.8]. (1) *For* $v \in \Omega$, $f^\vee(F(v)) = b_0 f(v)^{-1}$. (2) *For* $v^\vee \in \Omega^\vee$, $f(F^\vee(v^\vee)) = b_0 f^\vee(v^\vee)^{-1}$.

2.2. LEMMA [Gyo1, Theorem 1.18]. *Let* $(TO_1^\vee)^\perp$ *be the conormal bundle of* $O_1^\vee$, *i.e.,*

$$(TO_1^\vee)^\perp = \{(v, v^\vee) \in V \times V^\vee | v^\vee \in O_1^\vee, v \perp T_{v^\vee} O_1^\vee (\subset V^\vee)\}.$$

(1) *We have an isomorphism* Φ *making the following diagram commutative.*



Here $\Phi(v, v^\vee) := v + F^\vee(v^\vee)$.

(2) *The inverse of* Φ *is given by* $\Psi(v) := (v \Leftrightarrow F^\vee F(v), F(v))$ *for* $v \in \Omega$.
(3) *All these morphisms are* G*-equivariant.*
(4) *By these isomorphisms,* $O_1 \subset \Omega$ *corresponds to the zero section of* $(TO_1^\vee)^\perp$.
(5) *In particular,* F *induces an isomorphism* $O_1 \to O_1^\vee$, *whose inverse is given by* $F^\vee$.
(6) *The isotropy subgroup* $G_{v^\vee}$ *of* G *at* $v^\vee \in O_1^\vee$ *is reductive.*

Interchanging symbols with $^\vee$ and those without $^\vee$, we define $\Phi^\vee$ and $\Psi^\vee$.

2.3. DOUBLE COVERING $\tilde{O}_1 \to O_1$ AND LOCALLY CONSTANT SHEAF $L(\omega) = \mathbb{C}\,\omega$

For a local coordinate system $\{z_1, \dots, z_m\}$ of O_1, put

$$\omega^2 := \det\left(\left\langle F_*\left(\frac{\partial}{\partial z_i}\right), \frac{\partial}{\partial z_j}\right\rangle\right) \cdot (dz_1 \wedge \cdots \wedge dz_m)^{\otimes 2}. \qquad (1)$$

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Here $\partial/\partial z_j$ denotes the vector field on O_1 defined by z_j, and $F_*(\partial/\partial z_j)$ denotes the corresponding vector field on $O_1^\vee$. Then ω^2 is independent of the choice of the local coordinates, and gives rise to a global section of the line bundle $(\bigwedge^m T^*O_1)^{\otimes 2}$ which is everywhere non-vanishing [Gyo1, 3.12]. Let $\pi: \tilde{O}_1 \to O_1$ be the two-fold covering of O_1 determined by $\omega := \sqrt{\omega^2}$, cf. [Gyo1, 3.14]. The m-form ω on O_1 is defined only locally (with respect to the classical topology), but its pull-back $\pi^*\omega =: \tilde{\omega}$ is defined globally on $\tilde{O}_1$. Define $\omega^{\vee 2}$, $\omega^\vee$, $\tilde{\omega}^\vee$, and $\pi^\vee: \tilde{O}_1^\vee \to O_1^\vee$, replacing O_1 and f with $O_1^\vee$ and $f^\vee$. Let $L(\omega^\vee)$ denote the isotypic part of $\pi_*^\vee \mathbb{C}_{\tilde{O}_1^\vee}$ corresponding to the non-trivial character of Gal $(\tilde{O}_1^\vee/O_1^\vee)$. Then $L(\omega^\vee)$ is a locally constant sheaf on $O_1^\vee$, and $L(\omega^\vee) = \mathbb{C}\,\omega^\vee$. (Although $\omega^\vee$ is not globally defined, the ambiguity is only the multiplication by ± 1. Hence the totality of its scalar multiples $\mathbb{C}\,\omega^\vee$ is globally well-defined and gives a locally constant sheaf on $O_1^\vee$. The meaning of $\mathbb{C}\,f^\alpha$ etc. below should be understood in the same way.) Define $L(\omega) = \mathbb{C}\omega$ in the same way.

Let $O_1 \xrightarrow{i} \Omega \xrightarrow{j} V$ and $O_1^\vee \xrightarrow{i^\vee} \Omega^\vee \xrightarrow{j^\vee} V^\vee$ be the inclusion mappings.

2.4. THEOREM. *Let $\mathcal{F}_{\text{geom}}$ denote the Sato–Fourier transformation* [SatKasKaw] (*cf.* [BryMalVer], [HotKas], [KasSch]). Then

$$\mathcal{F}_{\text{geom}}(Rj_*\mathbb{C}f^\alpha[n]) = j_!^\vee i_*^\vee(\mathbb{C}f^{\vee\,-\alpha} \otimes L(\omega^\vee))[m], \tag{1}$$

$$\mathcal{F}_{\text{geom}}(j_!\mathbb{C}f^\alpha[n]) = Rj_*^\vee i_*^\vee(\mathbb{C}f^{\vee\,-\alpha} \otimes L(\omega^\vee))[m], \tag{2}$$

$$\mathcal{F}_{\text{geom}}(Rj_*^\vee i_*^\vee\mathbb{C}f^{\vee\,-\alpha}[m]) = j_!(\mathbb{C}f^\alpha \otimes F^*L(\omega^\vee))[n], \quad \text{and} \tag{3}$$

$$\mathcal{F}_{\text{geom}}(j_!^\vee i_*^\vee\mathbb{C}f^{\vee\,-\alpha}[m]) = Rj_*(\mathbb{C}f^\alpha \otimes F^*L(\omega^\vee))[n]. \tag{4}$$

See [Gyo1, 3.23] for (1) and (2). See [Gyo3, 6.22] for (3) and (4). We need (3) and (4) only in the proof of Theorem A2 in Section 7.

For the convenience of the readers, we recall that for a bounded complex K of sheaves on V the Sato–Fourier transform $\mathcal{F}_{\text{geom}}(K)$ of K is defined by $\mathcal{F}_{\text{geom}}(K) := Rpr_!^\vee(pr^*K \otimes \langle\,\rangle^*L_-)[n]$, where L_- is the sheaf on $\mathbb{A}^1_{\mathbb{C}}$ obtained by extending by zero the constant sheaf $\mathbb{C}_Z$ on $Z := \{z \in \mathbb{C} | \mathrm{Re}(z) \leqslant 0\}$, where pr and $pr^\vee$ are the projections of $V^\vee \times V$ on V and $V^\vee$, and where $\langle\,\rangle: V^\vee \times V \to \mathbb{A}^1_{\mathbb{C}}$ is the natural pairing.

3. Determinant of Frobenius action and monodromy

Let k be a field. We denote by $\bar{k}$ an algebraic closure of k. In all what follows ℓ is a prime number different from the characteristic of the base field we are working with (for the moment k). For any separated Noetherian scheme X over k we denote by $D_c^b(X, \overline{\mathbb{Q}_\ell})$ the category defined by Deligne [Del3, (1.1.1)–(1.1.3)]. Its objects are 'bounded complexes' whose cohomology are constructible $\overline{\mathbb{Q}_\ell}$-sheaves.

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We will use the standard notation from the theory of derived categories as in [Lau1]. In particular when C is a complex of vector spaces with bounded and finite dimensional cohomology and E an endomorphism of C, we use the following notation

$$\operatorname{rank} C = \sum_i (\Leftrightarrow 1)^i \dim H^i(C),$$

$$\operatorname{tr}(E, C) = \sum_i (\Leftrightarrow 1)^i \operatorname{tr}(E, H^i(C)),$$

$$\det(E, C) = \prod_i \det(E, H^i(C))^{(-1)^i},$$

where tr and det denote the trace and the determinant. Often a sheaf (complex) and a restriction of it will be denoted by the same symbol, without mentioning. As usual $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$ denote the n-dimensional affine and projective space over k, considered as schemes, and $\mathbb{G}_{m,k} = \mathbb{A}^1_k \backslash \{0\}$. We denote $\{0, 1, 2, \ldots\}$ (resp. $\{1, 2, 3, \ldots\}$) by **N** (resp. $\mathbf{N}_0$).

3.1. Laumon's product formula

3.1.1. Let k be any field. Let s be a closed point of $\mathbb{P}^1_k$, i.e. $s \in |\mathbb{P}^1_k|$, and $\bar{s}$ a geometric point of $\mathbb{P}^1_k$ with image s. The Henselization of $\mathbb{P}^1_k$ at s is denoted by T_s. Let η_s be the generic point of T_s, and $\bar{\eta}_s$ a generic geometric point of T_s.

We denote by G_s, resp. I_s, the fundamental group of $T_s \backslash \{0\}$, resp. $T_s \otimes \bar{k} \backslash \{0\}$, i.e. the arithmetic, resp. geometric, monodromy group at s. With a G_s, resp. I_s, module we always mean a module in the sense of [Lau1, 2.1.2], i.e. a smooth $\overline{\mathbb{Q}_\ell}$-sheaf on $T_s \backslash \{0\}$, resp. $T_s \otimes \bar{k} \backslash \{0\}$.

Let $K \in D^b_c(U, \overline{\mathbb{Q}_\ell})$, where $U = \mathbb{P}^1_k \backslash$ {finite number of points}. We denote by $[K_{\bar{\eta}_0}]$ the image of $\sum_i (\Leftrightarrow 1)^i H^i(K)_{\bar{\eta}_0}$ in the Grothendieck group $\mathcal{K}_{I_0}$ of I_0-modules. Similarly $[K_{\bar{\eta}_\infty}] := [(\pi_* K)_{\bar{\eta}_0}]$, where π is the map $x \mapsto x^{-1}$. Note that $[K_{\bar{\eta}_0}]$ and $[K_{\bar{\eta}_\infty}]$ are completely determined by the geometric monodromy at 0 and ∞ of the cohomology sheaves of K.

When $s \in |U|$, the total drop of K at s is defined by

$$a_{\bar{s}}(K) := \operatorname{rank}(K_{\bar{\eta}_s}) \Leftrightarrow \operatorname{rank}(K_{\bar{s}}) + \text{swan conductor}(K_{\bar{\eta}_s}),$$

see [Lau1, 3.1.5.2] where it is denoted by $a(T_s, K)$. For $N \in \mathbb{N}_0$, put $V_N = \pi_{N*}\overline{\mathbb{Q}_\ell}$, where π_N is the map $\mathbb{P}^1_k \to \mathbb{P}^1_k : x \mapsto x^N$, and denote the image of $(V_N)_{\bar{\eta}_0}$ in $\mathcal{K}_{I_0}$ by $[V_N]$.

3.1.2. Let $\psi \in \operatorname{Hom}(\mathbb{F}_q, \overline{\mathbb{Q}_\ell}^\times) \backslash \{1\}$ and $\chi \in \operatorname{Hom}(\mathbb{F}_q^\times, \overline{\mathbb{Q}_\ell}^\times)$. Denote by L_ψ, resp. L_χ, the associated $\overline{\mathbb{Q}_\ell}$-sheaf (= Lang torsor, [Del4, p. 171]) on $\mathbb{A}^1_{\mathbb{F}_q}$, resp. $\mathbb{G}_{m,\mathbb{F}_q}$. Thus the geometric Frobenius endomorphism Frob_q over $\mathbb{F}_q$ at $x \in \mathbb{F}_q$,

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resp. $x \in \mathbb{F}_q^\times$, acts on the stalk $L_{\psi,x}$, resp. $L_{\chi,x}$, as the multiplication by $\psi(x)$, resp. $\chi(x)$. We recall that

$$G(\chi,\psi) := \sum_{x \in \mathbb{F}_q^\times} \chi(x)\psi(x).$$

When $f\colon X \to \mathbb{G}_{m,\mathbb{F}_q}$ is a morphism of schemes we will sometimes denote f^*L_χ by $L(\chi(f))$, abusively.

3.1.3. For any separated scheme X of finite type over $\mathbb{F}_q$ and $K \in D_c^b(X, \overline{\mathbb{Q}_\ell})$ one defines

$$\varepsilon(X,K) := \det(-\mathrm{Frob}_q, R\Gamma(X \otimes \overline{\mathbb{F}_q}, K))^{-1},$$

and

$$\varepsilon_0(X,K) := \det(-\mathrm{Frob}_q, R\Gamma_c(X \otimes \overline{\mathbb{F}_q}, K))^{-1}.$$

3.1.4. Assume the notation of (3.1.1) with $k = \mathbb{F}_q$. Let $\omega \neq 0$ be a rational differential form on $\mathbb{P}^1_{\mathbb{F}_q}$ and ψ as in (3.1.2). For $s \in |\mathbb{P}^1_{\mathbb{F}_q}|$ and $K \in D_c^b(T_s, \overline{\mathbb{Q}_\ell})$ one defines the *local constant*

$$\varepsilon_\psi(T_s, K, \omega) \in \overline{\mathbb{Q}_\ell}^\times, \tag{1}$$

as in [Lau1, 3.1.5.4] (cf. also [Del5, 4.1]). It equals 1 when the cohomology sheaves of K are smooth and ω has no pole and no zero at s. Moreover for $K \in D_c^b(T_s \backslash \{0\}, \overline{\mathbb{Q}_\ell})$, e.g., a G_s-module, we put

$$\varepsilon_{\psi,0}(T_s, K, \omega) = \varepsilon_\psi(T_s, j_{s!}K, \omega), \tag{2}$$

where j_s is the immersion $T_s \backslash \{0\} \to T_s$. We have

$$\varepsilon_{\psi,0}(T_0, L_\chi, dx) = \chi(-1)G(\chi,\psi), \qquad \varepsilon_\psi(T_0, \overline{\mathbb{Q}_\ell}, dx) = 1, \tag{3}$$

where x is the standard coordinate on $\mathbb{A}^1_{\mathbb{F}_q}$, see [Lau1, p. 199]. Here and in Section 8 we will repeatedly use the fundamental properties of the local constant which are summarized in [Lau1, p. 186]. (Although there is some restriction on the additive character ψ in [Lau1, (0.2)], the results of loc. cit. remain valid for general ψ with obvious adaptation in loc. cit. (3.1.5.4, (iv)).) Moreover, if ω has a simple pole at 0 and if K, K' are tame G_0-modules having the same image in the Grothendieck group of I_0-modules, then

$$\varepsilon_{\psi,0}(T_0, K, \omega) = \varepsilon_{\psi,0}(T_0, K', \omega). \tag{4}$$

(See [Lau1, 2.1.4] for the definition of tameness.)

This follows directly from [Sai1, Lemma 1.(1)]. (Alternatively: Assertion (4) follows from [Lau1, 3.1.5.6] when rank $K = 1$. The general case reduces to this by an argument similar to [Lau1, p. 198 line 5–13].)

3.1.5. THEOREM (Laumon's Product Formula [Lau1, Thm 3.2.1.1]). *For any $K \in D_c^b(\mathbb{P}^1_{\mathbb{F}_q}, \overline{\mathbb{Q}_\ell})$ we have*

$$\varepsilon(\mathbb{P}^1_{\mathbb{F}_q}, K) = q^{\text{rank}\,(K_{\bar\eta_0})} \prod_{s \in |\mathbb{P}^1_{\mathbb{F}_q}|} \varepsilon_\psi(T_s, K, \omega).$$

3.1.6. Let X be any separated scheme of finite type over $\mathbb{F}_q$ and $K \in D_c^b(X, \overline{\mathbb{Q}_\ell})$. Suppose for all $x \in |X|$ that the characteristic polynomial of the Frobenius action on K_x has coefficients in $\mathbb{Q}$. Then $\varepsilon_0(X, K) \in \mathbb{Q}$. Indeed the associated L-function (see e.g. [Lau1, 3.1]) belongs then to $\mathbb{Q}(t)$. But $\varepsilon_0(X, K)$ can be rationally expressed in terms of that L-function.

3.2. FROBENIUS DETERMINANT OF A TWIST

The next Proposition goes back to Loeser [Loe, Cor. 5.5] which is similar, see also [DenLoe1, Prop. 2.4.1] and [DenLoe2]. It is also very much related to the material in [Sai1] which goes much deeper.

3.2.1. PROPOSITION. *Assume the notation of* (3.1.1) *with* $k = \mathbb{F}_q$.

Let $K \in D_c^b(\mathbb{G}_{m,\mathbb{F}_q}, \overline{\mathbb{Q}_\ell})$ and suppose $H^i(K)$ has tame ramification at 0 *and at* ∞*, for all i. Assume that*

$$[K_{\bar\eta_0}] = \sum_{\substack{N \in \mathbf{N} \\ (N,q)=1}} \alpha_N [V_N] \quad \text{and} \quad [K_{\bar\eta_\infty}] = \sum_{\substack{N \in \mathbf{N} \\ (N,q)=1}} \beta_N [V_N],$$

where the integers α_N, β_N are zero for almost all N. Then

$$\begin{aligned} &\varepsilon_0(\mathbb{G}_{m,\mathbb{F}_q}, K \otimes L_\chi)/\varepsilon_0(\mathbb{G}_{m,\mathbb{F}_q}, K) \\ &\quad = q^{\sum_{\chi^N \neq 1} \beta_N} \chi\left(\left(\prod_{s \in \overline{\mathbb{F}_q}^\times} s^{a_s(K)}\right) \prod_{\substack{N \in \mathbf{N} \\ (N,q)=1}} N^{-N(\alpha_N - \beta_N)}\right) \\ &\qquad \times \prod_{\substack{N \in \mathbf{N} \\ (N,q)=1}} (\Leftrightarrow G(\chi^N, \psi))^{\alpha_N - \beta_N}. \end{aligned}$$

Proof. From Laumon's Product Formula (3.1.5) for $j_!K$ and for $j_!(K \otimes L_\chi)$ with j the immersion of $\mathbb{G}_{m,\mathbb{F}_q}$ into $\mathbb{P}^1_{\mathbb{F}_q}$, and from [Lau1, 3.1.5.6] we get

$$\frac{\varepsilon_0(\mathbb{G}_{m,\mathbb{F}_q}, K \otimes L_\chi)}{\varepsilon_0(\mathbb{G}_{m,\mathbb{F}_q}, K)} = c_0 c_\infty \chi\left(\prod_{s \in \overline{\mathbb{F}_q}^\times} s^{a_s(K)}\right), \tag{1}$$

where

$$c_0 = \frac{\varepsilon_{\psi,0}(T_0, K \otimes L_\chi, x^{-1}\,\mathrm{d}x)}{\varepsilon_{\psi,0}(T_0, K, x^{-1}\,\mathrm{d}x)}$$

and

$$c_\infty = \frac{\varepsilon_{\psi,0}(T_\infty, K \otimes L_\chi, x^{-1}\,\mathrm{d}x)}{\varepsilon_{\psi,0}(T_\infty, K, x^{-1}\,\mathrm{d}x)}.$$

The Proposition follows now directly from (3.1.4, (4)), Lemma 3.2.2 below, and from the formula

$$G(\chi^{-1}, \psi) = q\chi(\Leftrightarrow 1)G(\chi, \psi)^{-1} \quad \text{if } \chi \neq 1. \tag{2}$$

□

3.2.2. LEMMA. *If* gcd $(N, q) = 1$, *then*

$$\frac{\varepsilon_{\psi,0}(T_0, V_N \otimes L_\chi, x^{-1}\,\mathrm{d}x)}{\varepsilon_{\psi,0}(T_0, V_N, x^{-1}\,\mathrm{d}x)} = \Leftrightarrow \chi^N(N^{-1})G(\chi^N, \psi),$$

and

$$\frac{\varepsilon_{\psi,0}(T_\infty, V_N \otimes L_\chi, x^{-1}\,\mathrm{d}x)}{\varepsilon_{\psi,0}(T_\infty, V_N, x^{-1}\,\mathrm{d}x)} = \Leftrightarrow \chi^{-N}(\Leftrightarrow N^{-1})G(\chi^{-N}, \psi).$$

Proof. The first formula follows from [Lau1, 3.1.5.4. (iv)] with K_1 the sum of $j_! L_{\chi^N}$ in degree 0 and $j_! \overline{\mathbb{Q}_\ell}$ in degree 1 and with f the map $x \mapsto x^N$, and from (3.1.4, (3)) and [Lau1, 3.1.5.5]. The second formula follows directly from the first by [Lau1, 3.1.5.5] with $a = \Leftrightarrow 1$. □

3.3. Deligne–Fourier transform of a χ-homogeneous complex

3.3.1. Let $V = \mathbb{A}^n_{\mathbb{F}_q}$, $V^\vee$ the dual space of V, and $\langle\ \rangle : V^\vee \times V \to \mathbb{A}^1_{\mathbb{F}_q}$ the natural pairing.

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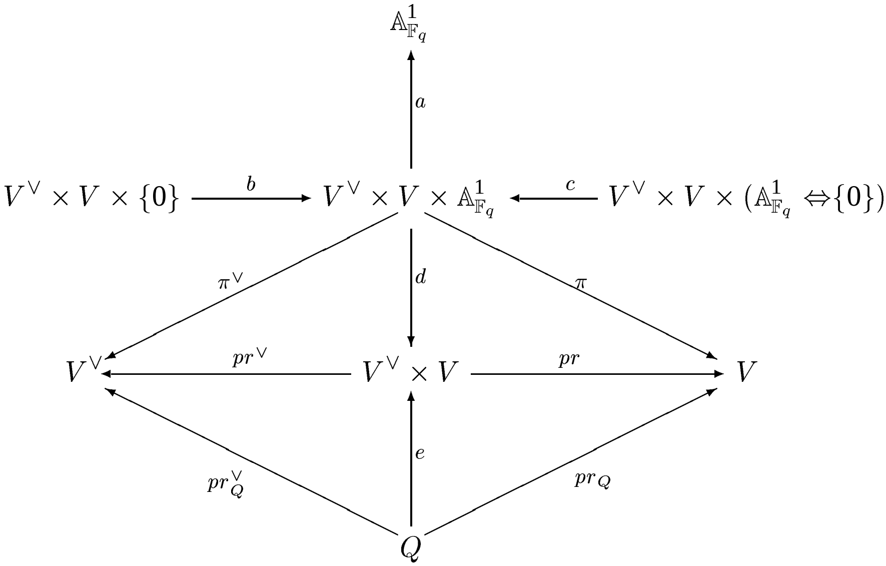
3.3.2. Consider the following diagram



where $Q = \{(v^\vee, v) \in V^\vee \times V | \langle v^\vee, v\rangle = 1\}$ and a is defined by $a(v^\vee, v, t) = t(\langle v^\vee, v\rangle \Leftrightarrow 1)$. The remaining morphisms are inclusions, projections, or their compositions.

3.3.3. Define a functor: $\mathcal{F}_\psi\colon D^b_c(V_{\mathbb{F}_q}, \overline{\mathbb{Q}_\ell}) \to D^b_c(V^\vee_{\mathbb{F}_q}, \overline{\mathbb{Q}_\ell})$ by $\mathcal{F}_\psi(\Leftrightarrow) = Rpr^\vee_!(pr^*(\Leftrightarrow) \otimes \langle\ \rangle^* L_\psi)[n]$, which is called the *Deligne–Fourier transformation*, see [KatLau].

3.3.4. Define $h\colon \mathbb{G}_{m,\mathbb{F}_q} \times V \to V$ by $h(t, v) = tv$. For $\chi \in \mathrm{Hom}(\mathbb{F}_q^\times, \overline{\mathbb{Q}_\ell}^\times)$, a complex $K \in D^b_c(V, \overline{\mathbb{Q}_\ell})$ is called χ-*homogeneous* if $h^*K = L_\chi \otimes K$.

3.3.5. Put $\tau(\chi, \psi) = R\Gamma_c(\mathbb{G}_{m,\mathbb{F}_q}, L_\chi \otimes L_\psi)[1] \cong \overline{\mathbb{Q}_\ell}$ (cf. [Del4, 4.2], note however that this isomorphism does not preserve the Frobenius action).

3.3.6. Assume that $K \in D^b_c(V, \overline{\mathbb{Q}_\ell})$ is χ-homogeneous. Consider the distinguished triangle

$$R\pi^\vee_!(\pi^*K \otimes c_!c^*a^*L_\psi) \to R\pi^\vee_!(\pi^*K \otimes a^*L_\psi)$$
$$\to R\pi^\vee_!(\pi^*K \otimes b_*b^*a^*L_\psi) \overset{+1}{\Leftrightarrow} . \tag{1}$$

Let us look at this triangle more closely. Let $\varphi\colon V(\mathbb{F}_q) \to \overline{\mathbb{Q}_\ell}$ be a function such that $\varphi(tv) = \chi(t)\varphi(v)$ for $t \in \mathbb{F}_q^\times$ and $v \in V(\mathbb{F}_q)$. Consider the following calculation:

$$\left(\sum_{v\in V(\mathbb{F}_q)} \varphi(v)\psi(\langle v^\vee, v\rangle)\right)\left(\sum_{t\in\mathbb{F}_q^\times} \bar{\chi}(t)\bar{\psi}(t)\right)$$

$$= \sum_{v\in V(\mathbb{F}_q),t\in\mathbb{F}_q^\times} \varphi(t^{-1}v)\psi(\langle v^\vee, v\rangle \Leftrightarrow t)$$

$$= \sum_{v\in V(\mathbb{F}_q),t\in\mathbb{F}_q^\times} \varphi(v)\psi(t(\langle v^\vee, v\rangle \Leftrightarrow 1)),$$

where $\bar{\chi} = \chi^{-1}$ and $\bar{\psi} = \psi^{-1}$. Following this calculation, we get

$$R\pi_!^\vee(\pi^*K \otimes c_!c^*a^*L_\psi)[n+1] = \mathcal{F}_\psi(K) \otimes R\Gamma_c(\mathbb{G}_{m,\mathbb{F}_q}, L_{\bar{\chi}} \otimes L_{\bar{\psi}})[1]. \quad (2)$$

Next consider the following calculation

$$\sum_{v\in V(\mathbb{F}_q),t\in\mathbb{F}_q} \varphi(v)\psi(t(\langle v^\vee, v\rangle \Leftrightarrow 1)) = q \sum_{v\in V(\mathbb{F}_q),\langle v^\vee, v\rangle=1} \varphi(v).$$

Following this calculation, we get

$$R\pi_!^\vee(\pi^*K \otimes a^*L_\psi) = R(pr_Q^\vee)_!pr_Q^*K(\Leftrightarrow 1)[\Leftrightarrow 2], \quad (3)$$

where $(\Leftrightarrow 1)$ is the Tate twist. Last, we can show that

$$R\pi_!^\vee(\pi^*K \otimes b_*b^*a^*L_\psi) = Rpr_!^\vee pr^*K. \quad (4)$$

By (1)–(4), we get the following

3.3.7. LEMMA. *Assume the above notation and let $K \in D_c^b(V, \overline{\mathbb{Q}_\ell})$ be χ-homogeneous. Then we have a distinguished triangle*

$$R(pr_Q^\vee)_!pr_Q^*K(\Leftrightarrow 1)[\Leftrightarrow 2]$$

$$\xrightarrow{r} Rpr_!^\vee pr^*K \to \mathcal{F}_\psi(K)[\Leftrightarrow n] \otimes \tau(\bar{\chi}, \bar{\psi}) \xrightarrow{+1}. \quad (1)$$

3.3.8. Let us describe the morphism r appearing in (3.3.7) independently of the additive character ψ. Note that r is the morphism

$$R\pi_!^\vee(\pi^*K \otimes a^*L_\psi) \to R\pi_!^\vee(\pi^*K \otimes b_*b^*a^*L_\psi),$$

induced from the natural morphism

$$a^*L_\psi \to b_*b^*a^*L_\psi. \quad (1)$$

Since

$$R\pi_!^\vee(\pi^*K \otimes a^*L_\psi) = Rpr_!^\vee(pr^*K \otimes Rd_!a^*L_\psi),$$

$$R\pi_!^\vee(\pi^*K \otimes b_*b^*a^*L_\psi) = Rpr_!^\vee(pr^*K \otimes Rd_!b_*b^*a^*L_\psi),$$

$$Rd_!a^*L_\psi = e_*\overline{\mathbb{Q}_\ell}_{,Q}(\Leftrightarrow 1)[\Leftrightarrow 2], \quad \text{and}$$

$$Rd_!b_*b^*a^*L_\psi = \overline{\mathbb{Q}_\ell}_{,V^\vee \times V},$$

(1) induces a morphism

$$e_!e^!\overline{\mathbb{Q}_\ell}_{,V^\vee\times V} = e_*\overline{\mathbb{Q}_\ell}_{,Q}(\Leftrightarrow 1)[\Leftrightarrow 2] \overset{r'}{\Leftrightarrow\!\!\to} \overline{\mathbb{Q}_\ell}_{,V^\vee\times V}, \tag{2}$$

and (2) induces r. Since

$$\begin{aligned} R\,\mathrm{Hom}_{V^\vee\times V}(e_!e^!\overline{\mathbb{Q}_\ell}, \overline{\mathbb{Q}_\ell}) &= R\,\mathrm{Hom}_Q(e^!\overline{\mathbb{Q}_\ell}, e^!\overline{\mathbb{Q}_\ell}) \\ &= R\,\mathrm{Hom}_Q(\overline{\mathbb{Q}_\ell}, \overline{\mathbb{Q}_\ell}) = \overline{\mathbb{Q}_\ell}, \end{aligned}$$

(2) is a scalar multiple of the morphism γ induced by the adjointness, i.e., the natural morphism $\gamma\colon e_!e^!\overline{\mathbb{Q}_l} = R\Gamma_Q(\overline{\mathbb{Q}_l}) \to \overline{\mathbb{Q}_l}$. If (2) is zero, then r is also zero for any K, and hence

$$\mathcal{F}_\psi(K)[\Leftrightarrow n] \otimes \tau(\bar\chi, \bar\psi) = R(pr_Q^\vee)_! pr_Q^* K(\Leftrightarrow 1)[\Leftrightarrow 1] \oplus Rpr_!^\vee pr^* K.$$

(The third term of the triangle in (3.3.7) is the mapping cone of $r = 0$.) If we take the constant sheaf $\overline{\mathbb{Q}_\ell}_{,V}$ as K, the left-hand side is supported by $\{0\}$, although the direct summand $Rpr_!^\vee pr^*K$ of the right-hand side is not. Thus we get a contradiction, and hence (2) is non-zero, i.e.,

$$r' = \gamma \times (\text{non-zero scalar}). \tag{3}$$

3.3.9. LEMMA. *Consider the morphism $\gamma\colon e_!e^!\overline{\mathbb{Q}_\ell} \to \overline{\mathbb{Q}_\ell}$ induced by the adjointness. Put $\omega = \mathrm{cone}(\gamma)$. Then for any χ-homogeneous $K \in D_c^b(V, \overline{\mathbb{Q}_\ell})$ we have $\mathcal{F}_\psi(K) \otimes \tau(\bar\chi, \bar\psi) \cong \mathcal{R}(K)$ where $\mathcal{R}(K) := Rpr_!^\vee(pr^*K \otimes \omega)[n]$ is the Radon transform.*

Proof.

$$\begin{aligned} &\mathcal{F}_\psi(K)[\Leftrightarrow n] \otimes \tau(\bar\chi, \bar\psi) \\ &= \mathrm{cone}(Rpr_!^\vee(pr^*K \otimes e_*\overline{\mathbb{Q}_\ell}_{,Q}(\Leftrightarrow 1)[\Leftrightarrow 2]) \overset{r}{\Leftrightarrow\!\!\to} Rpr_!^\vee pr^*K) \quad \text{by (3.3.7)} \\ &= \mathrm{cone}(Rpr_!^\vee(pr^*K \otimes e_!e^!\overline{\mathbb{Q}_\ell}) \overset{r}{\Leftrightarrow\!\!\to} Rpr_!^\vee(pr^*K \otimes \overline{\mathbb{Q}_\ell})) \\ &= Rpr_!^\vee(pr^*K \otimes \mathrm{cone}(e_!e^!\overline{\mathbb{Q}_\ell} \overset{r'}{\Leftrightarrow\!\!\to} \overline{\mathbb{Q}_\ell})) \\ &\cong Rpr_!^\vee(pr^*K \otimes \omega) \quad \text{by (3.3.8, (3)).} \qquad \square \end{aligned}$$

3.3.10. REMARK. If $\chi = 1$, our 'Radon transformation $\mathcal{R}(\ )$' given in (3.3.9) relates to the one given in [Bry, 9.13] as follows. Let $g\colon V^\times := V\backslash\{0\} \to P$ and

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$g^\vee\colon V^{\vee\times} := V^\vee\backslash\{0\} \to P^\vee$ be the natural morphisms to the projective spaces of V and $V^\vee$. For $\tilde{K} \in D_c^b(P,\overline{\mathbb{Q}_l})$, let $\Phi(\tilde{K}) \in D_c^b(P^\vee,\overline{\mathbb{Q}_l})$ be its Radon transform in the sense of loc. cit., and $K \in D_c^b(V,\overline{\mathbb{Q}_l})$ the zero extension of $g^*\tilde{K}$. Then we have a distinguished triangle

$$R\Gamma(P,\tilde{K})[n-1] \to \mathcal{R}(K) \to g^{\vee *}\Phi(\widetilde{K})(-1) \xrightarrow{+1}$$

in $D_c^b(V^{\vee\times},\overline{\mathbb{Q}_l})$, where the first term means the 'constant sheaf' in the obvious sense. Thus the difference between the two 'Radon transformations' is almost trivial.

3.4. Discrete Fourier transform of $\chi(f)$

3.4.1. We use the notation of (3.3.1). Let $f: V \to \mathbb{A}^1_{\mathbb{F}_q}$ be a homogeneous polynomial over $\mathbb{F}_q$ of degree $d := \deg f$. Put $U = V\backslash f^{-1}(0)$ and consider the open immersion $j\colon U \to V$. Let $K \in D_c^b(V,\overline{\mathbb{Q}_\ell})$ be ν-homogeneous for some $\nu \in \mathrm{Hom}(\mathbb{F}_q^\times,\overline{\mathbb{Q}_\ell}^\times)$, cf. (3.3.4).

For $v^\vee \in V^\vee(\overline{\mathbb{F}_q})\backslash\{0\}$, put

$$H(v^\vee) = \{v \in V | \langle v^\vee, v\rangle = 1\}.$$

3.4.2. Assume now also the notation of (3.1.2) and (3.3.3). We are interested in the character sum

$$S_K^\vee(\chi, v^\vee) := \sum_{v\in U(\mathbb{F}_q)} \mathrm{tr}(\mathrm{Frob}_q, K_v)\chi(f(v))\psi(\langle v^\vee, v\rangle), \tag{1}$$

for $v^\vee \in V^\vee(\mathbb{F}_q)$. By the Grothendieck–Lefschetz trace formula we have

$$S_K^\vee(\chi, v^\vee) = \mathrm{tr}(\mathrm{Frob}_q, \mathcal{F}_\psi(K\otimes j_! f^* L_\chi[-n])_{v^\vee}). \tag{2}$$

When $K = \overline{\mathbb{Q}_\ell}$, the sum $S_K^\vee(\chi, v^\vee)$ equals the discrete Fourier transform (multiplied with $q^{n/2}$) of the map $v \mapsto \chi(f(v))$ with the convention that $\chi(0) = 0$.

3.4.3. LEMMA. *Suppose that $R^i f_! K$ has tame ramification at* 0 *and at* ∞ *for all* i, *and that*

$$[(Rf_!K)_{\bar\eta_0}] = \sum_{\substack{N\in\mathbf{N}\\(N,q)=1}} \gamma_N[V_N],$$

where the integers γ_N are zero for almost all N. Then the rank of $R\Gamma_c(U\otimes\overline{\mathbb{F}_q}, K\otimes f^*L_\chi)$ is 0 *and*

$$\varepsilon_0(U, K\otimes f^*L_\chi) = q^{-\sum_{\chi^N=1}\gamma_N}.$$

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Proof. Note that $R\Gamma_c(U\otimes\overline{\mathbb{F}_q}, K\otimes f^*L_\chi) = R\Gamma_c(\mathbb{G}_{m,\overline{\mathbb{F}_q}}, (Rf_!K)\otimes L_\chi)$. By homogeneity, the cohomology sheaves of $Rf_!K$ are smooth on $\mathbb{G}_{m,\mathbb{F}_q}$. Moreover they are tame at 0 and ∞. Hence we get the first equality, and in the Grothendieck group of smooth $\overline{\mathbb{Q}_\ell}$-sheaves on $\mathbb{G}_{m,\overline{\mathbb{F}_q}}$ we have

$$\sum_i (\Leftrightarrow 1)^i R^i f_! K = \sum_{\substack{N\in\mathbf{N}\\(N,q)=1}} \gamma_N V_N.$$

(Indeed each I_0-module canonically determines a smooth $\overline{\mathbb{Q}_\ell}$-sheaf on $\mathbb{G}_{m,\overline{\mathbb{F}_q}}$ which is tame at ∞, cf. [Lau1, 2.2.2].) Thus by (3.1.5) and (3.1.4, (4)) with $\omega = x^{-1}\,\mathrm{d}x$ we get

$$\begin{aligned}\varepsilon_0(U, K\otimes f^*L_\chi) &= \prod_N \varepsilon_0(\mathbb{G}_{m,\mathbb{F}_q}, V_N\otimes L_\chi)^{\gamma_N}\\ &= \prod_N \varepsilon_0(\mathbb{G}_{m,\mathbb{F}_q}, L_{\chi^N})^{\gamma_N} = \prod_{\chi^N=1} q^{-\gamma_N},\end{aligned}$$

which yields the Lemma. □

3.4.4. LEMMA. *Assume the hypothesis of Lemma* (3.4.3) *and let* $0\neq v^\vee\in V^\vee(\mathbb{F}_q)$. *Then*

$$\begin{aligned}&\det(\mathrm{Frob}_q, \mathcal{F}_\psi(K\otimes j_!f^*L_\chi[\Leftrightarrow n])_{v^\vee})\\ &\quad= \varepsilon_0(U\cap H(v^\vee), K\otimes f^*L_\chi)q^{\rho\delta_\chi+\sum_{\chi^N=1}\gamma_N}\cdot G(\nu\chi^d,\psi)^\rho,\end{aligned}$$

where $\rho := \mathrm{rank}\,\mathcal{F}_\psi(K\otimes j_!f^*L_\chi[\Leftrightarrow n])_{v^\vee}$ *does not depend on* χ, *and* $\delta_\chi = 1$ *if* $\nu\chi^d = 1$ *and zero otherwise.*

Proof. Note that $K\otimes j_!f^*L_\chi$ is $\nu\chi^d$-homogeneous. Taking the stalk at $v^\vee$ of the triangle in Lemma (3.3.7) with K replaced by $K\otimes j_!f^*L_\chi$ we get a distinguished triangle

$$\begin{aligned}&R\Gamma_c(H(v^\vee)\otimes\overline{\mathbb{F}_q}, K\otimes j_!f^*L_\chi)(\Leftrightarrow 1)[\Leftrightarrow 2]\\ &\quad\to R\Gamma_c(U\otimes\overline{\mathbb{F}_q}, K\otimes f^*L_\chi)\\ &\quad\to \mathcal{F}_\psi(K\otimes j_!f^*L_\chi[\Leftrightarrow n])_{v^\vee}\otimes\tau(\bar\nu\bar\chi^d,\bar\psi)\overset{+1}{\Leftrightarrow}.\end{aligned}\tag{1}$$

Thus by (3.4.3) we have

$$\mathrm{rank}\,R\Gamma_c(H(v^\vee)\otimes\overline{\mathbb{F}_q}, K\otimes j_!f^*L_\chi) = \Leftrightarrow\rho,\tag{2}$$

and

$$\begin{aligned}&\varepsilon_0(U, K\otimes f^*L_\chi)\\ &\quad= \varepsilon_0(U\cap H(v^\vee), K\otimes f^*L_\chi)q^\rho\\ &\qquad\times\varepsilon_0(\{v^\vee\}, \mathcal{F}_\psi(K\otimes j_!f^*L_\chi[\Leftrightarrow n])_{v^\vee})(\Leftrightarrow G(\bar\nu\bar\chi^d,\bar\psi))^{-\rho}.\end{aligned}\tag{3}$$

The Lemma follows now from (3.4.3) and the formula

$$q/G(\bar{\nu}\bar{\chi}^d, \bar{\psi}) = G(\nu\chi^d, \psi)q^{\delta_\chi}.$$

In fact, we can see that ρ does not depend on χ, by applying the formula of Grothendieck–Ogg–Šafarevič to

$$R\Gamma_c(H(v^\vee) \otimes \overline{\mathbb{F}_q}, K \otimes j_! f^* L_\chi) = R\Gamma_c(\mathbb{G}_m \otimes \overline{\mathbb{F}_q}, R(f|_{H(v^\vee)})_! K \otimes L_\chi)$$

(cf. [Lau1, 3.1.5.3]). □

3.4.5. PROPOSITION. *Assume the notation of* (3.4.1) *and* (3.4.2), *and let* $0 \neq v^\vee \in V^\vee(\mathbb{F}_q)$. *Suppose that* $R^i(f|_{H(v^\vee)})_! K$ *and* $R^i f_! K$ *have both tame ramification at* 0 *and at* ∞, *for all* i, *and that*

$$[(R(f|_{H(v^\vee)})_! K)_{\bar{\eta}_0}] = \sum_{\substack{N \in \mathbf{N} \\ (N,q)=1}} \alpha_N [V_N], \tag{1}$$

$$[(R(f|_{H(v^\vee)})_! K)_{\bar{\eta}_\infty}] = \sum_{\substack{N \in \mathbf{N} \\ (N,q)=1}} \beta_N [V_N], \tag{2}$$

$$[(Rf_! K)_{\bar{\eta}_0}] = \sum_{\substack{N \in \mathbf{N}_0 \\ (N,q)=1}} \gamma_N [V_N], \tag{3}$$

where the integers $\alpha_N, \beta_N, \gamma_N$ *are zero for almost all* N. *Put* $\rho = \operatorname{rank} \mathcal{F}_\psi(K \otimes j_! f^* L_\chi[\Leftrightarrow n])_{v^\vee}$, *and*

$$a^\vee(v^\vee) = \prod_{s \in \overline{\mathbb{F}_q}^\times} s^{a_s(R(f|_{H(v^\vee)})_! K)} \in \mathbb{F}_q^\times. \tag{4}$$

Then ρ *does not depend on* χ, *and*

$$\begin{aligned} &\det(\mathrm{Frob}_q, \mathcal{F}_\psi(K \otimes j_! f^* L_\chi[\Leftrightarrow n])_{v^\vee}) \\ &\quad = (\Leftrightarrow 1)^{\sum_N (\alpha_N - \beta_N)} q^{\sum_N \beta_N} \varepsilon_0(U \cap H(v^\vee), K) \\ &\qquad \times q^{\rho\delta_\chi + \sum_{\chi^N = 1}(\gamma_N - \beta_N)} \chi\left(a^\vee(v^\vee) \prod_{\substack{N \in \mathbf{N} \\ (N,q)=1}} N^{-N(\alpha_N - \beta_N)} \right) \\ &\qquad \times G(\nu\chi^d, \psi)^\rho \prod_{\substack{N \in \mathbf{N} \\ (N,q)=1}} G(\chi^N, \psi)^{\alpha_N - \beta_N}, \end{aligned} \tag{5}$$

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where $\delta_\chi = 1$ if $\nu\chi^d = 1$ and zero otherwise.

Suppose moreover that j^*K is pure of weight zero, and that $\mathcal{F}_\psi(K \otimes j_! f^* L_\chi[\Leftrightarrow n])$ is locally at $v^\vee$ a smooth ℓ-adic sheaf shifted to degree m. Then for any embedding of $\overline{\mathbb{Q}_\ell}$ into $\mathbb{C}$ we have

$$|\varepsilon_0(U \cap H(v^\vee), K)| = \sqrt{q}^{\rho(m-1) - \sum_N (\alpha_N + \beta_N)}. \tag{6}$$

If in addition $\rho = \pm 1$, then $\rho = (\Leftrightarrow 1)^m$, and $(\Leftrightarrow 1)^m (S_K^\vee(\chi, v^\vee))^{(-1)^m}$ equals the right-hand side of (5).

REMARK. Unlike $a^\vee(v^\vee)$, the integers α_N and β_N (if they exist) are constant for $v^\vee$ in a suitable Zariski dense subset of $V^\vee(\overline{\mathbb{F}_q})$, when all data are obtained by reduction mod p from data over $\mathbb{Q}$ for $p \gg 0$. Indeed $(R^i(f|_{H(v^\vee)})_! K)_s$ is the stalk at $(s, v^\vee)$ of a constructible $\overline{\mathbb{Q}_\ell}$-sheaf on $\mathbb{A}^1_{\mathbb{F}_q} \times V^\vee$. Hence outside $\{0\} \times V^\vee$, the restriction of this sheaf to a suitable étale neighbourhood in $\mathbb{A}^1_{\mathbb{F}_q} \times V^\vee$ of a suitable point $(0, v_0^\vee)$ is smooth and equal to the pullback of a sheaf on $\mathbb{A}^1_{\mathbb{F}_q}$ (because of tame ramification at $(0, v_0^\vee)$ when $p \gg 0$). However this remark will not be used in the sequel.

Proof. The formula (5) follows directly from Lemma (3.4.4) and Proposition (3.2.1) with K replaced by $R(f|_{H(v^\vee)})_! K$. Clearly $j^*K \otimes f^* L_\chi$ is pure of weight 0. Moreover $j_!(j^*K \otimes f^* L_\chi) = Rj_*(j^*K \otimes f^* L_\chi)$ for sufficiently general χ. Indeed this follows from [KatLau, 6.5 and 6.5.2] applied to the direct images of the cohomology sheaves of j^*K under the map

$$U \to V \times \mathbb{G}_{m,\mathbb{F}_q} : x \mapsto (x, f(x)).$$

Thus for these $\chi, K \otimes j_! f^* L_\chi$ is pure of weight zero and hence also $\mathcal{F}_\psi(K \otimes j_! f^* L_\chi[\Leftrightarrow n])$, by [KatLau, 2.2.1]. We conclude that the absolute value of the left hand side of (5) equals $\sqrt{q}^{\rho m}$, when χ is sufficiently general. This yields (6), replacing if necessary $\mathbb{F}_q$ by a finite extension to have a general enough χ available. □

3.5. Application to prehomogeneous vector spaces

3.5.1. Assume the notation of (1.1) and (1.2), with all objects defined over $\mathbb{Q}$. In particular (G, ρ, V) is a prehomogeneous vector space defined over $\mathbb{Q}$ and $f \in \mathbb{Q}[V], f^\vee \in \mathbb{Q}[V^\vee]$ are corresponding relative invariants of (G, ρ, V) and its dual. Moreover $\Omega = V \backslash f^{-1}(0), \Omega^\vee = V^\vee \backslash f^{\vee -1}(0), O_1$, resp. $O_1^\vee$, is the closed G-orbit in Ω, resp. $\Omega^\vee$ and $n = \dim V, m = \dim O_1$.

If $p = \mathrm{char}(\mathbb{F}_q) \gg 0$, we denote by the subscript $\mathbb{F}_q$ the result of applying reduction mod p and base change $\mathbb{F}_p \to \mathbb{F}_q$, e.g. $V_{\mathbb{F}_q}$. If there is no fear for confusion we may omit the subscripts, e.g. $\Omega(\mathbb{F}_q)$ instead of $\Omega_{\mathbb{F}_q}(\mathbb{F}_q)$, or even Ω instead of $\Omega_{\mathbb{F}_q}$, etc.

In (2.3) and (1.2) we introduced the immersions $j\colon\Omega\to V$, $i\colon O_1\to\Omega$, $j^\vee\colon\Omega^\vee\to V^\vee$, $i^\vee\colon O_1^\vee\to\Omega^\vee$, and the map $F\colon\Omega\to O_1^\vee$.

3.5.2. Let $L(\omega^\vee)$ denote the smooth rank one $\overline{\mathbb{Q}_\ell}$-sheaf on $(O_1^\vee)_{\mathbb{F}_q}$ constructed in the same way as the sheaf $L(\omega^\vee)$ in (2.3), after reduction mod p. Moreover we define the sheaf $L(\omega)$ on $(O_1)_{\mathbb{F}_q}$ similarly, replacing each object by its dual (e.g. V by $V^\vee$, etc.).

3.5.3. THEOREM. *Assume the above notations. If the characteristic of* $\mathbb{F}_q$ *is sufficiently large, then we have for all* $\chi\in\operatorname{Hom}(\mathbb{F}_q^\times,\overline{\mathbb{Q}_\ell}^\times)$ *that*

$$\mathcal{F}_\psi(Rj_*f^*L_\chi[n])\cong j_!^\vee i_*^\vee(f^{\vee *}L_{\chi^{-1}}\otimes L(\omega^\vee))[m]\quad \textit{on } V_{\mathbb{F}_q}^\vee\otimes\overline{\mathbb{F}_q},\tag{1}$$

$$\mathcal{F}_\psi(j_!f^*L_\chi[n])\cong Rj_*^\vee i_*^\vee(f^{\vee *}L_{\chi^{-1}}\otimes L(\omega^\vee))[m]\quad \textit{on } V_{\mathbb{F}_q}^\vee\otimes\overline{\mathbb{F}_q},\tag{2}$$

$$\mathcal{F}_\psi(Rj_*^\vee i_*^\vee f^{\vee *}L_{\chi^{-1}}[m])\cong j_!(f^*L_\chi\otimes F^*L(\omega^\vee))[n]\quad \textit{on } V_{\mathbb{F}_q}\otimes\overline{\mathbb{F}_q},\tag{3}$$

$$\mathcal{F}_\psi(j_!^\vee i_*^\vee f^{\vee *}L_{\chi^{-1}}[m])\cong Rj_*(f^*L_\chi\otimes F^*L(\omega^\vee))[n]\quad \textit{on } V_{\mathbb{F}_q}\otimes\overline{\mathbb{F}_q},\tag{4}$$

where $\mathcal{F}_\psi$ *and* L_χ *are as in* (3.1.2) *and* (3.3.3). (Note that the above isomorphisms do not have to preserve the Frobenius action.)

Proof. In principle we want to obtain this theorem from (2.4) by reduction mod p. But since the definition of the Sato–Fourier transformation involves the halfspace, we cannot consider its reduction mod p. Also since the definition of the Deligne–Fourier transformation involves the Artin–Schreier sheaf L_ψ, we cannot obtain it as a result of reduction modulo p. We avoid these difficulties as follows: Since $j_!f^*L_\chi$ (resp. $Rj_*f^*L_\chi$, etc.) is χ^d-homogeneous, its Deligne–Fourier transform equals its Radon transform by Lemma (3.3.9). Thus it is enough to prove the

$$\text{‘Radon version on } (V_{\overline{\mathbb{F}_q}})_{\text{et}} \quad\text{and}\quad (V_{\overline{\mathbb{F}_q}}^\vee)_{\text{et}}\text{’}\tag{A}$$

of the above statement, where $(\ )_{\text{et}}$ denotes the étale site. Since the Radon transformation is compatible with reduction mod p for almost all p, it is enough to prove the

$$\text{‘Radon version on } (V_{\mathbb{C}})_{\text{et}} \quad\text{and}\quad (V_{\mathbb{C}}^\vee)_{\text{et}}\text{’.}\tag{B}$$

More precisely we obtain (A) from (B) as follows. Let $L_{\mathbb{C}}$ (resp. $R_{\mathbb{C}}$) be the left (resp. right)-hand side of the relevant equality of (B), and put $X_{\mathbb{C}}=V_{\mathbb{C}}^\vee$ (resp. $X_{\mathbb{C}}=V_{\mathbb{C}}$) if we consider (1) or (2) (resp. (3) or (4)). Consider the similar construction for S-schemes where $S=\operatorname{Spec}\mathbb{Z}[N^{-1}]$ with sufficiently divisible $N\geqslant 1$, and indicate the result by the suffix S. In particular we obtain in this way L_S (resp. R_S) in $D_c^b(X_S,\overline{\mathbb{Q}_l})$. Moreover we indicate the result of base change $S'\to S$ by replacing the suffix S by S'. Let $a_S\colon X_S\to S$ be the natural morphism. Note that

L_S and R_S are reflexive for sufficiently divisible N [KatLau, (3.2)]. By [KatLau, Sect. 3], enlarging N we may assume that $\mathcal{H}(\bullet,\bullet')_S := R^0(a_S)_*R\,\underline{\mathrm{Hom}}(\bullet_S,\bullet'_S)$ is smooth for all choices of $\bullet,\bullet' \in \{L,R\}$, and that $\mathcal{H}(\bullet,\bullet')_{S'} := b^*\mathcal{H}(\bullet,\bullet')_S = R^0(a_{S'})_*R\,\underline{\mathrm{Hom}}(\bullet_{S'},\bullet'_{S'})$ for any base change $b\colon S' \to S$ with S' 'bon' in the sense of [KatLau, (1.0)]. (Note that, for reflexive objects, $R\,\underline{\mathrm{Hom}}$ can be expressed in terms of $\otimes^{\mathbb{L}}$ and the relative dualizing functor over S, cf. [KatLau, (1.1.1)].) For a closed point $x \in S$ and a geometric point $\overline{x} \to x$, let $S_{\overline{x}}$ be the strict Henselization. Put $\overline{\eta} := \mathrm{Spec}(\mathbb{C})$, and lift $\overline{\eta} \to S$ to $\overline{\eta} \to S_{\overline{x}}$. Then we get $\mathcal{H}(L,R)_{\overline{\eta}} \overset{\cong}{\longleftarrow} \Gamma(S_{\overline{x}}, \mathcal{H}(L,R)_{S_{\overline{x}}}) \overset{\cong}{\longrightarrow} \mathcal{H}(L,R)_{\overline{x}}$ because $\mathcal{H}(L,R)_S$ is smooth. Moreover we have the following commutative diagram of natural maps

$$
\begin{array}{ccccc}
\mathrm{Hom}_{D^b_c(X_{\mathbb{C}},\overline{\mathbb{Q}_l})}(L_{\mathbb{C}},R_{\mathbb{C}}) & = & R^0\Gamma(X_{\mathbb{C}}, R\,\underline{\mathrm{Hom}}(L_{\mathbb{C}},R_{\mathbb{C}})) & = & \mathcal{H}(L,R)_{\overline{\eta}} \\
\uparrow & & \uparrow & & \uparrow\cong \\
\mathrm{Hom}_{D^b_c(X_{S_{\overline{x}}},\overline{\mathbb{Q}_l})}(L_{S_{\overline{x}}},R_{S_{\overline{x}}}) & = & R^0\Gamma(X_{S_{\overline{x}}}, R\,\underline{\mathrm{Hom}}(L_{S_{\overline{x}}},R_{S_{\overline{x}}})) & = & \Gamma(S_{\overline{x}}, \mathcal{H}(L,R)_{S_{\overline{x}}}) \\
\downarrow & & \downarrow & & \downarrow\cong \\
\mathrm{Hom}_{D^b_c(X_{\overline{x}},\overline{\mathbb{Q}_l})}(L_{\overline{x}},R_{\overline{x}}) & = & R^0\Gamma(X_{\overline{x}}, R\,\underline{\mathrm{Hom}}(L_{\overline{x}},R_{\overline{x}})) & = & \mathcal{H}(L,R)_{\overline{x}}
\end{array}
.
$$

To understand the last equality in the second row of this diagram, note that the functor $\Gamma(S_{\overline{x}}, -)$ is exact since $S_{\overline{x}}$ is strictly Henselian. Hence the isomorphism given by (B) induces the desired isomorphism $L_{\overline{x}} \cong R_{\overline{x}}$. Thus we get (A). Note that the above argument gives a uniform upper bound, independently of χ, for the set of 'bad primes'.

Next by the comparison theorem for the classical topology and the étale topology (cf. [BeiBerDel, 6.1.2]), we can reduce the proof to the

$$\text{'Radon version on } V(\mathbb{C})_{cl} \quad \text{and} \quad V^\vee(\mathbb{C})_{cl}\text{'}, \tag{C}$$

where $(\ )_{cl}$ denotes the classical site. Now we adapt the proof of Lemma (3.3.9) to the setting of the Sato–Fourier transform. Then the proof reduces to the

$$\text{'Fourier version on } V(\mathbb{C})_{cl} \quad \text{and} \quad V^\vee(\mathbb{C})_{cl}\text{'}, \tag{D}$$

which is nothing but (2.4). The adaptation goes as follows:

In (3.3.4), define 'α-homogeneity' ($\alpha \in \mathbb{C}$) replacing L_χ by the locally constant sheaf $L_\alpha := \mathbb{C}t^\alpha$ on $(\mathbb{C}^\times)_{cl}$.

In (3.3.5), replace $(\mathbb{G}_{m,\mathbb{F}_q})_{et}$ by $(\mathbb{C}^\times)_{cl}$; L_χ by L_α; and L_ψ by L_-. Here we use the notation of (2.4). Then we get

$$
\begin{aligned}
\tau(\alpha, L_-) &:= R\Gamma_c(\mathbb{C}^\times, L_\alpha \otimes L_-)[1] \\
&= R\Gamma_c(\mathbb{C}^\times \cap Z, L_\alpha)[1] \cong R\Gamma_c(\mathbb{C}^\times \cap Z, \mathbb{C})[1] \\
&= R\Gamma_c([-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi] \times \mathbb{R}_{>0}, \mathbb{C})[1] = R\Gamma_c(\mathbb{R}_{>0}, \mathbb{C})[1] \cong \mathbb{C}.
\end{aligned}
$$

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In (3.3.6)–(3.3.9), the necessary adaptation will be obvious once we note that

$$\begin{aligned} R\Gamma_c(\mathbb{C}, L_-) &\cong R\Gamma_c(Z, \mathbb{C}) = R\Gamma_c(\mathbb{R} \times \mathbb{R}_{\geqslant 0}, \mathbb{C}) \\ &\cong R\Gamma_c(\mathbb{R}_{\geqslant 0}, \mathbb{C})[-1] = 0. \end{aligned}$$ □

In Section 5 we will need the following

3.5.4. LEMMA. *Assume the above notations and let* $v^\vee \in O_1^\vee(\mathbb{F}_q)$. *Then*

$$\mathrm{tr}(\mathrm{Frob}_q, L(\omega^\vee)_{v^\vee}) = \chi_{1/2}(h^\vee(v^\vee)),$$

where $\chi_{1/2}$ *and* $h^\vee$ *are as in* (1.5).

Proof. This follows directly from the definitions by a straightforward calculation (see also (6.1.4, (4))). □

4. The Aomoto complex and Bernstein polynomials

4.1. Determinant of the Aomoto complex

4.1.1. Let X be a smooth quasi-projective algebraic variety over $\mathbb{C}$ of pure dimension dim X. Denote by $\mathcal{O}_X$, resp. $\mathcal{D}_X$, the sheaf of regular rational functions, resp. algebraic differential operators, on X, and by $\Omega^\cdot_X$ the complex of sheaves of regular rational differential forms on X. Let $D_h^b(\mathcal{D}_X)$ be the derived category of bounded complexes of quasi-coherent $\mathcal{D}_X$-modules with holonomic cohomology, and $D_c^b(X(\mathbb{C}), \mathbb{C})$ the derived category of bounded complexes of sheaves of $\mathbb{C}$-vector spaces on $X(\mathbb{C})$ with (algebraically) constructible cohomology. We denote by DR the de Rham functor $DR: D_h^b(\mathcal{D}_X) \to D_c^b(X(\mathbb{C}), \mathbb{C})$ appearing in the Riemann–Hilbert correspondence, so normalized that $DR(\mathcal{O}_X) = \mathbb{C}[\dim X]$, cf. [Bor1, Chap. VIII]. Note that this normalization differs form [Gyo1] and [Gyo2] by a shift.

4.1.2. Let $f: X \to \mathbb{G}_{m,\mathbb{C}}$ be a morphism, $\mathcal{M}$ a holonomic $\mathcal{D}_X$-module and $\mathcal{M}^\cdot \in D_h^b(\mathcal{D}_X)$. Denote by $\mathcal{M}f^s$ the holonomic $\mathbb{C}(s) \otimes_\mathbb{C} \mathcal{D}_X$-module $\mathbb{C}(s) \otimes_\mathbb{C} \mathcal{M}$ where the action of $\mathcal{D}_X$ is twisted by f^s in the obvious way, i.e.

$$\frac{\partial}{\partial x_i}(\varphi(s) \otimes m) = \varphi(s) \otimes \frac{\partial m}{\partial x_i} + s\varphi(s) \otimes f^{-1}\frac{\partial f}{\partial x_i} m, \tag{1}$$

when $\varphi(s) \in \mathbb{C}(s), m \in \mathcal{M}$. For ease of notation we will denote $\varphi(s) \otimes m$ by $\varphi(s) m f^s$. Similarly we define $\mathcal{M}^\cdot f^s \in D_h^b(\mathcal{D}_{X \otimes \mathbb{C}(s)})$ and $\mathcal{M}^\cdot f^\alpha \in D_h^b(\mathcal{D}_X)$ for $\alpha \in \mathbb{C}$, in the obvious way.

Let

$$p_X: X \times \mathrm{Spec}(\mathbb{C}(s)) \to \mathrm{Spec}(\mathbb{C}(s))$$

be the natural projection.

The Aomoto complex $\mathcal{A}_f(\mathcal{M}^{\cdot})$ is the complex of $\mathbb{C}(s)$-vector spaces defined [And], [LoeSab] by

$$\mathcal{A}_f(\mathcal{M}^{\cdot}) = (p_X)_+\mathcal{M}^{\cdot} f^s, \tag{2}$$

where $(p_X)_+$ denotes as usual the direct image under p_X, cf. [Bor1, Chap. VI]. This complex has finite dimensional cohomology (over $\mathbb{C}(s)$) (cf. [Bor1, VII 10.1]), and is quasi-isomorphic to the complex

$$R\Gamma(X \otimes \mathbb{C}(s), \Omega^{\cdot}_X \otimes_{\mathcal{O}_X} \mathcal{M}^{\cdot} f^s[\dim X]), \tag{3}$$

see [Bor1, VI 5.3.2]. Here $\Omega^{\cdot}_X \otimes_{\mathcal{O}_X} \mathcal{M}^{\cdot} f^s$ denotes the algebraic de Rham complex of $\mathcal{M}^{\cdot} f^s$; its differential is the one of $\Omega^{\cdot}_X$ twisted by $\mathcal{M}^{\cdot} f^s$. In particular when X is affine (3) implies that $\mathcal{A}_f(\mathcal{M}^{\cdot})$ is quasi-isomorphic to

$$\Gamma(X \otimes \mathbb{C}(s), \Omega^{\cdot}_X \otimes_{\mathcal{O}_X} \mathcal{M}^{\cdot} f^s)[\dim X]. \tag{4}$$

The translation $\tau\colon s \mapsto s+1$ acts on $\mathcal{M}^{\cdot} f^s$ by

$$\tau(\varphi(s) m f^s) = \varphi(s+1)(fm) f^s.$$

This action is linear over $\mathbb{C}$, but only semi-linear over $\mathbb{C}(s)$. It induces an action on the Aomoto complex $\mathcal{A}_f(\mathcal{M}^{\cdot})$ via its action on $\Omega^{\cdot}_X \otimes_{\mathcal{O}_X} \mathcal{M}^{\cdot} f^s$ (which commutes with the differential), cf. [LoeSab, Sect. 1.3]. Put

$$\det(\tau, \mathcal{A}_f(\mathcal{M}^{\cdot})) := \prod_i \det(\tau, H^i(\mathcal{A}_f(\mathcal{M}^{\cdot})))^{(-1)^i}.$$

Note however that $\det(\tau, \mathcal{A}_f(\mathcal{M}^{\cdot}))$ is only defined up to a factor $h(s+1)/h(s)$, with $h(s) \in \mathbb{C}(s)^{\times}$, because τ is only semi-linear over $\mathbb{C}(s)$, see [LoeSab].

4.1.3. We use the notation of (3.1.1), with $k = \mathbb{C}$. Let $K \in D^b_c(X, \overline{\mathbb{Q}_\ell})$ and suppose that we can write

$$[(Rf_!K)_{\bar{\eta}_0}] = \sum_{N \in \mathbf{N}_0} \alpha_N [V_N] \quad \text{and} \quad [(Rf_!K)_{\bar{\eta}_\infty}] = \sum_{N \in \mathbf{N}_0} \beta_N [V_N],$$

where $\alpha_N, \beta_N \in \mathbb{Z}$. This is for example the case when the pull back of K to a suitable degree 2 cover of X is a geometrically constant sheaf. Indeed the eigenvalues of the monodromy action are roots of unity which are permuted by Galois conjugation, since we can work in this case with $\mathbb{Q}$ instead of $\overline{\mathbb{Q}_\ell}$.

After choosing an embedding of $\overline{\mathbb{Q}_\ell}$ into $\mathbb{C}$, K determines a complex $K^a \in D^b_c(X(\mathbb{C}), \mathbb{C})$ and hence via the Riemann–Hilbert correspondence a regular holonomic $\mathcal{K}^{\cdot} \in D^b_h(\mathcal{D}_X)$ with $DR(\mathcal{K}^{\cdot}) = K^a$. The next theorem is a direct consequence of a result of Anderson [And] and Loeser and Sabbah [LoeSab]. (Indeed

$[(Rf_*K)_{\bar\eta_s}] = [(Rf_!K)_{\bar\eta_s}]$ for $s \in |\mathbb{P}^1_{\mathbb{C}}|$, by [Lau2].) It can be used to determine the $\alpha_N \Leftrightarrow \beta_N$ in terms of det $(\tau, \mathcal{A}_f(\mathcal{K}^\cdot))$.

4.1.4. THEOREM ([And], [LoeSab]). *Assume the notation of* (4.1.2) *and* (4.1.3). *Then we have*

$$\det(\tau, \mathcal{A}_f(\mathcal{K}^\cdot))^{-1} = \frac{h(s+1)}{h(s)} c \prod_{N \in \mathbf{N}_0} \prod_{k=1}^{N} \left(s \Leftrightarrow \frac{k}{N}\right)^{\alpha_N - \beta_N} \tag{1}$$

with $h(s) \in \mathbb{C}(s)^\times$ *and*

$$c = \prod_{t \in \mathbb{C}^\times} t^{a_t(Rf_!K)}. \tag{2}$$

See (3.1.1) for the definition of a_t. In [LoeSab] one assumes that X is affine, but this is not necessary. Note that the right-hand side of (1) completely determines the $\alpha_N \Leftrightarrow \beta_N$ and c.

4.1.5. LEMMA. *Let* $\mathcal{M}^\cdot \in D^b_h(\mathcal{D}_X)$ *be regular holonomic. Then for all but countably many* $\alpha \in \mathbb{C}$ *we have for any* $i \in \mathbb{Z}$

$$\dim_{\mathbb{C}(s)} H^i(\mathcal{A}_f(\mathcal{M}^\cdot)) = \dim_{\mathbb{C}} H^i(X, DR(\mathcal{M}^\cdot) \otimes \mathbb{C}f^\alpha). \tag{1}$$

Proof. There exists a countable algebraically closed subfield k of $\mathbb{C}$ such that X, f and $\mathcal{M}^\cdot$ are obtained by base change from a variety X_k over k, a morphism $f_k\colon X_k \to \mathbb{G}_{m,k}$ and a complex $\mathcal{M}^\cdot_k$ of $\mathcal{D}_{X_k}$-modules. Let $\alpha \in \mathbb{C}$ be transcendental over k. Then we have

$$\begin{aligned}
&\dim_{\mathbb{C}(s)} H^i(\mathcal{A}_f(\mathcal{M}^\cdot))\\
&\quad = \dim_{\mathbb{C}(s)} H^i(X \otimes \mathbb{C}(s), \Omega^\cdot_X \otimes_{\mathcal{O}_X} \mathcal{M}^\cdot f^s[\dim X]), \text{by } (4.1.2,\ (3))\\
&\quad = \dim_{k(s)} H^i(X_k \otimes k(s), \Omega^\cdot_{X_k} \otimes_{\mathcal{O}_{X_k}} \mathcal{M}^\cdot_k f^s_k[\dim X])\\
&\quad = \dim_{\mathbb{C}} H^i(X, \Omega^\cdot_X \otimes_{\mathcal{O}_X} \mathcal{M}^\cdot f^\alpha[\dim X]), \text{by considering } k(s) \to \mathbb{C}\colon s \mapsto \alpha\\
&\quad = \dim_{\mathbb{C}} H^i(\pi_+(\mathcal{M}^\cdot f^\alpha)) \text{where } \pi \text{ is the projection } X \to \operatorname{Spec}(\mathbb{C}).
\end{aligned}$$

This implies the Lemma because the Riemann–Hilbert correspondence yields

$$\begin{aligned}
\pi_+(\mathcal{M}^\cdot f^\alpha) &= DR(\pi_+(\mathcal{M}^\cdot f^\alpha)) = R\pi_*(DR(\mathcal{M}^\cdot f^\alpha))\\
&= R\Gamma(X, DR(\mathcal{M}^\cdot) \otimes \mathbb{C}f^{-\alpha}).
\end{aligned}$$

□

REMARK. The above Lemma appears implicitly in [LoeSab, p. 471] with a different proof yielding the stronger result that (1) holds for α in the complement of the set of all integer translates of a suitable finite subset of $\mathbb{C}$.

4.1.6. Let $g\colon X\to\mathbb{A}^1_{\mathbb{C}}$ be a morphism, $\mathcal{M}$ a holonomic $\mathcal{D}_X$-module, and $\mathcal{M}^{\cdot}\in D^b_h(\mathcal{D}_X)$. Denote by $\mathcal{M}e^g$ the holonomic $\mathcal{D}_X$-module obtained from $\mathcal{M}$ by twisting the action of $\mathcal{D}_X$ by e^g in the obvious way, i.e.

$$\frac{\partial}{\partial x_i}me^g=\left(\frac{\partial m}{\partial x_i}\right)e^g+m\frac{\partial g}{\partial x_i}e^g.$$

Similarly we define $\mathcal{M}^{\cdot}e^g\in D^b_h(\mathcal{D}_X)$.

4.1.7. Assume the notation of (4.1.2) and (4.1.6). Let Y be a smooth quasi-projective algebraic variety over $\mathbb{C}$, and $\pi\colon Y\to X$ a morphism. Then

$$\pi^!(\mathcal{M}^{\cdot}e^gf^s)=\pi^!(\mathcal{M}^{\cdot})e^{g\circ\pi}(f\circ\pi)^s, \tag{1}$$

$$\mathbb{D}(\mathcal{M}^{\cdot}e^gf^s)=\mathbb{D}(\mathcal{M}^{\cdot})e^{-g}(f^{-1})^s, \tag{2}$$

where $\mathbb{D}$ denotes the duality functor, and

$$\pi^+(\mathcal{M}^{\cdot}e^gf^s)=\pi^+(\mathcal{M}^{\cdot})e^{g\circ\pi}(f\circ\pi)^s. \tag{3}$$

Indeed (1) follows directly from the definitions of the concepts involved, because $\mathcal{F}e^gf^s$ is a free $\mathcal{D}_{X\otimes\mathbb{C}(s)}$-module whenever $\mathcal{F}$ is a free $\mathcal{D}_X$-module. Moreover (2) follows from the next lemma with X (resp. $\mathcal{F}$) replaced by $X\otimes\mathbb{C}(s)$ (resp. $\mathcal{O}_{X\otimes\mathbb{C}(s)}e^gf^s$), and (3) follows directly from (1) and (2), since $\pi^+=\mathbb{D}\circ\pi^!\circ\mathbb{D}$.

4.1.8. LEMMA. *For a smooth quasi-projective variety X, let $D^b_{\mathrm{coh}}(\mathcal{D}_X)$ be the bounded derived category of quasi-coherent left $\mathcal{D}_X$-modules whose cohomology sheaves are coherent. Then for a left $\mathcal{D}_X$-module $\mathcal{F}$ which is coherent over $\mathcal{O}_X$, and for $\mathcal{M}^{\cdot}\in D^b_{\mathrm{coh}}(\mathcal{D}_X)$, we have a canonical isomorphism*

$$\Phi\colon\mathbb{D}(\mathcal{M}^{\cdot})\otimes_{\mathcal{O}_X}\mathbb{D}(\mathcal{F})\leftarrow\mathbb{D}(\mathcal{M}^{\cdot}\otimes_{\mathcal{O}_X}\mathcal{F}).$$

Proof. In general, for a morphism $\varphi:Z\to Y$ of smooth quasi-projective varieties, and for $\mathcal{N}^{\cdot}\in D^b_{\mathrm{coh}}(\mathcal{D}_Y)$ with respect to which φ is non-characteristic (cf. [KasSch, 11.2.11]), we have a canonical isomorphism $\Phi:{}^L\varphi^*(\mathbb{D}\mathcal{N}^{\cdot})\leftarrow\mathbb{D}({}^L\varphi^*\mathcal{N}^{\cdot})$. Here φ^* is the usual pull-back of $\mathcal{O}$-modules, and hence ${}^L\varphi^*=\varphi^![\dim Y\Leftrightarrow\dim Z]$. (For the isomorphism Φ, see [SatKasKaw, Theorem 3.5.6, pp. 414–417], replacing $\mathcal{C}_{Z|Y}$ by $R\Gamma_{[Z]}(\mathcal{O}_Y)[\dim Y\Leftrightarrow\dim Z]$, $\mathcal{P}_{Z\to Y}$ by $\mathcal{D}_{Z\to Y}$, $\mathcal{P}_Y$ by $\mathcal{D}_Y$ etc., with obvious adaptation in loc. cit. pp. 406–417.) Then applying this to the diagonal morphism $i:X\to X\times X$ and $\mathcal{N}^{\cdot}=\mathcal{M}^{\cdot}\boxtimes\mathcal{F}$, we get the desired isomorphism by [Bor1, VIII Sect. 21.1, (4), p. 346]. □

4.2. The exponential Aomoto complex

4.2.1. Let $V=\mathbb{A}^n_{\mathbb{C}}, V^\vee$ the dual space of V, and $\langle\ \rangle\colon V^\vee\times V\to\mathbb{A}^1_{\mathbb{C}}$ the natural pairing. For $v^\vee\in V^\vee(\mathbb{C})\backslash\{0\}$, put

$$H(v^\vee)=\{v\in V|\langle v^\vee,v\rangle=1\}.$$

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Choose a basis $\{v_1^\vee, \dots, v_n^\vee\}$ for the vector space $V^\vee$, consider $v_1^\vee, \dots, v_n^\vee$ as linear functions on V and denote them by $x_1, \dots, x_n$. Thus $x_1, \dots, x_n$ are affine coordinates for V.

Let $f: V \to \mathbb{A}^1_{\mathbb{C}}$ be a nonzero polynomial over $\mathbb{C}$. Put $U = V \backslash f^{-1}(0)$, and consider the immersion $j: U \to V$.

4.2.2. For $\alpha \in \mathbb{C}$ and f homogeneous, we say that $\mathcal{M}^\cdot \in D_h^b(\mathcal{D}_U)$ is *α-homogeneous* if $h^+\mathcal{M}^\cdot = \mathcal{O}_{\mathbb{G}_{m,\mathbb{C}}} t^\alpha \boxtimes \mathcal{M}^\cdot[\Leftrightarrow 1]$, where h is the map from $\mathbb{G}_{m,\mathbb{C}} \times U$ to U given by $h(t,x) = tx$. This is equivalent with requiring that $h^!\mathcal{M}^\cdot = \mathcal{O}_{\mathbb{G}_{m,\mathbb{C}}} t^\alpha \boxtimes \mathcal{M}^\cdot[1]$, because h is smooth.

4.2.3. For $d \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$, let $\Gamma_{d,\alpha}$ be the $\mathbb{C}(s)$ vector space generated by the classical gamma function $\Gamma(ds + \alpha + 1)$. The action of the τ operator on Γ_d is given by

$$\tau(\Gamma(ds+\alpha+1)) := (ds+\alpha+d)(ds+\alpha+d \Leftrightarrow 1) \\ \dots (ds+\alpha+1)\Gamma(ds+\alpha+1).$$

Denoting by t^d the map $\mathbb{G}_{m,\mathbb{C}} \to \mathbb{G}_{m,\mathbb{C}} : t \mapsto t^d$, we see that $\mathcal{A}_{t^d}(\mathcal{O}_{\mathbb{G}_{m,\mathbb{C}}} t^\alpha e^{-t})$ is quasi-isomorphic to $\Gamma_{d,\alpha}$ respecting τ. Indeed, this exponential Aomoto complex is represented by the complex $A^\cdot = (\cdots \to 0 \to A^{-1} \to A^0 \to 0 \to \cdots)$ with $A^{-1} = A^0 = \mathbb{C}(s)[t, t^{-1}]t^{ds+\alpha}e^{-t}$, which is quasi-isomorphic to $H^0(A^\cdot)$, and the latter is easily shown to be isomorphic to $\Gamma_{d,\alpha}$ by the usual argument to prove the functional equation $\Gamma(s+1) = s\Gamma(s)$.

4.2.4. PROPOSITION. *Assume the notation of* (4.2.1) *and* (4.2.3), *with f homogeneous of degree d. Suppose that $\mathcal{M}^\cdot \in D_h^b(\mathcal{D}_U)$ is regular holonomic and α-homogeneous for some $\alpha \in \mathbb{C}$. Let $0 \neq v^\vee \in V^\vee(\mathbb{C})$, and denote by γ the immersion $\gamma: U \cap H(v^\vee) \to U$. Then there exists a quasi-isomorphism*

$$\mathcal{A}_{f|_U}(\mathcal{M}^\cdot e^{-v^\vee}) \to \mathcal{A}_{f|_{U \cap H(v^\vee)}}(\gamma^+\mathcal{M}^\cdot[\Leftrightarrow 1]) \otimes_{\mathbb{C}(s)} \Gamma_{d,\alpha}, \tag{1}$$

which respects the τ action. Moreover for all but countably many $\beta \in \mathbb{C}$ we have for any $i \in \mathbb{Z}$

$$\dim_{\mathbb{C}(s)} H^i(\mathcal{A}_{f|_U}(\mathcal{M}^\cdot e^{-v^\vee})) \\ = \dim_{\mathbb{C}} H^{-i}(\mathcal{F}_{\text{geom}}(j_!(\mathbb{D}(DR(\mathcal{M}^\cdot)) \otimes \mathbb{C}f^\beta[\Leftrightarrow n]))_{v^\vee}), \tag{2}$$

where $\mathbb{D}(\Leftrightarrow)$ denotes the Verdier dual and $\mathcal{F}_{\text{geom}}$ the Sato–Fourier transformation.

Proof. We may suppose that $v^\vee = v_1^\vee$ in the notation of (4.2.1). Put

$$Z := \{v \in U | \langle v_1^\vee, v\rangle = 0\} = \{(x_1, \dots, x_n) \in V | x_1 = 0, f(x) \neq 0\}$$

and consider the immersions $a: U\backslash Z \to U$ and $b: Z \to U$. We have an exact triangle [Bor1, VI 8.3]

$$b_+b^!(\mathcal{M}^\cdot e^{-x_1}f^s) \to \mathcal{M}^\cdot e^{-x_1}f^s \to a_+((\mathcal{M}^\cdot e^{-x_1}f^s)|_{U\backslash Z}) \overset{+1}{\Leftrightarrow} .$$

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Let $p_U, p_{U\backslash Z}, p_Z$ be the natural projections of $U \otimes \mathbb{C}(s), (U\backslash Z) \otimes \mathbb{C}(s), Z \otimes \mathbb{C}(s)$ on Spec$(\mathbb{C}(s))$. Applying $(p_U)_+$ to the above triangle we see that in order to prove (1) it suffices to show that

$$(p_Z)_+ b^!(\mathcal{M}^{\cdot} e^{-x_1} f^s) = 0, \quad \text{and} \tag{3}$$

$$(p_{U\backslash Z})_+((\mathcal{M}^{\cdot} e^{-x_1} f^s)|_{U\backslash Z}) = \mathcal{A}_{f|_{U\cap H(v_1^{\vee})}}(\gamma^+\mathcal{M}^{\cdot}[\Leftrightarrow 1]) \otimes_{\mathbb{C}(s)} \Gamma_{d,\alpha}. \tag{4}$$

From (4.1.7) we obtain

$$(p_Z)_+ b^!(\mathcal{M}^{\cdot} e^{-x_1} f^s) = (p_Z)_+((b^!\mathcal{M}^{\cdot})(f|_Z)^s) = \mathcal{A}_{f|_Z}(b^!\mathcal{M}^{\cdot}). \tag{5}$$

Put $K := R(f|_Z)_* DR(b^!\mathcal{M}^{\cdot})$. The α-homogeneity of $\mathcal{M}^{\cdot}$ and the homogeneity of f imply that K is locally constant on $\mathbb{G}_{m,\mathbb{C}}$. Hence for all but countably many $\beta \in \mathbb{C}$

$$R\Gamma(Z, DR(b^!\mathcal{M}^{\cdot}) \otimes \mathbb{C}f^{\beta}) = R\Gamma(\mathbb{G}_{m,\mathbb{C}}, K \otimes \mathbb{C}t^{\beta}) = 0. \tag{6}$$

Together with (5) and Lemma (4.1.5), this implies (3). We now turn to the proof of (4). Consider the isomorphism

$$h_0\colon \mathbb{G}_{m,\mathbb{C}} \times (U \cap H(v_1^{\vee})) \xrightarrow{\cong} U\backslash Z$$
$$: (t, (1, x_2, \ldots, x_n)) \mapsto (t, tx_2, \ldots, tx_n)$$

and the morphism $h\colon \mathbb{G}_{m,\mathbb{C}} \times U \to U\colon (t, x) \mapsto tx$. We have

$$\begin{aligned}
&h_0^+((\mathcal{M}^{\cdot} e^{-x_1} f^s)|_{U\backslash Z}) \\
&\quad = (\mathrm{Id}, \gamma)^+ h^+(\mathcal{M}^{\cdot} e^{-x_1} f^s) \\
&\quad = ((\mathrm{Id}, \gamma)^+ h^+ \mathcal{M}^{\cdot}) e^{-t}(t^d f)^s, \quad \text{by } (4.1.7,\ (3)) \\
&\quad = ((\mathrm{Id}, \gamma)^+(\mathcal{O}_{\mathbb{G}_{m,\mathbb{C}}} t^{\alpha} \boxtimes \mathcal{M}^{\cdot}[\Leftrightarrow 1])) e^{-t}(t^d f)^s \\
&\quad = \mathcal{O}_{\mathbb{G}_{m,\mathbb{C}}} t^{\alpha} e^{-t} t^{ds} \boxtimes (\gamma^+\mathcal{M}^{\cdot}[\Leftrightarrow 1]) f^s.
\end{aligned} \tag{7}$$

Thus since h_0 is an isomorphism we conclude by (4.1.2, (4)) that

$$\begin{aligned}
&\mathcal{A}_{f|_{U\backslash Z}}((\mathcal{M}^{\cdot} e^{-x_1})|_{U\backslash Z}) \\
&\quad = \mathcal{A}_{t^d}(\mathcal{O}_{\mathbb{G}_{m,\mathbb{C}}} t^{\alpha} e^{-t}) \otimes_{\mathbb{C}(s)} \mathcal{A}_{f|_{U\cap H(v^{\vee})}}(\gamma^+\mathcal{M}^{\cdot}[\Leftrightarrow 1]).
\end{aligned}$$

This proves (4) and hence also (1).

We now turn to the proof of (2). Replacing h_0^+, h^+, γ^+ by $h_0^!, h^!, \gamma^!$ in the argument leading to (7), and comparing with (7), we get

$$\gamma^!\mathcal{M}^{\cdot}[1] = \gamma^+\mathcal{M}^{\cdot}[\Leftrightarrow 1]. \tag{8}$$

(This follows also from the α-homogeneity and [KasSch, 5.4.13].)

The reasoning that gave (6) also shows that for all but countably many $\beta \in \mathbb{C}$

$$R\Gamma_c(V, j_!(\mathbb{D}(DR(\mathcal{M}^{\cdot})) \otimes \mathbb{C}f^{\beta})) = 0.$$

Hence the analogue of the triangle (3.3.7) for the Sato–Fourier transform yields

$$\mathcal{F}_{\text{geom}}(j_!(\mathbb{D}(DR(\mathcal{M}^{\cdot})) \otimes \mathbb{C}f^{\beta}))_{v^\vee}[-n]$$
$$\cong R\Gamma_c(U \cap H(v^\vee), \gamma^*(\mathbb{D}(DR(\mathcal{M}^{\cdot})) \otimes \mathbb{C}f^{\beta}))[-1].$$

Applying Verdier duality theorem and taking cohomology we get

$$H^{-i}(\mathcal{F}_{\text{geom}}\, j_!(\mathbb{D}(DR(\mathcal{M}^{\cdot})) \otimes \mathbb{C}f^{\beta}[-n])_{v^\vee})$$
$$\cong (H^i(U \cap H(v^\vee), \gamma^! DR(\mathcal{M}^{\cdot})[1] \otimes \mathbb{C}f^{-\beta}))^\vee.$$

The assertion (2) follows now directly from (1), (8) and Lemma 4.1.5 with $X = U \cap H(v^\vee)$. □

4.2.5. PROPOSITION. *Assume the notation of* (4.2.1). *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_U$-module generated by the single element $w \in \Gamma(U, \mathcal{M})$, i.e. $\mathcal{M} = \mathcal{D}_U w$. Suppose that there exists a polynomial $f^\vee: V^\vee \to \mathbb{A}^1_{\mathbb{C}}$ and $B(s) \in \mathbb{C}(s)^\times$ such that*

$$f^\vee(\text{grad}_x)(wf^{s+1}) = B(s)wf^s \quad \text{in} \quad \Gamma(U, \mathcal{M}f^s). \tag{1}$$

Fix $v^\vee \in V^\vee(\mathbb{C})$ with $f^\vee(v^\vee) \neq 0$. Assume that $\mathcal{A}_{f|_U}(\mathcal{M}e^{-v^\vee})$ is concentrated in degree 0 *and has dimension* 1 *over $\mathbb{C}(s)$. Then*

$$\det(\tau, \mathcal{A}_{f|_U}(\mathcal{M}e^{-v^\vee})) = \frac{h(s+1)}{h(s)} \frac{B(s)}{f^\vee(v^\vee)}, \tag{2}$$

with $h(s) \in \mathbb{C}^\times(s)$.

Note the analogy between $B(s)$ and Bernstein's polynomial.

Proof. Since U is affine we have by (4.1.2, (4)) that

$$\mathcal{A}_{f|_U}(\mathcal{M}e^{-v^\vee}) = \Gamma(U, \Omega^{\cdot}_U \otimes_{\mathcal{O}_U} \mathcal{M}e^{-v^\vee} f^s)[n]. \tag{3}$$

Every global section $\gamma e^{-v^\vee} f^s$ of $\mathcal{M}e^{-v^\vee} f^s$ determines an element $\gamma e^{-v^\vee} f^s dx_1 \wedge \ldots \wedge dx_n$ of $\Gamma(U, \Omega^n_U \otimes_{\mathcal{O}_U} \mathcal{M}e^{-v^\vee} f^s)$. When two such global sections $\gamma e^{-v^\vee} f^s$ and $\gamma' e^{-v^\vee} f^s$ determine elements with the same cohomology class in $H^0(\mathcal{A}_{f|_U}(\mathcal{M} e^{-v^\vee}))$ we will write $\gamma e^{-v^\vee} f^s \sim \gamma' e^{-v^\vee} f^s$.

If $\gamma e^{-v^\vee} f^s \in (\partial/\partial x_i)\mathcal{M}e^{-v^\vee} f^s$ for some i, then $\gamma e^{-v^\vee} f^s \sim 0$. Because of our hypothesis on $\mathcal{A}_{f|_U}(\mathcal{M}e^{-v^\vee})$ it suffices to prove that there exists a global section γ of $\mathcal{M}$ such that $\gamma e^{-v^\vee} f^s \not\sim 0$ and

$$\gamma e^{-v^\vee} f^{s+1} \sim f^\vee(v^\vee)^{-1} B(s) \gamma e^{-v^\vee} f^s. \tag{4}$$

We claim that there exists $g \in \mathbb{C}[x_1, \ldots, x_n]$ such that $gwe^{-v^\vee} f^s \not\sim 0$. Choose such a g with minimal degree. Let $P = f^\vee(\text{grad}_x)$ and P^* the adjoint differential operator of P, i.e. $P^* = f^\vee(\Leftrightarrow\text{grad}_x)$. Then

$$P^*(ge^{-v^\vee}) = f^\vee(v^\vee)ge^{-v^\vee} + he^{-v^\vee},$$

with $h \in \mathbb{C}[x_1, \ldots, x_n]$ having degree smaller than deg g. Thus $hwe^{-v^\vee} f^s \sim 0$, and hence also $hwe^{-v^\vee} f^{s+1} \sim 0$, by formally replacing s by $s+1$. So we obtain

$$\begin{aligned} gwe^{-v^\vee} f^{s+1} &\sim f^\vee(v^\vee)^{-1} w f^{s+1} P^*(ge^{-v^\vee}) \\ &\sim f^\vee(v^\vee)^{-1} P(wf^{s+1}) ge^{-v^\vee} \\ &\sim f^\vee(v^\vee)^{-1} B(s) gwe^{-v^\vee} f^s, \quad \text{by (1)}. \end{aligned}$$

This yields (4) for $\gamma = gw$.

It remains to prove the claim. Because of our hypothesis on $\mathcal{A}_{f|_U}(\mathcal{M}e^{-v^\vee})$, there exists a global section γ of $\mathcal{M}$ such that $\gamma e^{-v^\vee} f^s \not\sim 0$. We can write $\gamma = Rw$, with $R \in \Gamma(U, \mathcal{D}_U)$. Then $\gamma e^{-v^\vee} f^s \sim wR^*(e^{-v^\vee} f^s)$ where R^* is the adjoint differential operator of R. This yields the claim because $R^*(e^{-v^\vee} f^s)$ is a $\mathbb{C}(s)$-linear combination of elements of the form $ge^{-v^\vee} f^{s-k}$ with $g \in \mathbb{C}[x_1, \ldots, x_n], k \in \mathbb{N}$, and because $gwe^{-v^\vee} f^{s-k} \not\sim 0$ implies $gwe^{-v^\vee} f^s \not\sim 0$. □

4.3. Application to prehomogeneous vector spaces

4.3.1. Assume the notation of (1.1), (1.2) and (2.3). In particular (G, ρ, V) is a prehomogeneous vector space and $f, f^\vee$ are corresponding relative invariants of (G, ρ, V) and its dual. Moreover $\Omega = V \backslash f^{-1}(0), \Omega^\vee = V^\vee \backslash f^{\vee-1}(0), O_1$, resp. $O_1^\vee$, is the closed G-orbit in Ω, resp. $\Omega^\vee$, and $n = \dim V, m = \dim O_1$. We also have the immersions $j\colon \Omega \to V, i\colon O_1 \to \Omega, j^\vee\colon \Omega^\vee \to V^\vee, i^\vee\colon O_1^\vee \to \Omega^\vee$. For $v^\vee \in V^\vee(\mathbb{C}), v^\vee \neq 0$, put $H(v^\vee) = \{v \in V | \langle v^\vee, v\rangle = 1\}$. Finally let $L(\omega) = \mathbb{C}\,\omega$ be the sheaf on $O_1(\mathbb{C})$ introduced in (2.3).

4.3.2. The sheaf $L(\omega)$ is homogeneous, meaning that $h^* L(\omega) = \mathbb{C} \boxtimes L(\omega)$, where $h: \mathbb{G}_{m,\mathbb{C}} \times O_1 \to O_1$ is given by $h(t, v) = tv$. Indeed from the homogeneity of f and the definition of ω^2 it follows that $(dt \wedge h^*\omega)^2 = (dt \wedge \pi^*\omega)^2$, where π is the projection onto O_1. This implies the claim. Thus by the Riemann–Hilbert correspondence, if $\mathcal{M}$ is the regular holonomic $\mathcal{D}_\Omega$-module with $DR(\mathcal{M}) = i_* L(\omega)[m]$, then $\mathcal{M}$ is 0-homogeneous in the sense of (4.2.2). Moreover the dual of $\mathcal{M}$ equals $\mathcal{M}$ because the pullback of $L(\omega)$ to a suitable degree 2 cover of O_1 is constant, cf. (2.3).

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4.3.3. THEOREM. *Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_\Omega$-module with $DR(\mathcal{M}) = i_*L(\omega)[m]$. For any $v^\vee \in \Omega^\vee(\mathbb{C})$ we have*

$$\det(\tau, \mathcal{A}_{f|_\Omega}(\mathcal{M}e^{-v^\vee})) = \frac{h(s+1)}{h(s)}(\Leftrightarrow 1)^d \frac{b(\Leftrightarrow s \Leftrightarrow 1)}{f^\vee(v^\vee)},$$

where the Bernstein polynomial b is as in (1.2, (3)), *and $d = \deg f$.*

Proof. For $\alpha \in \mathbb{C}$ let $Df^{\vee\alpha}$ be the $\mathcal{D}_{V^\vee}$-module introduced in [Gyo1, 2.3.1], with V, f replaced by $V^\vee, f^\vee$. (Actually in loc. cit. $Df^{\vee\alpha}$ is a $\Gamma(V^\vee, \mathcal{D}_{V^\vee})$-module, but we consider it here as a $\mathcal{D}_{V^\vee}$-module in the obvious way.) It is regular holonomic [Gyo1, 2.8.6] and generated by a single element which is denoted in [Gyo1, 2.3.1] by $f^{\vee\alpha}$. In particular for $\alpha = 0$, we have the $\mathcal{D}_{V^\vee}$-module $Df^{\vee 0}$ generated by $f^{\vee 0}$ (not to be confused with 1). The Fourier transform $\mathcal{F}(Df^{\vee 0})$ of $Df^{\vee 0}$ is a regular holonomic $\mathcal{D}_V$-module [Gyo1, 3.19], generated by the single element $u := \mathcal{F}(f^{\vee 0})$, cf. [Gyo1, 2.7.1]. By Theorem 3.23 of [Gyo1] and the Riemann–Hilbert correspondence we have $Df^{\vee 0} = j_!^\vee \mathcal{O}_{\Omega^\vee}$ and $\mathcal{F}(Df^{\vee 0}) = j_+\mathcal{M}$. (Indeed $0 \in A_+$, where A_+ is defined in loc. cit. 2.3.6.). Thus $\mathcal{M} = \mathcal{F}(Df^{\vee 0})|_\Omega$ and $\mathcal{M}$ is generated by the restriction of u to Ω which we denote by w. Moreover by [Gyo1, 3.1] we have the functional equation

$$f^\vee(\text{grad}_x)(wf^{s+1}) = (\Leftrightarrow 1)^d b(\Leftrightarrow s \Leftrightarrow 1)wf^s \quad \text{in} \quad \Gamma(\Omega, \mathcal{M}f^s).$$

Applying the Sato–Fourier transformation to both sides of (2.4, (1)) yields

$$\mathcal{F}_{\text{geom}}(j_!(DR(\mathcal{M}) \otimes \mathbb{C}f^\alpha[\Leftrightarrow n])) = Rj_*^\vee \mathbb{C}f^{\vee -\alpha},$$

for any $\alpha \in \mathbb{C}$. Thus Proposition (4.2.4, (2)) and (4.3.2) imply that $\mathcal{A}_{f|_\Omega}(\mathcal{M}e^{-v^\vee})$ is concentrated in degree 0 and has dimension 1 over $\mathbb{C}(s)$. The Theorem follows now from Proposition (4.2.5). □

4.3.4. Assume the notation of (4.3.1) and (3.1.1). Let $0 \neq v^\vee \in V^\vee(\mathbb{C})$. There exist integers α_N, β_N such that

$$[(R(f|_{H(v^\vee)})_! j_! i_* L(\omega))_{\bar\eta_0}] = \sum_{N \in \mathbf{N}_0} \alpha_N [V_N], \quad \text{and} \tag{1}$$

$$[(R(f|_{H(v^\vee)})_! j_! i_* L(\omega))_{\bar\eta_\infty}] = \sum_{N \in \mathbf{N}_0} \beta_N [V_N], \tag{2}$$

because the pullback of $L(\omega)$ to a suitable degree 2 cover of O_1 is constant, cf. (4.1.3) and (2.3). Put

$$a^\vee(v^\vee) = \prod_{t \in \mathbb{C}^\times} t^{a_t(R(f|_{H(v^\vee)})_! j_! i_* L(\omega))}, \tag{3}$$

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where a_t is as in (3.1.1). Unlike $a^\vee(v^\vee)$, the integers α_N and β_N are constant for $v^\vee$ in a dense open subset of $V^\vee(\mathbb{C})$.

Theorem (4.1.4) with $X = \Omega \cap H(v^\vee), K = i_* L(\omega)|_{\Omega\cap H(v^\vee)}$ and f replaced by $f|_{\Omega\cap H(v^\vee)}$, together with Proposition (4.2.4) for $\alpha = 0$ and Lemma (4.3.3), can now be used to compute the $\alpha_N \Leftrightarrow \beta_N$ and $a^\vee(v^\vee)$. This yields the following

4.3.5. PROPOSITION. *Assume the notation of* (4.3.1) *and* (4.3.4). *Suppose that* $v^\vee \in \Omega^\vee(\mathbb{C})$. *Then for any* $N \in \mathbb{N}_0$ *we have*

$$(\Leftrightarrow 1)^m(\alpha_N \Leftrightarrow \beta_N) = \begin{cases} e(N) & \text{if } N \neq d, \\ e(N) \Leftrightarrow 1 & \text{if } N = d, \end{cases}$$

and

$$a^\vee(v^\vee) = \left(b_0 d^{-d} f^\vee(v^\vee)^{-1}\right)^{(-1)^m},$$

where $d = \deg f$ *and* $e(N), b_0$ *are as in* (1.2, (4), (5)).

5. Proof of Theorem A1

From now on we choose a field isomorphism between $\mathbb{C}$ and $\overline{\mathbb{Q}_\ell}$. Then the complex valued characters χ and ψ appearing in Section 1 determine $\overline{\mathbb{Q}_\ell}$-valued characters again denoted by χ, ψ, and we can study the character sums in Section 1 by ℓ-adic methods.

5.1. The monodromy at infinity

5.1.1. Let $V = \mathbb{A}^n_{\mathbb{C}}$, $V^\vee$ the dual space of V, and $\langle\ \rangle : V^\vee \times V \to \mathbb{A}^1_{\mathbb{C}}$ the natural pairing. For $v^\vee \in V^\vee$ put

$$H(v^\vee) := \{v \in V | \langle v^\vee, v\rangle = 1\}, \quad \text{and} \quad H_0(v^\vee) := \{v \in V | \langle v^\vee, v\rangle = 0\}.$$

Let $\alpha \in \mathbb{Q}$, and denote by L_α the sheaf $\mathbb{C}x^\alpha$ on $\mathbb{G}_{m,\mathbb{C}}$ or the corresponding $\overline{\mathbb{Q}_\ell}$-sheaf on $\mathbb{G}_{m,\mathbb{C}}$. We say that $K \in D^b_c(V, \overline{\mathbb{Q}_\ell})$ is *α-homogeneous* if $h^*K \cong L_\alpha \boxtimes K$ where h is the map from $\mathbb{G}_{m,\mathbb{C}} \times V$ to V given by $h(t, v) = tv$.

5.1.2. PROPOSITION. *Let* $f : V \to \mathbb{A}^1_{\mathbb{C}}$ *be a homogeneous polynomial over* $\mathbb{C}$ *of degree* d, *and let* $K \in D^b_c(V, \overline{\mathbb{Q}_\ell})$ *be* α*-homogeneous. Assume that* $v^\vee \in V^\vee(\mathbb{C})$ *is general enough. Then with the notation of* (3.1.1)

$$[(R(f|_{H(v^\vee)})_! K)_{\bar\eta_\infty}] = [(Rf_! K)_{\bar\eta_\infty}] \Leftrightarrow \kappa[(V_d \otimes L_{\alpha/d})_{\bar\eta_\infty}], \tag{1}$$

where κ *is the Euler characteristic* $\chi(H(v^\vee)\backslash f^{-1}(0), K)$ *for cohomology with coefficients in* K.

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Proof. By Lemma (5.1.4) below, it suffices to prove that

$$[(R(f|_{H_0(v^\vee)})_! K)_{\bar\eta_\infty}] = [(Rf_! K)_{\bar\eta_\infty}] \Leftrightarrow \kappa[(V_d \otimes L_{\alpha/d})_{\bar\eta_\infty}]. \tag{2}$$

Actually we will show that (2) even holds without assuming that $v^\vee$ is general enough. By Lemma (5.1.5) below, applied to K and to the extension by zero of $K|_{H_0(v^\vee)}$ we see it is sufficient to prove that

$$d^{-1}\chi(f^{-1}(1) \cap H_0(v^\vee), K) = d^{-1}\chi(f^{-1}(1), K) \Leftrightarrow \kappa,$$

which is equivalent to

$$d^{-1}\chi(f^{-1}(1)\backslash H_0(v^\vee), K) = \kappa := \chi(H(v^\vee)\backslash f^{-1}(0), K). \tag{3}$$

We claim that the map

$$\pi \colon f^{-1}(1)\backslash H_0(v^\vee) \to H(v^\vee)\backslash f^{-1}(0) \colon v \mapsto v/\langle v^\vee, v\rangle$$

is an unramified cover of degree d. Moreover $\pi^* K$ is locally (for the étale topology) isomorphic with the restriction of K to $f^{-1}(1)\backslash H_0(v^\vee)$, because K is α-homogeneous. Hence

$$\begin{aligned}\chi(H(v^\vee)\backslash f^{-1}(0), K) &= d^{-1}\chi(f^{-1}(1)\backslash H_0(v^\vee), \pi^* K)\\ &= d^{-1}\chi(f^{-1}(1)\backslash H_0(v^\vee), K),\end{aligned}$$

which yields (3).

It remains to prove the claim. If $v \in f^{-1}(1)\backslash H_0(v^\vee), w \in H(v^\vee)\backslash f^{-1}(0)$ and $\pi(v) = w$, then $w = v/\langle v^\vee, v\rangle, v = \lambda w$ with $\lambda \in \mathbb{C}^\times$, $1 = f(v) = f(\lambda w) = \lambda^d f(w)$, $\lambda = f(w)^{-1/d}$ and hence $v = w f(w)^{-1/d}$. This proves the claim. □

5.1.3. LEMMA. *Let X be a proper scheme over $\mathbb{C}$, $f : X \to \mathbb{P}^1_{\mathbb{C}}$ a morphism, and $K \in D^b_c(X, \overline{\mathbb{Q}_\ell})$. Let L be a general member of a linear system on X which has no base points in $f^{-1}(\infty)$. Then $[(R(f|_L)_! K)_{\bar\eta_\infty}]$ only depends on X, f, K and the set $L \cap f^{-1}(\infty)$.*

Concerning linear systems we use the terminology of [GriHar, p. 137] (working instead with Cartier divisors when X is not smooth).

Proof. We may suppose that X is reduced and K is zero on $f^{-1}(\infty)$. By restriction to the support of $H^i(K)$ and stratification, we may further suppose that $K = j_! \mathcal{F}$, with $j \colon Y \hookrightarrow X$ an open immersion, Y smooth, $f^{-1}(\infty) \subset X\backslash Y$, and $\mathcal{F}$ a smooth sheaf on Y. Considering an embedded resolution of singularities [Hir] of $X\backslash Y$ in X we may moreover assume that X is smooth and that $X\backslash Y$ has normal crossings in X (meaning that the irreducible components of $X\backslash Y$ are smooth and intersect transversally). Then L is smooth and $L \cup (X\backslash Y)$ has normal crossings in

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a neighbourhood of $f^{-1}(\infty)$, by Bertini's Theorem (cf. [GriHar, p. 137]). Hence for any $a \in L(\mathbb{C}) \cap f^{-1}(\infty)$ we have

$$\Psi_{f|_L,a}(j_!\mathcal{F}) \cong \Psi_{f,a}(j_!\mathcal{F}), \tag{1}$$

where $\Psi_f(K)$ denotes the complex of nearby cycles on $f^{-1}(\infty)$ of K, cf. [Del1]. Indeed X, resp. Y, is locally at a isomorphic with $L \times \mathbb{A}^1_{\mathbb{C}}$, resp. $(Y \cap L) \times \mathbb{A}^1_{\mathbb{C}}$, and f corresponds under this isomorphism to the projection of $L \times \mathbb{A}^1_{\mathbb{C}}$ onto L composed with $f|_L$. The lemma follows now from (1) and the isomorphism (cf. [Del1, (2.1.7.1) and (2.1.8)])

$$(R(f|_L)_!K)_{\bar{\eta}_\infty} \cong R\Gamma(L \cap f^{-1}(\infty), \Psi_{f|_L}(K)). \qquad \square$$

5.1.4. LEMMA. *Let $f : V \to \mathbb{A}^1_{\mathbb{C}}$ be a polynomial over $\mathbb{C}$, and $K \in D^b_c(V, \overline{\mathbb{Q}_\ell})$. Assume that $v^\vee \in V^\vee(\mathbb{C})$ is general enough. Then*

$$[(R(f|_{H_0(v^\vee)})_!K)_{\bar{\eta}_\infty}] = [(R(f|_{H(v^\vee)})_!K)_{\bar{\eta}_\infty}].$$

Proof. There exists a proper scheme X over $\mathbb{C}$ and morphisms

$$\mathbb{P}^n_{\mathbb{C}} \xleftarrow{\pi} X \xrightarrow{\hat{f}} \mathbb{P}^1_{\mathbb{C}}$$

such that π is an isomorphism over $V = \mathbb{A}^n_{\mathbb{C}} \subset \mathbb{P}^n_{\mathbb{C}}$, and $\hat{f}(x) = f(\pi(x))$ when $\pi(x) \in V$. Indeed take e.g. for X the closure in $\mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ of the graph of f in $V \times \mathbb{A}^1_{\mathbb{C}}$, and for $\pi, \hat{f}$ the projections.

Put $H_\infty := \mathbb{P}^n_{\mathbb{C}} \backslash V$ and let $\overline{H_0}$, resp. $\overline{H}$, be the closure of $H_0(v^\vee)$, resp. $H(v^\vee)$, in $\mathbb{P}^n_{\mathbb{C}}$. Note that $\pi^{-1}(\overline{H_0})$, resp. $\pi^{-1}(\overline{H})$, is a general member of a linear system on X with no base points in $\pi^{-1}(H_\infty)$. (Indeed the same holds when we omit π^{-1}.) Since $\hat{f}^{-1}(\infty) \subset \pi^{-1}(H_\infty)$ we have

$$\begin{aligned} \pi^{-1}(\overline{H_0}) \cap \hat{f}^{-1}(\infty) &= \pi^{-1}(\overline{H_0}) \cap \pi^{-1}(H_\infty) \cap \hat{f}^{-1}(\infty) \\ &= \pi^{-1}(\overline{H_0} \cap H_\infty) \cap \hat{f}^{-1}(\infty), \end{aligned}$$

and similarly

$$\pi^{-1}(\overline{H}) \cap \hat{f}^{-1}(\infty) = \pi^{-1}(\overline{H} \cap H_\infty) \cap \hat{f}^{-1}(\infty).$$

Thus $\pi^{-1}(\overline{H_0}) \cap \hat{f}^{-1}(\infty) = \pi^{-1}(\overline{H}) \cap \hat{f}^{-1}(\infty)$, because $\overline{H_0} \cap H_\infty = \overline{H} \cap H_\infty$. The Lemma follows now from Lemma (5.1.3) with f replaced by $\hat{f}$, K replaced by the extension by zero of the pullback of K on $\pi^{-1}(V)$, and L by $\pi^{-1}(\overline{H_0})$ and $\pi^{-1}(\overline{H})$. $\square$

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5.1.5. LEMMA. *Let* $f: V \to \mathbb{A}^1_{\mathbb{C}}$ *be a homogeneous polynomial over* $\mathbb{C}$ *of degree* d*, and let* $K \in D^b_c(V, \overline{\mathbb{Q}_\ell})$ *be* α*-homogeneous. Then*

$$[(Rf_!K)_{\bar\eta_\infty}] = \nu[(V_d \otimes L_{\alpha/d})_{\bar\eta_\infty}],$$

where $\nu = d^{-1}\chi(f^{-1}(1), K)$.

Proof. Tensoring K with $j_!f^*L_{-\alpha/d}$, where j is the immersion $V \backslash f^{-1}(0) \to V$, we may assume that K is 0-homogeneous. By the homogeneity of f and K it suffices to prove that

$$[(Rf_!K)_{\bar\eta_0}] = \nu[V_d].$$

The monodromy action around 0 on $H^{\cdot}_c(f^{-1}(1), K)$ coincides with the endomorphism induced by

$$\gamma: V \to V : v \mapsto e^{2\pi i/d}v$$

and a suitable morphism $\varphi: \gamma^*K \to K$ obtained from the 0-homogeneity of K. Thus it suffices to show for all $j \in \mathbb{N}$ that

$$\mathrm{tr}(\gamma^j, R\Gamma_c(f^{-1}(1), K)) = 0 \quad \text{if } d \nmid j, \quad \text{and} \tag{1}$$

$$= \nu d \quad \text{if } d|j. \tag{2}$$

Indeed the right-hand side of (1), resp. (2), equals the trace of the jth power of the monodromy action on $V_d^{\oplus\nu}$. Assertion (2) is clear and (1) follows from a generalization of the Lefschetz Fixed Point Theorem, see e.g. [KasSch, (9.6.2) and (9.6.16)]. Indeed if $d \nmid j$, then $\gamma^j : V \to V$ has no fixed points different from 0 and extends to an endomorphism of the compactification $(\mathbb{P}^1_{\mathbb{C}})^n$ of V whose graph intersects the diagonal transversally. Thus assertion (1) follows by applying the Fixed Point Theorem to this endomorphism and the extension by zero of $K|_{f^{-1}(1)}$ to $(\mathbb{P}^1_{\mathbb{C}})^n$. □

5.2. Proof of Theorem A1

5.2.0. *Notation and conventions*

(1) From now on till the end of the paper we assume the notation of Section 1 and of (3.5.1).
(2) We always assume that the characteristic p of $\mathbb{F}_q$ is sufficiently large.
(3) We have chosen an isomorphism between $\mathbb{C}$ and $\overline{\mathbb{Q}_\ell}$. Thus the complex valued characters ψ and χ in Section 1 become $\overline{\mathbb{Q}_\ell}$-valued and we have the sheaves $L_\psi, L_\chi, f^*L_\chi = L(\chi(f))$ from (3.1.2) and the Deligne–Fourier transformation $\mathcal{F}_\psi$. Moreover the sheaves $L(\omega^\vee)$ and $L(\omega)$ are defined in (3.5.2).

(4) For a constant $c \in \overline{\mathbb{Q}_\ell}^\times$, we denote by the same letter c the Frob_q-module $\overline{\mathbb{Q}_\ell}$ on which Frob_q acts as the multiplication by c.
(5) Put $E = E(f, f^\vee) := b_0 / \prod_{j \geqslant 1} (j^j)^{e(j)}$, with b_0 and $e(j)$ as in (1.2). Recall that $r := \sum_{j \geqslant 1} e(j)$, cf. (1.4).
(6) Put

$$\tau_\chi = \tau_\chi(f) := (\Leftrightarrow 1)^m q^{-m/2} \prod_{j \geqslant 1} \left(\frac{G(\chi^j, \psi)}{\sqrt{q}} \right)^{e(j)},$$

where m and $G(\chi, \psi)$ are as in (1.2, (2)) and (1.3).
(7) The dimension shift, resp. Tate twist, is denoted by $[n]$, resp. (n).
(8) $\bar{\chi} = \chi^{-1}$ and $\bar{\psi} = \psi^{-1}$.
(9) $H(v^\vee) = \{v \in V | \langle v^\vee, v \rangle = 1\}$ and $H^\vee(v) = \{v^\vee \in V^\vee | \langle v^\vee, v \rangle = 1\}$.

5.2.1. *Evaluation of the character sum on O_1 twisted by $\chi_{1/2}(h(v))$*

5.2.1.0. Our proof of Theorem A1 is somewhat indirect, first evaluating

$$S_h^\vee(\chi, v^\vee) := \sum_{v \in O_1(\mathbb{F}_q)} \chi_{1/2}(h(v)) \chi(f(v)) \psi(\langle v^\vee, v \rangle),$$

where $\chi_{1/2}$ and h are as in (1.5).

5.2.1.1. Note that the sheaf $L(\omega)$ is ν-homogeneous with ν the trivial character (compare with (4.3.2)). We will use Proposition (3.4.5) and its notation with $K = j_! i_* L(\omega)$. Thus $S_K^\vee(\chi, v^\vee)$ from (3.4.2) equals $S_h^\vee(\chi, v^\vee)$ by Lemma (3.5.4), and we will use the notations $\alpha_N, \beta_N, \gamma_N, \rho, a^\vee(v^\vee), \delta_\chi$ from (3.4.5). Fix $\tilde{v}^\vee \in \Omega^\vee(\mathbb{Q})$ in the open orbit $O_0^\vee$ of G, and let $v^\vee \in \Omega^\vee(\mathbb{F}_q)$ be the reduction mod p of $\tilde{v}^\vee$. Applying the Deligne–Fourier transformation to both sides of (3.5.3, (1)) we obtain, forgetting the Frobenius action

$$\mathcal{F}_\psi(j_! i_*(L(\omega) \otimes f^* L_\chi[\Leftrightarrow n])) \cong R j_*^\vee f^{\vee *} L_{\chi^{-1}}[\Leftrightarrow m] \quad \text{on } V^\vee \otimes \overline{\mathbb{F}_q}, \tag{1}$$

cf. [Lau1, 1.2.2.1]. This together with (4.1.3) implies that the hypotheses of Proposition (3.4.5) are satisfied and that $\rho = (\Leftrightarrow 1)^m$. Moreover

$$\chi(H(v^\vee) \backslash f^{-1}(0), K) = \Leftrightarrow \rho, \tag{2}$$

by (3.4.4, (2)).

5.2.1.2. Using the notation of (3.1.3), put

$$\varepsilon_0^\vee(v^\vee) := \varepsilon_0(O_1 \cap H(v^\vee), L(\omega)). \tag{1}$$

From (3.1.6) and Lemma (3.5.4) it follows that

$$\varepsilon_0^\vee(v^\vee) \in \mathbb{Q}^\times. \tag{2}$$

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Thus we can consider the sign of $\varepsilon_0^\vee(v^\vee)$ and denote it by sign $(\varepsilon_0^\vee(v^\vee)) \in \{1, -1\}$. From Proposition (3.4.5, (6)) we then get

$$\varepsilon_0^\vee(v^\vee) = \mathrm{sign}(\varepsilon_0^\vee(v^\vee))\sqrt{q}^{(-1)^m(m-1)-\sum_N(\alpha_N-\beta_N)} q^{-\sum_N \beta_N}. \tag{3}$$

Combining this with (2) yields

$$\sum_N (\alpha_N - \beta_N) \equiv m - 1 \bmod 2. \tag{4}$$

Together with (3) and Proposition (4.3.5) we get

$$\varepsilon_0^\vee(v^\vee) = \mathrm{sign}(\varepsilon_0^\vee(v^\vee))\sqrt{q}^{(-1)^m(m-r)} q^{-\sum_N \beta_N}, \tag{5}$$

and also the following

5.2.1.3. LEMMA. *Assume the notation of* (1.2, (2)) *and* (1.4). *Then* $m \equiv r \bmod 2$.

5.2.1.4. Because $O_0^\vee(\mathbb{C})$ is a homogeneous space under the action of G, we easily deduce from (5.2.1.1, (2)) and Proposition (5.1.2) that

$$\rho\delta_\chi + \sum_{\chi^N=1} (\gamma_N - \beta_N) = 0, \tag{1}$$

with the notation of (3.4.5), especially $\delta_\chi = 1$ if $\chi^d = 1$ and zero otherwise. Now we apply Proposition (3.4.5,(5)) and obtain by (5.2.1.2, (4) and (5)) and (1) that

$$\begin{aligned} S_h^\vee(\chi, v^\vee) \;=\; & -\mathrm{sign}(\varepsilon_0^\vee(v^\vee))\sqrt{q}^{m-r}\chi((a^\vee(v^\vee))^{(-1)^m} \\ & \times\chi\left(\prod_N N^{-N(-1)^m(\alpha_N-\beta_N)}\right) \\ & \times G(\chi^d, \psi)\prod_N G(\chi^N, \psi)^{(-1)^m(\alpha_N-\beta_N)}. \end{aligned}$$

Together with Proposition (4.3.5) and the notation (5.2.0, (5) and (6)) one concludes that

$$q^{-m} S_h^\vee(\chi, v^\vee) = -(-1)^m\, \mathrm{sign}(\varepsilon_0^\vee(v^\vee))\tau_\chi \chi(Ef^\vee(v^\vee)^{-1}), \tag{2}$$

for $v^\vee$ as in (5.2.1.1).

5.2.2. *The Deligne–Fourier transform of* $j_! i_*(L(\omega) \otimes f^* L_\chi)$

5.2.2.0. It is not possible to prove Theorem A1 by taking the Fourier transform of both sides of equation (5.2.1.4,(2)) because this equation may not hold when

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$f^\vee(v^\vee) = 0$. To overcome this problem we will determine $\mathcal{F}_\psi(j_! i_*(L(\omega) \otimes f^* L_\chi))$ on $V^\vee$. In order to reduce the complexity of the expression, we shall use the abusive notation $L(\chi(\varphi)) := \varphi^* L_\chi$ for a morphism $\varphi\colon X \to \mathbb{G}_{m,\mathbb{F}_q}$ of schemes.

5.2.2.1. Let X be a scheme of finite type over $\mathbb{F}_q$ such that $X \otimes \overline{\mathbb{F}_q}$ is connected. Then a smooth sheaf on X is uniquely determined by specifying the Frobenius action on the stalk of a single $\mathbb{F}_q$-rational point and by knowing the restriction of the sheaf to $X \otimes \overline{\mathbb{F}_q}$. Hence (5.2.1.1, (1)) implies that there exists a constant $c = c(\chi) \in \overline{\mathbb{Q}_\ell}$ such that

$$\mathcal{F}_\psi(j_! i_*(L(\omega) \otimes L(\chi(f))(m)[m])) \cong c \otimes L(\chi(f^\vee(\ \ )^{-1}))[n] \quad \text{on}\, \Omega^\vee. \quad (1)$$

Here and below, we always assume that the isomorphisms are compatible with the Frobenius action unless otherwise stated. Considering the trace of the Frobenius action on both sides of (1), and comparing with (5.2.1.4, (2)) we get $c = ⇔\mathrm{sign}(\varepsilon_0^\vee(v^\vee))\tau_\chi \chi(E)$. Hence sign $(\varepsilon_0^\vee(v^\vee))$ is constant on $\Omega^\vee(\mathbb{F}_q)$. Put

$$\sigma^\vee := ⇔\mathrm{sign}(\varepsilon_0^\vee(v^\vee)) \text{ for any } v^\vee \in \Omega^\vee(\mathbb{F}_q). \quad (2)$$

Then we can write (1) as

$$\mathcal{F}_\psi(j_! i_*(L(\omega) \otimes L(\chi(f))(m)[m])) \\ \cong \sigma^\vee \tau_\chi \otimes L(\chi(E f^\vee(\ )^{-1}))[n] \quad \text{on}\, \Omega^\vee. \quad (3)$$

5.2.2.2. The above isomorphism (5.2.2.1, (3)) extends to an isomorphism

$$\varphi\colon \mathcal{F}_\psi(j_! i_*(L(\omega) \otimes L(\chi(f))(m)[m])) \\ \to \sigma^\vee \tau_\chi \otimes R j_*^\vee L(\chi(E f^\vee(\ \ )^{-1}))[n] \quad \text{on}\, V^\vee, \quad (1)$$

because of Lemma 5.2.2.3 below, since both members of (1) are isomorphic to each other if we forget the Frobenius action, by (5.2.1.1, (1)).

5.2.2.3. LEMMA. *Let X be a scheme of finite type over $\mathbb{F}_q$, $j : U \to X$ an open immersion, $K \in D_c^b(X, \overline{\mathbb{Q}_\ell})$, and $F \in D_c^b(U, \overline{\mathbb{Q}_\ell})$. Assume that K and Rj_*F are isomorphic when we forget the Frobenius action, i.e. $K|_{X \otimes \overline{\mathbb{F}_q}} \cong (Rj_*F)|_{X \otimes \overline{\mathbb{F}_q}}$. Then any isomorphism $\varphi_0\colon j^*K \to F$ extends to an isomorphism $\varphi\colon K \to Rj_*F$.*

Proof. Certainly φ_0 induces a morphism $\varphi : K \to Rj_*F$. By Verdier duality it suffices to show that the induced morphism $\mathbb{D}(\varphi) : \mathbb{D}(Rj_*F) \to \mathbb{D}(K)$ is an isomorphism. Note that the restriction of $\mathbb{D}(\varphi)$ to U equals $\mathbb{D}(\varphi_0)$, which is an isomorphism since φ_0 is an isomorphism. Moreover $\mathbb{D}(Rj_*F) = j_!\mathbb{D}F$ and $(\mathbb{D}K)|_{X \otimes \overline{\mathbb{F}_q}} \cong (\mathbb{D}Rj_*F)|_{X \otimes \overline{\mathbb{F}_q}}$. Hence cohomologies of $\mathbb{D}(Rj_*F)$ and $\mathbb{D}K$ are zero on $X \backslash U$. Thus $\mathbb{D}(\varphi)$ is an isomorphism. □

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5.2.3. *Proof of Theorem A1*

5.2.3.1. Applying Verdier duality and the Deligne–Fourier transformation to the above isomorphism φ we obtain that

$$\begin{aligned}&\mathcal{F}_{\psi}(j_!^{\vee} L(\chi(f^{\vee}))(n)[n])\\ &\quad\cong \sigma^{\vee}\tau_{\chi} \otimes Rj_* i_*(L(\chi(Ef(\ \ )^{-1})) \otimes L(\omega))[m] \quad \text{on } V,\end{aligned} \tag{1}$$

cf. [Lau1, 1.3.2.2 and 1.2.2.1]. Replacing the triple (G, ρ, V) by its dual $(G, \rho^{\vee}, V^{\vee})$ yields

$$\begin{aligned}&\mathcal{F}_{\psi}(j_! L(\chi(f))(n)[n])\\ &\quad\cong \sigma\tau_{\chi} \otimes Rj_*^{\vee} i_*^{\vee}(L(\chi(Ef^{\vee}(\ \ )^{-1})) \otimes L(\omega^{\vee}))[m] \quad \text{on } V^{\vee},\end{aligned} \tag{2}$$

where (with the notation of (3.1.3))

$$\sigma := -\text{sign}(\varepsilon_0(O_1^{\vee} \cap H^{\vee}(v), L(\omega^{\vee}))) \text{ for any } v \in \Omega(\mathbb{F}_q). \tag{3}$$

Note that σ does not depend on the choice of $v \in \Omega(\mathbb{F}_q)$, cf. (5.2.2.1). Theorem A1 follows now directly from (2) and Lemma (3.5.4), with

$$\kappa^{\vee}(v^{\vee}) = \sigma\chi_{1/2}(h^{\vee}(v^{\vee})). \tag{4}$$

It is convenient to put

$$L(\kappa^{\vee}) := \sigma \otimes L(\omega^{\vee}), \tag{5}$$

so that

$$\text{tr}(\text{Frob}_q, L(\kappa^{\vee})_{v^{\vee}}) = \kappa^{\vee}(v^{\vee}), \quad \text{for all } v^{\vee} \in O_1^{\vee}(\mathbb{F}_q). \tag{6}$$

With this notation we reformulate (2) as Theorem A1† which refines both (3.5.3, (2)) and Theorem A1

5.2.3.2. THEOREM A1†. *Assume the notation of* (3.1.2) *and* (5.2.3.1,(5)). *If the characteristic of* $\mathbb{F}_q$ *is sufficiently large, then we have for all* $\chi \in \text{Hom}(\mathbb{F}_q^{\times}, \overline{\mathbb{Q}_\ell}^{\times})$ *that*

$$\begin{aligned}&\mathcal{F}_{\psi}(j_! L(\chi(f))(n)[n])\\ &\quad\cong (-1)^m q^{-(m+r)/2} \prod_{j \geqslant 1} G(\chi^j, \psi)^{e(j)}\\ &\qquad \otimes Rj_*^{\vee} i_*^{\vee} \left\{ L\left(\chi\left(\frac{b_0}{\prod_{j\geqslant 1}(j^j)^{e(j)}} f^{\vee}(\ \ )^{-1}\right)\right) \otimes L(\kappa^{\vee}) \right\}[m] \quad \text{on } V^{\vee}.\end{aligned}$$

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5.2.3.3. REMARK. Assume that the characteristic of $\mathbb{F}_q$ is sufficiently large and let $\chi \in \mathrm{Hom}(\mathbb{F}_q^\times, \mathbb{C}^\times)$. Suppose that

$$(\text{order of } \chi)^{-1} \not\equiv \alpha_j \bmod \mathbb{Z}, \tag{1}$$

for each α_j in (1.2, (4)). (This is equivalent with the requirement that the order of χ is different from the order in $\mathbb{Q}/\mathbb{Z}$ of each α_j, because of (1.2, (5)).) *Then the character sum in Theorem* A1 *vanishes for* $v^\vee \in (V^\vee \backslash \Omega^\vee)(\mathbb{F}_q)$. This follows from the following argument due to N. Kawanaka: By the Plancherel formula, the norm in $L^2(V^\vee(\mathbb{F}_q))$ of the discrete Fourier transform $\mathcal{F}_{\text{discr}}(\chi \circ f)$ of $\chi \circ f$ equals the norm in $L^2(V(\mathbb{F}_q))$ of $\chi \circ f$, and is thus equal to $|\Omega(\mathbb{F}_q)|^{1/2}$. But on the other hand the norm in $L^2(\Omega^\vee(\mathbb{F}_q))$ of $\mathcal{F}_{\text{discr}}(\chi \circ f)$ can be calculated by Theorem A1 and equals $(q^{n-m}|O_1^\vee(\mathbb{F}_q)|)^{1/2} = |\Omega(\mathbb{F}_q)|^{1/2}$, because $\sum_{\chi^j=1} e(j) = 0$ by (1), and because of Lemma (2.2, (1)). Thus the norms of $\mathcal{F}_{\text{discr}}(\chi \circ f)$ in $L^2(V^\vee(\mathbb{F}_q))$ and in $L^2(\Omega^\vee(\mathbb{F}_q))$ are the same, and we conclude that $\mathcal{F}_{\text{discr}}(\chi \circ f)$ vanishes on $V^\vee \backslash \Omega^\vee$. Actually even more is true: *If* (1) *holds then* $\mathcal{F}_\psi(j_! L(\chi(f)))$ *is zero on* $V^\vee \backslash \Omega^\vee$. This is a direct consequence of Theorem (3.5.3, (1)) and the fact that $Rj_* f^* L_\chi = j_! f^* L_\chi$ which follows from [Gyo1, (3.23, (5) and (6))]. Indeed if $\alpha \in \mathbb{Q}$ has order in $\mathbb{Q}/\mathbb{Z}$ equal to the order of χ, then (1) implies that $\Leftrightarrow\alpha \in A_+ \cap A_-$ with the notation of loc. cit., and hence $Rj_* \mathbb{C} f^\alpha = j_! \mathbb{C} f^\alpha$.

6. Proof of Theorem B

In this section, we prove Theorem B. The basic idea of our proof is (6.2, (9)), which gives an expression of $(\Leftrightarrow 1)^{s(G_{v^\vee})}$ for $v^\vee \in O_1^\vee(\mathbb{F}_q)$ in terms of $|G_{v^\vee}(\mathbb{F}_q)|$ and the operation '$q \mapsto q^{-1}$'. We keep the notation and the assumption of Section 1, and suppose that all geometric objects are defined over a finite field $\mathbb{F}_q$. Moreover, we fix an algebraic closure $\overline{\mathbb{F}_q}$ of $\mathbb{F}_q$ and identify any algebraic variety over $\mathbb{F}_q$ with 'the set of $\overline{\mathbb{F}_q}$-rational points together with the Frobenius action $\sigma(x) = \mathrm{Frob}_q^{-1} := x^q$ $(x \in \overline{\mathbb{F}_q})$'.

6.1. PROOF OF THEOREM B (FIRST STEP)

Put

6.1.1. $\quad \eta^\vee(v^\vee) := (\Leftrightarrow 1)^{r(v^\vee) - s(v^\vee)} \quad (v^\vee \in O_1^\vee(\mathbb{F}_q))$.

Theorem B asserts that $\kappa^\vee \equiv \eta^\vee$ on $O_1^\vee(\mathbb{F}_q)$, where $\kappa^\vee$ is the function appearing in Theorem A1. As a first step, we shall prove here that

6.1.2. $\quad \kappa^\vee \equiv C\eta^\vee \quad \text{on } O_1^\vee(\mathbb{F}_q)$,

with $C = +1$ or $C = \Leftrightarrow 1$. As for $\kappa^\vee$, the following equality is already proved (5.2.3.1, (4)).

6.1.3. $\quad \chi_{1/2}(h^\vee) \equiv \kappa^\vee \quad \text{or} \quad \equiv \Leftrightarrow\kappa^\vee \quad \text{on } O_1^\vee(\mathbb{F}_q)$.

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6.1.4. First, let us study how $\chi_{1/2}(h^\vee(v^\vee))$ varies when $v^\vee \in O_1^\vee(\mathbb{F}_q)$ moves. Take $v^\vee \in O_1^\vee(\mathbb{F}_q)$ and put $v := F^\vee(v^\vee) \in O_1$. Take linear bases $\{v_1, \ldots, v_m\}$ of T_vO_1, and $\{v_{m+1}, \ldots, v_n\}$ of $(T_{v^\vee}O_1^\vee)^\perp$. Assume that all the v_i's are $\mathbb{F}_q$-rational. Since $(F_*)_v : T_vO_1 \to T_{v^\vee}O_1^\vee$ is an isomorphism (2.2, (5)), and since $(F_*)_v|(T_{v^\vee}O_1^\vee)^\perp \equiv 0$ (2.2, (1)), we have

$$V = (T_vO_1) \oplus (T_{v^\vee}O_1^\vee)^\perp. \tag{1}$$

(Here $(F_*)_v : T_v\Omega(= V) \to T_{F(v)}V^\vee(= V^\vee)$ denotes the linear mapping induced by $F : \Omega \to V^\vee$.) In particular, $\{v_1, \ldots, v_n\}$ gives a linear basis of V. Let $\{v_1^\vee, \ldots, v_n^\vee\}$ be its dual basis of $V^\vee$. Then $\{v_1^\vee, \ldots, v_m^\vee\}$ (resp. $\{v_{m+1}^\vee, \ldots, v_n^\vee\}$) gives a linear basis of $T_{v^\vee}O_1^\vee$ (resp. $(T_vO_1)^\perp$). Consider $v_i^\vee$ (resp. v_j) as a linear function on V (resp. $V^\vee$), and denote it by x_i (resp. y_j). Then $\{x_1, \ldots, x_n\}$ (resp. $\{y_1, \ldots, y_n\}$) is a linear coordinate system of V (resp. $V^\vee$), and

$$\begin{aligned}
&\langle (F_*^\vee)_{v^\vee}(v_i^\vee), v_j^\vee\rangle \\
&\quad = \left\langle (F_*^\vee)_{v^\vee}\left(\frac{\partial}{\partial y_i}\right), \frac{\partial}{\partial y_j}\right\rangle \\
&\quad = \left\langle \sum_{k=1}^{n} \frac{\partial(\log f^\vee)_{y_k}}{\partial y_i}(v^\vee)\cdot\frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_j}\right\rangle \\
&\quad = \frac{\partial^2 \log f^\vee}{\partial y_i \partial y_j}(v^\vee).
\end{aligned} \tag{2}$$

Note that the most left member vanishes if $i > m$ or $j > m$;

$$\frac{\partial^2 \log f^\vee}{\partial y_i \partial y_j}(v^\vee) = 0 \quad \text{if } i > m \text{ or } j > m. \tag{3}$$

(Indeed, $(F_*^\vee)_{v^\vee}|(T_vO_1)^\perp \equiv 0$ and, especially, $(F_*^\vee)_{v^\vee}(v_i^\vee) = 0$ for $i > m$.) Hence

$$h^\vee(v^\vee) = \det\left(\frac{\partial^2 \log f^\vee}{\partial y_i \partial y_j}(v^\vee)\right)_{1\leqslant i,j\leqslant m} \quad \text{in } \mathbb{F}_q^\times/\mathbb{F}_q^{\times 2}. \tag{4}$$

(See (1.5) for $h^\vee$.) Take another point $v^{\vee\prime}$ of $O_1^\vee(\mathbb{F}_q)$. Then there exists $g \in G(\overline{\mathbb{F}_q})$ such that $v^{\vee\prime} = gv^\vee$. Let $\{y_1', \ldots, y_n'\}$ be a linear coordinate system constructed as above using $v^{\vee\prime}$ instead of $v^\vee$. Denote the morphism $(V^\vee, v^\vee) \to (V^\vee, gv^\vee)$ induced by the action of g by the same letter g. Since $(g^*f^\vee)(v^\vee) = f^\vee(gv^\vee) = \phi(g)^{-1}f^\vee(v^\vee)$,

$$\frac{\partial^2 \log f^\vee}{\partial y_i \partial y_j} = \frac{\partial^2 \log(g^* f^\vee)}{\partial y_i \partial y_j}$$

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$$= \sum_{i'} \frac{\partial^2 (g^* y'_{i'})}{\partial y_i \partial y_j} \cdot g^* \left(\frac{\partial \log f^\vee}{\partial y'_{i'}} \right)$$
$$+ \sum_{i',j'} \frac{\partial g^* y'_{i'}}{\partial y_i} \cdot \frac{\partial g^* y'_{j'}}{\partial y_j} \cdot g^* \left(\frac{\partial^2 \log f^\vee}{\partial y'_{i'} \partial y'_{j'}} \right). \tag{5}$$

Note also that

$$\frac{\partial^2 (g^* y'_{i'})}{\partial y_i \partial y_j}(v^\vee) = 0. \tag{6}$$

(Recall that y_i and $y'_{i'}$ are linear functions on V.) By (3), (5) and (6), we get

$$\left(\frac{\partial^2 \log f^\vee}{\partial y_i \partial y_j}(v^\vee) \right)_{i,j}$$
$$= \left(\frac{\partial g^* y'_{i'}}{\partial y_i}(v^\vee) \right)_{i,i'} \left(\frac{\partial^2 \log f^\vee}{\partial y'_{i'} \partial y'_{j'}}(gv^\vee) \right)_{i',j'} \left(\frac{\partial g^* y'_{j'}}{\partial y_j}(v^\vee) \right)_{j',j}, \tag{7}$$

where i, j, i' and j' run over $\{1, \ldots, m\}$. Now take any linear bases β of $T_{v^\vee} O_1^\vee$ and β' of $T_{gv^\vee} O_1^\vee$ which are $\mathbb{F}_q$-rational. Consider the dual bases $\beta^\vee$ and $\beta'^\vee$ of the dual spaces $(T_{v^\vee} O_1^\vee)^\vee$ and $(T_{gv^\vee} O_1^\vee)^\vee$, respectively. Then we can define

$$\begin{aligned} \Delta(g) &:= \det(g | T_{v^\vee} O_1^\vee \to T_{gv^\vee} O_1^\vee) \\ &= \det(g^* | (T_{v^\vee} O_1^\vee)^\vee \leftarrow (T_{gv^\vee} O_1^\vee)^\vee) \in \overline{\mathbb{F}_q}^\times, \end{aligned} \tag{8}$$

where g^* denotes the transpose of g and $(\Leftrightarrow)^\vee$ denotes the dual space. If we change β or β', then the value of $\Delta(g)$ is multiplied by some element of $\mathbb{F}_q^\times$. Hence

$$(\Delta(g) \bmod \mathbb{F}_q^\times) \text{ is well-defined.} \tag{9}$$

If we take $\beta^\vee = \{(dy_1)_{v^\vee}, \ldots, (dy_m)_{v^\vee}\}$ and $\beta'^\vee = \{(dy'_1)_{gv^\vee}, \ldots, (dy'_m)_{gv^\vee}\}$, then

$$\Delta(g) \equiv \det \left(\frac{\partial g^* y'_{i'}}{\partial y_i}(v^\vee) \right)_{1 \leqslant i,i' \leqslant m} \bmod \mathbb{F}_q^\times. \tag{10}$$

By (4), (7) and (10), we get

$$h^\vee(v^\vee) \equiv \Delta(g)^2 h^\vee(gv^\vee) \bmod \mathbb{F}_q^{\times 2}. \tag{11}$$

Since $h^\vee(v^\vee), h^\vee(gv^\vee) \in \mathbb{F}_q^\times / \mathbb{F}_q^{\times 2}$,

$$\Delta(g)^2 \in \mathbb{F}_q^\times / \mathbb{F}_q^{\times 2}. \tag{12}$$

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Therefore, $\chi_{1/2}(\Delta(g)^2)$ is well-defined, and

$$\chi_{1/2}(h^\vee(v^\vee)) = \chi_{1/2}(\Delta(g)^2)\chi_{1/2}(h^\vee(gv^\vee)), \tag{13}$$

whenever $v^\vee, gv^\vee \in O_1^\vee(\mathbb{F}_q)$ and $g \in G(\overline{\mathbb{F}_q})$.

6.1.5. Next, let us show that $\eta^\vee(v^\vee)$ varies in the same way as $\chi_{1/2}(h^\vee(v^\vee))$ when $v^\vee \in O_1^\vee(\mathbb{F}_q)$ moves. More precisely, we shall show that

$$\eta^\vee(v^\vee) = \chi_{1/2}(\Delta(g)^2)\eta^\vee(gv^\vee) \tag{1}$$

whenever $v^\vee, gv^\vee \in O_1^\vee(\mathbb{F}_q)$ and $g \in G(\overline{\mathbb{F}_q})$. (See (6.1.1) for $\eta^\vee$.)

6.1.6. In general, for a (not necessarily connected) reductive group G' over $\mathbb{F}_q$, let $\mathcal{B} = \mathcal{B}(G')$ be the totality of Borel subgroups of $(G')^0$. (Here and below $(\ )^0$ denotes the identity component.) Let $\mathrm{Mor}(B_1, B_2)$ $(B_1, B_2 \in \mathcal{B})$ be the totality of $\varphi \in \mathrm{Hom}(B_1, B_2)$ which come from inner automorphisms of $(G')^0$, and consider $\mathcal{B}$ as a category. Put

$$X = X(G') := \varprojlim_{B\in\mathcal{B}(G')} \mathrm{Hom}(B, \mathbb{G}_m) = \varinjlim_{B\in\mathcal{B}(G')} \mathrm{Hom}(B, \mathbb{G}_m). \tag{1}$$

(Note that all elements of Mor (B_1, B_2) induce the same morphism $\mathrm{Hom}(B_2, \mathbb{G}_m) \to \mathrm{Hom}(B_1, \mathbb{G}_m)$ by [Bor2, IV, 11.16].) Thus an element $\phi \in X$ is of the form

$$\phi = (\phi_B)_{B\in\mathcal{B}}, \quad \phi_B \in \mathrm{Hom}(B, \mathbb{G}_m). \tag{2}$$

Take a Borel subgroup B_0 of $(G')^0$ defined over $\mathbb{F}_q$, and a maximal torus T' of B_0 also defined over $\mathbb{F}_q$ [SprSte, I, 2.9]. Then $X(G')$ is canonically isomorphic to Hom $(B_0, \mathbb{G}_m) = X(T')$. (We need $X(G')$ to work without specific choice of B_0 and T'.) Now $\sigma = \sigma_q := \mathrm{Frob}_q^{-1}$, where Frob_q is the geometric Frobenius, acts naturally on $\mathcal{B}(G')$, and it also acts on $X(G')$ by the transpose: if $\phi = (\phi_B)_{B\in\mathcal{B}}$ and $b \in B \in \mathcal{B}$, then $(\sigma^*\phi)_B(b) = \phi_{\sigma B}(\sigma b)$. We have

$$\det(\sigma^*|X(G')) = (-1)^{r(G')-s(G')}q^{r(G')}. \tag{3}$$

Indeed, the eigenvalues of $q^{-1}\sigma^*$ on $X(G')$ are roots of unity among which 1 appears with multiplicity $s(G')$, cf. [Bor2, III, 8.15].

6.1.7. LEMMA. *Let G' be a connected reductive group defined over an algebraically closed field k, and $A \in \mathrm{Aut}(G')$. Denote by the same letter the induced automorphism of $\mathrm{Lie}(G') =: \mathfrak{g}'$.*

(1) *If $g \in G'$ and $A = A_g$ ($:=$ inner automorphism by g), then* $\det(A|\mathfrak{g}') = 1$ *and* $\det(A^*|X(G') \otimes k) = 1$.
(2) *Generally*, $\det(A|\mathfrak{g}') = \det(A^*|X(G') \otimes k)$.

Proof. (1) Define a rational character ϕ' of G' by $\phi'(g) = \det(A_g|\mathfrak{g}')$. Let T' be a maximal torus of G' and $\mathfrak{t}' := \mathrm{Lie}(T')$. Considering the root space decomposition $\mathfrak{g}' = \mathfrak{t}' \oplus (\bigoplus_\alpha \mathfrak{g}'(\alpha))$, we can see that $\phi'|_{T'} \equiv 1$. Hence $\phi' \equiv 1$ on G' by [Bor2, IV, 11.10]. The second equality is obvious.

(2) Take a Borel subgroup $B' \subset G'$ and a maximal torus $T' \subset B'$. We may assume by (1) that $A(B') = B'$ and $A(T') = T'$. Define an automorphism A of the dual space of $\mathfrak{t}'$ so that $\langle A\lambda, AX\rangle = \langle \lambda, X\rangle$ for $\lambda \in \mathrm{Hom}(\mathfrak{t}', k)$ and $X \in \mathfrak{t}'$. (In particular, we can consider $A\alpha$ for any root α.) Take $0 \neq X_\alpha \in \mathfrak{g}'(\alpha)$ for each root α so that $[X_\alpha, X_{-\alpha}] = \alpha^\vee$ ($\in \mathfrak{t}'$), where $\alpha^\vee$ denotes the coroot associated with α. (If α is a root, the corresponding coroot $\alpha^\vee$ is characterized as follows: there is a homomorphism $u\colon SL_2 \to G'$, which maps the algebraic subgroups $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$ isomorphically onto the root subgroups U_α and $U_{-\alpha}$, respectively. Then $du\colon sl_2 \to \mathfrak{g}'$ maps $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ to $\alpha^\vee$.) Then $A(\mathfrak{g}'(\alpha)) = \mathfrak{g}'(A\alpha)$, and hence $AX_\alpha = c_\alpha X_{A\alpha}$ with some $0 \neq c_\alpha \in k$. Since

$$\begin{aligned}(A\alpha)^\vee &= A(\alpha^\vee) = [AX_\alpha, AX_{-\alpha}] \\ &= [c_\alpha X_{A\alpha}, c_{-\alpha} X_{-A\alpha}] = c_\alpha c_{-\alpha} (A\alpha)^\vee,\end{aligned}$$

it follows

$$c_\alpha c_{-\alpha} = 1. \tag{3}$$

If we define an order of roots so that $\mathrm{Lie}(B') \supset \mathfrak{g}'(\alpha)$ ($\alpha > 0$), then

$$A\alpha > 0 \quad \text{whenever} \quad \alpha > 0. \tag{4}$$

By (3) and (4), $\det(A|\mathfrak{g}') = \det(A|\mathfrak{t}') = \det(A^*|\mathrm{Hom}(\mathfrak{t}', k)) = \det(A^*|X(G') \otimes k)$. □

6.1.8. *Proof of* (6.1.5, (1)). Since $(\sigma g)v^\vee = \sigma(gv^\vee) = gv^\vee$, it follows $c := g^{-1} \cdot (\sigma g) \in G_{v^\vee}$ and $\sigma(gbg^{-1}) = gc(\sigma b)c^{-1}g^{-1}$ for $b \in B \in \mathcal{B}(G_{v^\vee})$. Put $A(x) = A_g(x) := gxg^{-1}$ for $x \in G$. Then A induces $\mathcal{B}(G_{v^\vee}) \overset{\cong}{\Leftrightarrow} \mathcal{B}(G_{gv^\vee})$, which we denote by the same letter $A = A_g$. Put $\sigma_g = A_g^{-1} \cdot \sigma \cdot A_g$. Then $\sigma_g = A_g^{-1} \cdot A_{\sigma g} \cdot \sigma = A_c \cdot \sigma$ and hence we get the commutative diagrams

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$$\begin{array}{ccc} \mathcal{B}(G_{v^\vee}) & \xrightarrow[A_g]{\cong} & \mathcal{B}(G_{gv^\vee}) \\ {\scriptstyle A_c\cdot\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ \mathcal{B}(G_{v^\vee}) & \xrightarrow[A_g]{\cong} & \mathcal{B}(G_{gv^\vee}) \end{array} \quad \text{and} \quad \begin{array}{ccc} X(G_{v^\vee}) & \xleftarrow[A_g^*]{\cong} & X(G_{gv^\vee}) \\ {\scriptstyle \sigma^*\cdot A_c^*}\uparrow & & \uparrow{\scriptstyle\sigma^*} \\ X(G_{v^\vee}) & \xleftarrow[A_g^*]{\cong} & X(G_{gv^\vee}). \end{array} \qquad (1)$$

Hence

$$\frac{\det(\sigma^*|X(G))}{\det(\sigma^*|X(G_{gv^\vee}))} = \eta^\vee(gv^\vee)q^{r'} \quad \text{by } (6.1.6,\ (3))$$

$$= \frac{\det(\sigma^*|X(G))}{\det(\sigma^*|X(G_{v^\vee}))\det(A_c^*|X(G_{v^\vee}))} \quad \text{by } (1)$$

$$= \eta^\vee(v^\vee)q^{r'}\cdot\det(A_c^*|X(G_{v^\vee}))^{-1} \quad \text{by } (6.1.6,\ (3)) \text{ again,} \qquad (2)$$

where $r' = r(v^\vee) = r(gv^\vee)$. (Cf. (1.4) for $r(v^\vee)$.) Thus in order to prove (6.1.5, (1)), it suffices to show that

$$\det(A_c^*|X(G_{v^\vee})) = \chi_{1/2}(\Delta(g)^2). \qquad (3)$$

We have

$$\det(A_c^*|X(G_{v^\vee})) = \det(A_c|\mathrm{Lie}(G_{v^\vee})) \quad \text{by } (6.1.7,\ (2))$$

$$= \frac{\det(A_c|\mathrm{Lie}(G))}{\det(c|T_{v^\vee}O_1^\vee)} = \det(c|T_{v^\vee}O_1^\vee)^{-1} \quad \text{by } (6.1.7,\ (1)). \qquad (4)$$

(Note that G is connected, but $G_{v^\vee}$ is not in general.) Take linear bases, say β and β', of $T_{v^\vee}O_1^\vee$ and $T_{gv^\vee}O_1^\vee$, respectively, which are $\mathbb{F}_q$-rational. Then (the inverse of) the last member of (4) is equal to

$$\det(g^{-1}\cdot\sigma(g)|T_{v^\vee}O_1^\vee)$$

$$= \det(g^{-1}|(T_{v^\vee}O_1^\vee,\beta) \leftarrow (T_{gv^\vee}O_1^\vee,\beta'))$$

$$\times \det(\sigma(g)|(T_{v^\vee}O_1^\vee,\beta) \to (T_{gv^\vee}O_1^\vee,\beta'))$$

$$= \Delta(g)^{-1}\cdot\sigma(\Delta(g)) = \Delta(g)^{q-1} = (\Delta(g)^2)^{(q-1)/2}$$

$$= \chi_{1/2}(\Delta(g)^2) \quad \text{by } (6.1.4,\ (12)). \qquad (5)$$

Now (3) follows from (4) and (5), and hence the proof of (6.1.5, (1)) is now complete. □

6.1.9. *Proof of* (6.1.2). By (6.1.3), it suffices to show that

$$\chi_{1/2}(h^\vee) \equiv \eta^\vee \text{ or } \Leftrightarrow\eta^\vee \quad \text{on } O_1^\vee(\mathbb{F}_q). \tag{1}$$

Fix $v_1^\vee \in O_1^\vee(\mathbb{F}_q)$. Then any $v^\vee \in O_1^\vee(\mathbb{F}_q)$ can be expressed as $v^\vee = gv_1^\vee$ with some $g \in G(\overline{\mathbb{F}_q})$. By (6.1.4, (13)) and (6.1.5, (1)), we get

$$\frac{\chi_{1/2}(h^\vee(v^\vee))}{\eta^\vee(v^\vee)} = \frac{\chi_{1/2}(h^\vee(v_1^\vee))}{\eta^\vee(v_1^\vee)}, \tag{2}$$

where the right-hand side is $= \pm 1$, and independent of $v^\vee$. Hence we get the result. □

Before concluding (6.1), let us record a result which can be proved using the technique given in (6.1.4).

6.1.10. LEMMA. *For* $v^\vee \in O_1^\vee(\mathbb{F}_q)$, $h^\vee(v^\vee) = h(F^\vee(v^\vee))$ *in* $\mathbb{F}_q^\times/\mathbb{F}_q^{\times 2}$, *where* $h^\vee$ *and* h *are as in* (1.5).

Proof. Take linear coordinates $\{x_1, \ldots, x_n\}$ (resp. $\{y_1, \ldots, y_n\}$) of V (resp. $V^\vee$) as in (6.1.4). Then on $O_1^\vee$, we have

$$\begin{aligned}
\frac{\partial^2 \log f^\vee}{\partial y_i \partial y_j} &= \frac{\partial^2 \log F^{\vee *}(b_0 f^{-1})}{\partial y_i \partial y_j} \text{ by (2.1)} \\
&= \Leftrightarrow \frac{\partial^2 \log F^{\vee *} f}{\partial y_i \partial y_j} \\
&= \Leftrightarrow \Bigg\{ \sum_{i'} \frac{\partial^2 F^{\vee *} x_{i'}}{\partial y_i \partial y_j} \cdot F^{\vee *}\left(\frac{\partial \log f}{\partial x_{i'}}\right) \\
&\qquad + \sum_{i',j'} \frac{\partial F^{\vee *} x_{i'}}{\partial y_i} \cdot \frac{\partial F^{\vee *} x_{j'}}{\partial y_j} \cdot F^{\vee *}\left(\frac{\partial^2 \log f}{\partial x_{i'} \partial x_{j'}}\right) \Bigg\} \\
&= \Leftrightarrow \Bigg\{ \sum_{i'} \frac{\partial^3 \log f^\vee}{\partial y_i \partial y_j \partial y_{i'}} \cdot y_{i'} \\
&\qquad + \sum_{i',j'} \frac{\partial^2 \log f^\vee}{\partial y_i \partial y_{i'}} \cdot \frac{\partial^2 \log f^\vee}{\partial y_j \partial y_{j'}} \cdot F^{\vee *}\left(\frac{\partial^2 \log f}{\partial x_{i'} \partial x_{j'}}\right) \Bigg\} \quad \text{by (2.2, (5))}
\end{aligned}$$

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$$= 2\frac{\partial^2 \log f^\vee}{\partial y_i \partial y_j} \Leftrightarrow \sum_{i',j'} \frac{\partial^2 \log f^\vee}{\partial y_i \partial y_{i'}} \cdot \frac{\partial^2 \log f^\vee}{\partial y_j \partial y_{j'}} \cdot F^{\vee *}\left(\frac{\partial^2 \log f}{\partial x_{i'} \partial x_{j'}}\right)$$

by Euler′s identity,

where i, j, i' and j' run over $\{1, \ldots, n\}$. However, as far as we are concerned with the value at $v^\vee$, we may assume that they run over $\{1, \ldots, m\}$. (Cf. (6.1.4, (3)).) Hence

$$\left(\frac{\partial^2 \log f^\vee}{\partial y_i \partial y_j}(v^\vee)\right)_{i,j}$$
$$= \left(\frac{\partial^2 \log f^\vee}{\partial y_i \partial y_{i'}}(v^\vee)\right)_{i,i'} \left(\frac{\partial^2 \log f}{\partial x_{i'} \partial x_{j'}}(F^\vee(v^\vee))\right)_{i',j'} \left(\frac{\partial^2 \log f^\vee}{\partial y_{j'} \partial y_j}(v^\vee)\right)_{j',j},$$

where i, j, i' and j' run over $\{1, \ldots, m\}$. By (6.1.4, (4)), we get the result. □

6.2. Proof of Theorem B (Second Step)

Taking the trivial character as χ in Theorem A1† (5.2.3.2), and considering the Verdier dual, we get

$$\mathcal{F}_\psi(Rj_*\overline{\mathbb{Q}_\ell}(n)[n]) \cong (\Leftrightarrow 1)^{m+r} q^{(m+r)/2} \otimes j^\vee_! i^\vee_* L(\kappa^\vee)(m)[m]. \tag{1}$$

Cf. [Lau1, 1.3.2.2]. Put

$$\varphi'_0(v^\vee) = \mathrm{tr}(\mathrm{Frob}_q, \mathcal{F}_\psi(Rj_*\overline{\mathbb{Q}_\ell}(n)[n])_{v^\vee})$$

for $v^\vee \in V^\vee(\mathbb{F}_q)$. Then by (1), we have

$$\varphi'_0(v^\vee) = 0 \quad \text{if } v^\vee \notin O_1^\vee(\mathbb{F}_q), \tag{2}$$

and (6.1.2) yields

$$\varphi'_0(v^\vee) = C \times q^{(-m+r)/2} (\Leftrightarrow 1)^r (\Leftrightarrow 1)^{r(v^\vee) - s(v^\vee)} \quad \text{if } v^\vee \in O_1^\vee(\mathbb{F}_q). \tag{3}$$

Since G and $G_{v^\vee}$ ($v^\vee \in O_1^\vee$) are reductive (cf. (2.2, (6)), we have $\dim G = r(G) + 2N$ with N the number of positive roots, and a similar formula holds for $G_{v^\vee}$ as well. Hence

$$r(v^\vee) := r(G) \Leftrightarrow r(G_{v^\vee}) \equiv \dim G \Leftrightarrow \dim G_{v^\vee} =: m \bmod 2. \tag{4}$$

By (5.2.1.3) (or alternatively, by [Gyo2, 7.6]), and by (4), we get

$$r(v^\vee) \equiv r \bmod 2. \tag{5}$$

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(Later in (6.3), we shall see that $r(v^\vee) = r$ without using (5).) Hence (3) can be written as follows

$$\varphi_0'(v^\vee) = C \times q^{(-m+r)/2} (\Leftrightarrow 1)^{s(v^\vee)} \quad \text{if } v^\vee \in O_1^\vee(\mathbb{F}_q). \tag{6}$$

Let $\langle \varphi_1, \varphi_2 \rangle_X := \sum_{x \in X} \varphi_1(x)\varphi_2(x)$ for any set X and functions φ_1, φ_2 on it. By (2) and (6), we get

$$\begin{aligned} \langle \varphi_0', 1 \rangle_{V^\vee(\mathbb{F}_q)} &= C q^{(-m+r)/2} \sum_{v^\vee \in O_1^\vee(\mathbb{F}_q)} (\Leftrightarrow 1)^{s(v^\vee)} \\ &= C q^{(-m+r)/2} \sum_{v^\vee \in O_1^\vee(\mathbb{F}_q)/G(\mathbb{F}_q)} (\Leftrightarrow 1)^{s(v^\vee)} \frac{|G(\mathbb{F}_q)|}{|G_{v^\vee}(\mathbb{F}_q)|}. \end{aligned} \tag{7}$$

We know that $|G(\mathbb{F}_q)|$ can be expressed as

$$|G(\mathbb{F}_q)| = q^N \prod_i \phi_i(q) \times (q \Leftrightarrow 1)^{s(G)}, \tag{8}$$

where N is the number of positive roots, and the $\phi_i(q)$'s are some cyclotomic polynomials $\neq q \Leftrightarrow 1$, cf. [Ste, 11.16] and (6.4) below. This polynomial expression of $|G(\mathbb{F}_q)|$ may depend on q, but its polynomial degree is always equal to dim G. Hence

$$|G(\mathbb{F}_q)|_{q \mapsto q^{-1}} = (\Leftrightarrow 1)^{s(G)} |G(\mathbb{F}_q)| q^{-\dim G - N}, \tag{9}$$

and similarly for $|G_{v^\vee}(\mathbb{F}_q)|$. (This argument is motivated by [Kaw2]. Justification concerning $|_{q \mapsto q^{-1}}$ will be given in (6.4), where we shall understand every expression $(\ )|_{q \mapsto q^{-1}}$ appearing in (6.2) except for (10e) as a specialization of some polynomial which we shall explicitly specify. In particular, we *do not* need that $(\ )|_{q \mapsto q^{-1}}$ has a canonical meaning. However, if the reader wants, this operation in (9) and (10b)–(10d) can be understood as follows. In these places, $(\ )|_{q \mapsto q^{-1}}$ is applied to quantities each of which has a natural polynomial expression, say $f(q)$, and which has the same expression $f(q^e)$ if $\mathbb{F}_q$ is replaced by $\mathbb{F}_{q^e}$, whenever e is close enough to 1 in $\varprojlim \mathbb{Z}/n\mathbb{Z}$. Therefore the polynomial f is uniquely determined, and hence $f(q)|_{q \mapsto q^{-1}} = f(q^{-1})$ has a canonical meaning.) Postponing the justification until (6.4), we get from (7) and (9) that

$$\langle \varphi_0', 1 \rangle_{V^\vee(\mathbb{F}_q)} \tag{10a}$$

$$= C q^{(-m+r)/2} \left\{ \sum_{v^\vee \in O_1^\vee(\mathbb{F}_q)/G(\mathbb{F}_q)} \left(\frac{|G(\mathbb{F}_q)|}{|G_{v^\vee}(\mathbb{F}_q)|} \right)_{q \mapsto q^{-1}} \right\}$$

$$\times q^{m+(m-r(v^\vee))/2} \tag{10b}$$

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$$= Cq^{(r-r(v^\vee))/2} q^m \{|O_1^\vee(\mathbb{F}_q)|_{q \mapsto q^{-1}}\} \tag{10c}$$

$$= Cq^{(r-r(v^\vee))/2} q^m \{(q^{-n+m}|\Omega(\mathbb{F}_q)|)_{q \mapsto q^{-1}}\} \quad \text{by } (2.2,\ (1)) \tag{10d}$$

$$= Cq^{(r-r(v^\vee))/2} q^n \operatorname{tr}(\mathrm{Frob}_q, R\Gamma_c(\Omega \otimes \overline{\mathbb{F}_q}, \overline{\mathbb{Q}_\ell}))_{q \mapsto q^{-1}} \tag{10e}$$

$$= Cq^{(r-r(v^\vee))/2} \operatorname{tr}(\mathrm{Frob}_q, R\Gamma(\Omega \otimes \overline{\mathbb{F}_q}, \overline{\mathbb{Q}_\ell})) \text{ by the Poincaré duality.} \tag{10f}$$

(Recall that $\Omega = V \backslash f^{-1}(0)$.) Put

$$\varphi_0(v) := \operatorname{tr}(\mathrm{Frob}_q, (Rj_*\overline{\mathbb{Q}_\ell})_v) = (-1)^n q^n \operatorname{tr}(\mathrm{Frob}_q, (Rj_*\overline{\mathbb{Q}_\ell}(n)[n])_v)$$

for $v \in V(\mathbb{F}_q)$. Thus by [Lau1, (1.2.1.2)], $\varphi_0'(v^\vee) = q^{-n} \sum_{v \in V(\mathbb{F}_q)} \varphi_0(v)\psi(\langle v^\vee, v\rangle)$, and hence

$$\langle \varphi_0', 1\rangle_{V^\vee(\mathbb{F}_q)} = \varphi_0(0) = \operatorname{tr}(\mathrm{Frob}_q, (Rj_*\overline{\mathbb{Q}_\ell})_0). \tag{11}$$

Because of (10) and (11), in order to prove $C = 1$, it suffices to show that

$$\operatorname{tr}(\mathrm{Frob}_q, R\Gamma(V \otimes \overline{\mathbb{F}_q} \backslash f^{-1}(0), \overline{\mathbb{Q}_l})) = \operatorname{tr}(\mathrm{Frob}_q, (Rj_*\overline{\mathbb{Q}_l})_0), \tag{12}$$

and that

$$\sum_{v^\vee \in O_1^\vee(\mathbb{F}_q)} (-1)^{s(v^\vee)} \neq 0 \tag{13}$$

(i.e., (7), (10) and (11) are non-zero). Here recall that $C = \pm 1$. The proof of (12) and (13) will be given later in (6.5) and (6.6). Note that we have obtained the following theorem in the same time, which was originally conjectured by N. Kawanaka [Kaw2, (3.4.7), (ii)], [GyoKaw, 3, Remark].

6.3. THEOREM. *The number r of integer roots of $b(s)$ (counting multiplicity) equals $r(v^\vee)$ for any $v^\vee \in O_1^\vee$.*

(Our argument to obtain (6.3) would seem to depend on (6.2, (5)), but in reality it does not. In fact, if we do not use (6.2, (5)), we should replace C by $(-1)^{r+r(v^\vee)}C$ everywhere in (6.2, (6)–(10)), but it does not affect the absolute values of (10) and (11).)

6.3.1. REMARK. By [Gyo5], the most important case would be the case where $\dim G = \dim V$. In this case, we can construct a special type of relative invariant following [SatKim, Sect. 4, Prop. 16] as follows. Fix linear bases of $\mathfrak{g} = \mathrm{Lie}(G)$

and V, and put $f(v) := \det(\mathfrak{g} \to V; A \mapsto Av)$ for $v \in V$, and $\phi_0(g) := \det(V \to V; v \mapsto gv)$ for $g \in G$. Then

$$\begin{aligned} f(gv) &= \det(\mathfrak{g} \to V; A \mapsto Agv) \\ &= \det(\mathfrak{g} \to \mathfrak{g}; A \mapsto g^{-1}Ag) \cdot \det(\mathfrak{g} \to V; A \mapsto Av) \\ &\quad \cdot \det(V \to V; v \mapsto gv) \\ &= 1 \cdot f(v) \cdot \phi_0(g). \end{aligned}$$

Cf. (6.1.7, (1)). (For example, if (G, ρ, V) is irreducible, then such a relative invariant is irreducible, and every relative invariant can be obtained as a scalar multiple of a power of f.) By the definition of f, $O_0 = V \backslash f^{-1}(0)$, and especially $O_0 = \Omega = O_1$. (See (1.1) and (1.2) for notation.) Hence, in [Gyo1, (3.11, (5))], the defining equations of Du''_α become

$$\Leftrightarrow A u''_\alpha = (\alpha + 1)\phi_0(A) u''_\alpha \quad \text{for all } A \in \mathfrak{g}.$$

Hence $(Du''_\alpha)[f^{\vee -1}] = (Df^{\vee -\alpha -1})[f^{\vee -1}]$, and by [Gyo1, (3.11, (5))] again, we get

$$\mathcal{F}(Df^{\alpha})[f^{\vee -1}] \overset{\cong}{\Leftrightarrow\!\!\to} (Df^{\vee -\alpha -1})[f^{\vee -1}], \qquad \mathcal{F}(f^{\alpha}) \mapsto f^{\vee -\alpha -1}. \tag{1}$$

By [Gyo1, (3.1)],

$$f(\text{grad})(f^{\vee s+1}\mathcal{F}(f^{-1})) = (\Leftrightarrow 1)^d b(\Leftrightarrow s \Leftrightarrow 2) f^{\vee s}\mathcal{F}(f^{-1}). \tag{2}$$

By (1) and (2), we get

$$f(\text{grad}) f^{\vee s+1} = (\Leftrightarrow 1)^d b(\Leftrightarrow s \Leftrightarrow 2) f^{\vee s}, \quad \text{i.e.,}$$

$$b(s) = (\Leftrightarrow 1)^d b(\Leftrightarrow s \Leftrightarrow 2). \tag{3}$$

Cf. [SatM, Chap. 2, Thm. 4, (ii)]. Since $b(s) = b_0 \prod_{j=1}^{d}(s + \alpha_j)$ with $b_0 \in \mathbb{C}^\times$ and $\alpha_j \in \mathbb{Q}_{>0}$, (3) implies that $0 < \alpha_j < 2$. Hence, in this special case, (6.3) implies that

$$\#\{j | \alpha_j = 1\} = \operatorname{rank} G \Leftrightarrow \operatorname{rank} G_{v_1} = \operatorname{rank} G \quad \text{for } v_1 \in O_1. \tag{4}$$

Now, regard the b-function as a kind of ζ-function, and (3) as the functional equation satisfied by $b(s)$. (This standpoint would be justifiable by the resemblance of (3) to the usual functional equation of ζ-functions, and also by the deep relation of $b(s)$ with the ζ-functions of the prehomogeneous vector space. In fact, (1) is the D-module version of the functional equation of the ζ-function in the sense of M. Sato and T. Shintani.) Since f^{-s}, instead of f^{+s}, is used in the definition of ζ-function, let us consider $\xi(s) := b(-s)$. Then

$$\xi(s) = (-1)^d \xi(2 - s), \tag{3'}$$

and the reflection point of (3′) (i.e., the fixed point of $s \leftrightarrow 2 - s$) is 1. Moreover, (4) can be read as

$$\begin{aligned}&(\text{the order of zero of } \xi(s) \text{ at } s = 1)\\ &\quad = \operatorname{rank} G - \operatorname{rank} G_{v_1}\\ &\quad =: (\text{rank of the prehomogeneous vector space}).\end{aligned} \tag{4'}$$

It is amusing to note the resemblance between $(3') + (4')$ and the famous conjecture of B. Birch and H. P. F. Swinnerton–Dyer, which says the order of zero at $s = 1$ of the zeta function of an elliptic curve, say E, over $\mathbb{Q}$ would be equal to the rank of $E(\mathbb{Q})$.

6.4. Justification of ‘$q \mapsto q^{-1}$’

6.4.1. First, we need to prepare some notation. For any set X on which σ acts, X^σ denotes the set of σ-fixed points. (Recall that $\sigma = \mathrm{Frob}_q^{-1}$.) Put $W := (\mathcal{B}(G) \times \mathcal{B}(G))/G$ (= the Weyl group of G). In order to specify the dependence of the split rank $s(G)$ on the $\mathbb{F}_q$-structure, we sometimes write $s(G, \sigma)$ for $s(G)$. Fix $v^\vee \in O_1^\vee(\mathbb{F}_q)$, recall that $G^0_{v^\vee}$ is the identity component of $G_{v^\vee}$, and put $\pi_0(G_{v^\vee}) := G_{v^\vee}/G^0_{v^\vee}$. For a group Γ on which σ acts as an automorphism, let $H^1(\sigma, \Gamma)$ denote the quotient set of Γ by the equivalence relation

$$a \sim b \Leftrightarrow \text{‘}a = c^{-1} \cdot b \cdot \sigma(c) \text{ for some } c \in \Gamma\text{’}$$

for any $a, b \in \Gamma$.

6.4.2. Next, let us review the (natural) one-to-one correspondence between $O_1^\vee(\mathbb{F}_q)/G(\mathbb{F}_q)$ and $H^1 := H^1(\sigma, \pi_0(G_{v^\vee}))$. For $c \in G_{v^\vee}$, let $[c]$ denote its class in H^1. Let $c_0 = 1, c_1, \ldots, c_l \in G_{v^\vee}$ be a complete set of representatives of H^1. Take $g_i \in G$ so that $g_i^{-1} \cdot \sigma(g_i) = c_i$ [SprSte, I, 2.2]. Then $\{v_i^\vee := g_i v^\vee \mid 0 \leqslant i \leqslant l\}$ is a complete set of representatives of $O_1^\vee(\mathbb{F}_q)/G(\mathbb{F}_q)$ [SprSte, I, 2.7].

6.4.3. Now we explain the justification concerning ‘$q \mapsto q^{-1}$’. Put

$$\varphi(t; G, \sigma) := (-1)^{s(G,\sigma)} \sum_{w \in W^\sigma} t^{l(w)+N} \cdot \det(1 - tq^{-1}\sigma^* \mid X \otimes \mathbb{Q}),$$

where N is the number of positive roots and $l(w)$ is the length of $w \in W$, especially $l(w) + N$ is the dimension of the G-orbit $w \subset \mathcal{B}(G) \times \mathcal{B}(G)$. See (6.1.6, (1)) for $X = X(G)$. Then

$$|G(\mathbb{F}_q)| = \varphi(q; G, \sigma) \quad (\text{cf. [Ste, 11.10 and 11.11]}).$$

This follows from the Bruhat decomposition and the fact that for a maximal torus T of G defined over $\mathbb{F}_q$, we have

$$|T(\mathbb{F}_q)| = |\det(1 \Leftrightarrow \sigma^*|X \otimes \mathbb{Q})| = (\Leftrightarrow 1)^{s(G)}\det(1 \Leftrightarrow \sigma^*|X \otimes \mathbb{Q}),$$

the first equality following from the duality between T and X, and the second from the argument which gave (6.1.6, (3)). Put $G_i := G_{v_i^\vee}$, $\mathcal{B}_i := \mathcal{B}(G_i)$, $X_i := X(G_i)$, $W_i := \mathcal{B}_i \times \mathcal{B}_i/G_i$, and

$$\psi(t;\sigma) := \sum_{i=0}^{l} \frac{1}{|\pi_0(G_i)^\sigma|} \cdot \frac{\varphi(t;G,\sigma)}{\varphi(t;G_i^0,\sigma)}.$$

(A priori, we only know that $\psi(t;\sigma) \in \mathbb{Q}(t)$, but after (8) and (11) below, we can easily see that $\psi(t;\sigma) \in \mathbb{Q}[t]$, and therefore we can substitute any number for t.) Then by [SprSte, I, 2.11],

$$|O_1^\vee(\mathbb{F}_q)| = \psi(q;\sigma).$$

Take $M \in \mathbb{N}_0$ $(= \{1, 2, \ldots\})$ so that the following conditions are satisfied

$$G \text{ and } G_i \ (0 \leqslant i \leqslant l) \text{ split over } \mathbb{F}_{q^M}, \tag{1}$$

$$\sigma^M \text{ acts trivially on } W \text{ and } W_i \ (0 \leqslant i \leqslant l), \tag{2}$$

$$\sigma^M\ (g_i) = g_i\ (0 \leqslant i \leqslant l), \tag{3}$$

$$\sigma^M \text{ acts trivially on } \pi_0(G_i)\ (0 \leqslant i \leqslant l). \tag{4}$$

By (1), $\sigma^{*M} = q^M$ $(\in \operatorname{End}(X))$. Hence

$$\det(1 \Leftrightarrow tq^{-Mk-1}\sigma^{*Mk+1}|X \otimes \mathbb{Q}) = \det(1 \Leftrightarrow tq^{-1}\sigma^*|X \otimes \mathbb{Q}) \tag{5}$$

for any $k \in \mathbb{N}$ $(= \{0, 1, 2, \ldots\})$. Since, by (6.1.6, (3)), $(\Leftrightarrow 1)^{s(G,\sigma)}$ is the leading coefficient of $\det(1 \Leftrightarrow tq^{-1}\sigma^*) \in \mathbb{Q}[t]$,

$$(\Leftrightarrow 1)^{s(G,\sigma)} = (\Leftrightarrow 1)^{s(G,\sigma^{Mk+1})}. \tag{6}$$

By (2),

$$W^\sigma = W^{\sigma^{Mk+1}}. \tag{7}$$

By (5)–(7),

$$\varphi(t;G,\sigma) = \varphi(t;G,\sigma^{Mk+1}). \tag{8}$$

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Hence

$$|G(\mathbb{F}_{q^{Mk+1}})| = \varphi(q^{Mk+1}; G, \sigma^{Mk+1}) = \varphi(q^{Mk+1}; G, \sigma). \tag{9}$$

By (3), $g_i^{-1} \cdot \sigma^{Mk+1}(g_i) = g_i^{-1} \cdot \sigma(g_i) = c_i$. Hence

$$\begin{aligned}&\{v_i^\vee | 0 \leqslant i \leqslant l\} \text{ is also a complete set of representatives}\\ &\quad \text{of } O_1^\vee(\mathbb{F}_{q^{Mk+1}})/G(\mathbb{F}_{q^{Mk+1}}).\end{aligned} \tag{10}$$

Applying (8) to G_i^0, we get

$$\varphi(t; G_i^0, \sigma) = \varphi(t; G_i^0, \sigma^{Mk+1}) \quad (0 \leqslant i \leqslant l). \tag{11}$$

By (4), (8), (10) and (11), we get

$$\psi(t; \sigma) = \psi(t; \sigma^{Mk+1}). \tag{12}$$

Hence

$$|O_1^\vee(\mathbb{F}_{q^{Mk+1}})| = \psi(q^{Mk+1}; \sigma^{Mk+1}) = \psi(q^{Mk+1}; \sigma). \tag{13}$$

The precise meaning of the left side of (6.2, (9)) is $\varphi(q^{-1}; G, \sigma)$. The meaning of the inside of { } in (6.2, (10b) and (10c)) is $\psi(q^{-1}; \sigma)$. Since

$$\begin{aligned}|\Omega(\mathbb{F}_{q^{Mk+1}})| &= (q^{Mk+1})^{n-m} |O_1^\vee(\mathbb{F}_{q^{Mk+1}})|\\ &= (q^{Mk+1})^{n-m} \psi(q^{Mk+1}; \sigma),\end{aligned} \tag{14}$$

($n = \dim V$, $m = \dim O_1^\vee$) by (2.2, (1)), we can understand the meaning of (6.2, (10d)) similarly as above; the inside of { } means $\psi(q^{-1}; \sigma)$ again. As for (6.2, (10e)), we need to understand it in two ways. On the one hand, we understand it as

$$\mathrm{tr}(\mathrm{Frob}_q, R\Gamma_c(\Omega, \overline{\mathbb{Q}_l}))|_{q \mapsto q^{-1}} = q^{-(n-m)} \psi(q^{-1}; \sigma). \tag{15}$$

On the other hand, in order to get '(10e) = (10f)', we need to understand it as

$$\mathrm{tr}(\mathrm{Frob}_q, R\Gamma_c(\Omega, \overline{\mathbb{Q}_l}))|_{q \mapsto q^{-1}} = \sum \alpha_i^{-1} \Leftrightarrow \sum \beta_j^{-1}, \tag{16}$$

where the α_i's (resp. β_j's) are the eigenvalues of Frob_q on $H_c^{\text{even}}(\Omega, \overline{\mathbb{Q}_l})$ (resp. H_c^{odd}) including multiplicities. Then our task is to show that (15) is equal to (16). By (14),

$$\sum \alpha_i^{Mk+1} \Leftrightarrow \sum \beta_j^{Mk+1} = (q^{n-m} \psi(q; \sigma))|_{q \mapsto q^{Mk+1}},$$

and our task is to show that

$$\sum \alpha_i^{-1} \Leftrightarrow \sum \beta_j^{-1} = (q^{n-m}\psi(q;\sigma))|_{q\mapsto q^{-1}}.$$

Thus it is enough to prove the following lemma.

6.4.4. LEMMA. *Let* $c_i \in \mathbb{R}$, $0 \neq \alpha_i \in \mathbb{C}$ $(1 \leqslant i \leqslant N)$, $M \in \mathbb{N}_0$, *and assume that*

$$\sum_{i=1}^{N} c_i \alpha_i^{Mk+1} = 0 \quad \textit{for any } k \in \mathbb{N}. \tag{1}$$

Then

$$\sum_{i=1}^{N} c_i \alpha_i^{-1} = 0. \tag{2}$$

Proof. Express (1) as $\sum_{i=1}^{N} c_i\alpha_i(\alpha_i^M)^k = 0$. Considering the Vandermonde determinant, we get

$$\sum_{\alpha_i^M=\beta} c_i\alpha_i = 0 \quad \text{for any } 0 \neq \beta \in \mathbb{C}. \tag{3}$$

Put $\gamma_i := \alpha_i|\beta|^{-1/M}$. Then $|\gamma_i| = 1$ and $\overline{\gamma_i} = \gamma_i^{-1}$. Considering

$$(\text{complex conjugate of } ((3) \times |\beta|^{-1/M})) \times |\beta|^{-1/M},$$

we get (2). □

6.5. PROOF OF (6.2, (13))

We use the notation of (6.4). Since $O_1^\vee$ is absolutely irreducible, we have $\lim_{q\to\infty} q^{-m}|\, O_1^\vee(\mathbb{F}_q)| = 1$, and similarly for G and G_i^0. Considering (6.4.3, (13)) for $k \to \infty$, we see that

$$\sum_{i=0}^{l} \frac{1}{|\pi_0(G_i)^\sigma|} = 1. \tag{1}$$

From (6.2, (8)), we get

$$q^{-N}|G(\mathbb{F}_q)| \equiv (\Leftrightarrow 1)^{s(G)} \operatorname{mod} q\mathbb{Z}_p, \tag{2}$$

where $p = \operatorname{char}(\mathbb{F}_q)$, and similarly for G_i^0. Suppose now that q does not divide $|\pi_0(G_{v^\vee})|$ for one (and hence for all) $v^\vee \in O_1^\vee$. (We may assume this because $p \gg$

0.) Put $n_i := |\pi_0(G_i)^\sigma|$ and $A_i := (-1)^{s(v_i)}(q^{-N}|G(\mathbb{F}_q)|) \times (q^{-N_0}|G_i^0(\mathbb{F}_q)|)^{-1}$, where N (resp. N_0) is the number of positive roots of G (resp. G_v). Then q does not divide n_i, $A_i = 1 + qa_i$ with some $a_i \in \mathbb{Z}_p$ by (2), and

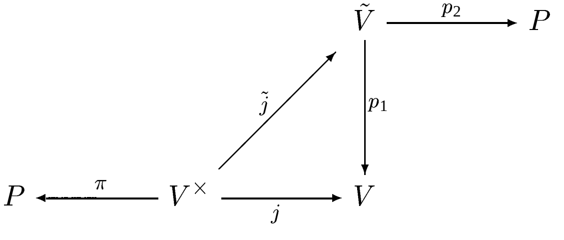
$$\begin{aligned} & q^{-(N-N_0)} \sum_{v^\vee \in O_1^\vee(\mathbb{F}_q)} (-1)^{s(v^\vee)} \\ & = \sum_{i=1}^{l} \frac{A_i}{n_i} \quad \text{by } (6.2,\ (7)) \text{ and } (6.4.2) \\ & = \sum_{i=1}^{l} \frac{1}{n_i} + \sum_{i=1}^{l} \frac{q}{n_i} a_i \equiv 1 \bmod p\mathbb{Z}_p \quad \text{by } (1). \end{aligned}$$

Now the proof of (6.2, (13)) is complete. □

6.6. PROOF OF (6.2, (12))

In this paragraph, the notation is not compatible with the remainder of Section 6. The content must be well known, but is included for the sake of convenience of the readers.

6.6.1. Let $V := \mathbb{A}^n_{\overline{\mathbb{F}_q}}$, $V^\times := V \backslash \{0\}$, P be the projective space consisting of lines passing through $0 \in V$, $\tilde{V}$ the blow-up of V with center $\{0\}$, i.e., $\tilde{V} := \{(v, L) \in V \times P | v \in L\}$, and



the natural morphisms. Then (6.2, (12)) follows immediately from the next lemma.

6.6.2. LEMMA. *Assume the above notation and let $K \in D_c^b(V^\times, \overline{\mathbb{Q}_l})$. If $j_! K$ is χ-homogeneous for some $\chi \in \mathrm{Hom}(\mathbb{F}_q^\times, \overline{\mathbb{Q}_l}^\times)$, then the natural morphism*

$$R\Gamma(V^\times, K) = R\Gamma(V, Rj_*K) \to R\Gamma(\{0\}, Rj_*K) = (Rj_*K)_0$$

is an isomorphism.

Proof. First consider the case where $n = 1$ and $K = \overline{\mathbb{Q}_l}$. In this case, we can get the result, comparing the spectral sequences $E_2^{rs} = H^r(X, R^s j_* \overline{\mathbb{Q}_l})$ for $X = V$

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and $X = \{0\}$, and noting that $j_*\overline{\mathbb{Q}_l} = \overline{\mathbb{Q}_l}$ and that supp $R^s j_*\overline{\mathbb{Q}_l} \subset \{0\}$ $(s > 0)$. Second, if $n = 1$ and $K = L_\chi$ with $\chi \neq 1$, then both members vanish.

The general case can be reduced to these two cases as follows. By the proper base change theorem, we have

$$(Rj_*K)_0 = (R(p_1)_*R\tilde{j}_*K)_0 = R\Gamma(p_1^{-1}(0), R\tilde{j}_*K|_{p^{-1}(0)}).$$

On the other hand, we have

$$R\Gamma(V^\times, K) = R\Gamma(P, R(p_2)_*R\tilde{j}_*K).$$

Hence it suffices to prove that the natural morphism

$$R(p_2)_*R\tilde{j}_*K \to R\tilde{j}_*K|_{p_1^{-1}(0)} \tag{1}$$

is an isomorphism, where $p_1^{-1}(0)$ is identified with P via p_2. Since $p_2 : \tilde{V} \to P$ is a line bundle, we may identify

$$V^\times \overset{\tilde{j}}{\Leftrightarrow} \tilde{V} \overset{p_2}{\Leftrightarrow} P, \quad p_1^{-1}(0), \quad \text{and } K \text{ on } V^\times$$

with

$$\mathbb{G}_m \times P \overset{\tilde{j}}{\Leftrightarrow} \mathbb{A}^1 \times P \overset{p_2}{\Leftrightarrow} P, \quad \{0\} \times P, \quad \text{and}$$

$$K \cong L_\chi \boxtimes s^*K \text{ on } \mathbb{G}_m \times P,$$

locally on P. Here s is a local section of $V^\times \overset{p_2\tilde{j}}{\Leftrightarrow} P$. Thus the proof of (1) reduces to the first two cases. □

7. Proof of Theorem A2

In this section, we show that Theorem A2 can be obtained from Theorem A1.

7.1. We keep the notation and the conventions of (5.2.0). We record Theorem A1† in (5.2.3.2) in the following form

$$\mathcal{F}_\psi(j_!L(\chi^{-1}(Ef(\ )^{-1}))(n)[n])$$
$$= \tau_\chi \otimes Rj_*^\vee i_*^\vee(L(\chi(f^\vee(\ )^{-1})) \otimes L(\kappa^\vee))[m] \tag{1$_\chi$}$$

on $V^\vee$.

7.2. From (3.5.3, (4)) and (5.2.3.1, (5) and (6)), we can show that there exists a constant $C = C(\chi) \neq 0$ such that

$$q^{-m} \sum_{v^\vee \in O_1^\vee(\mathbb{F}_q)} \chi(f^\vee(v^\vee))\psi(\langle v^\vee, v\rangle)$$
$$= C \cdot (\Leftrightarrow 1)^m \tau_\chi \cdot \chi(Ef(v)^{-1}) \cdot \kappa^\vee(F(v)) \tag{1}$$

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for $v \in \Omega(\mathbb{F}_q)$, and

$$\mathcal{F}_\psi(j_!^\vee i_*^\vee L(\chi(f^\vee))(m)[m]) \cong C\tau_\chi \otimes L(\chi(Ef(\,)^{-1})) \otimes F^* L(\kappa^\vee)[n] \qquad (2)$$

on Ω. We can extend the isomorphism (2) to the whole space V in a similar fashion as (5.2.2.2), using (3.5.3, (4)):

$$\begin{aligned}&\mathcal{F}_\psi(j_!^\vee i_*^\vee L(\chi(f^\vee))(m)[m])\\ &\quad = C\tau_\chi \otimes Rj_*(L(\chi(Ef(\,)^{-1})) \otimes F^* L(\kappa^\vee))[n] \qquad (3)\end{aligned}$$

on V. Our purpose is to prove that $C = 1$. Applying $\mathcal{F}_{\bar{\psi}}$ to both members of (3), and changing $\chi \to \chi'$, we get by [Lau1, 1.2.2.1]

$$\begin{aligned}&C\tau_{\chi'} \otimes \mathcal{F}_{\bar{\psi}}(Rj_*(L(\chi'(Ef(\,)^{-1})) \otimes F^* L(\kappa^\vee))(n)[n])\\ &\quad = j_!^\vee i_*^\vee L(\chi'(f^\vee))(m)[m] \qquad (4)_{\chi'}\end{aligned}$$

on $V^\vee$.

7.3. Now consider the $\overline{\mathbb{Q}_\ell}$-valued functions on $V^\vee$ obtained from (7.1, $(1)_\chi$) and (7.2, $(4)_{\chi'}$) by taking the trace of Frob_q. Next consider their L^2-inner product using the fact that the Fourier transformation preserves the L^2-inner product. Then we get

$$\begin{aligned}&C\tau_{\chi'} q^{-n} \sum_{v \in \Omega(\mathbb{F}_q)} (\chi^{-1}\chi')(Ef(v)^{-1})\kappa^\vee(F(v))\\ &\quad = \tau_\chi q^{-m} \sum_{v^\vee \in O_1^\vee(\mathbb{F}_q)} (\chi^{-1}\chi')(f^\vee(v^\vee))\kappa^\vee(v^\vee). \qquad (1)\end{aligned}$$

(Remark. In (1), except for the factor $\tau_\chi/\tau_{\chi'}$, everything depends only on $\chi^{-1}\chi'$. So, it might seem strange at first glance, but for almost all $\chi^{-1}\chi'$, both sums vanish, and it is not absurd.) In particular, taking $\chi = \chi'$, we get

$$Cq^{-n+m} \sum_{v \in \Omega(\mathbb{F}_q)} \kappa^\vee(F(v)) = \sum_{v^\vee \in O_1^\vee(\mathbb{F}_q)} \kappa^\vee(v^\vee). \qquad (2)$$

By (2.2, (1)), (2) can be written as

$$(1 \Leftrightarrow C) \sum_{v^\vee \in O_1^\vee(\mathbb{F}_q)} \kappa^\vee(v^\vee) = 0. \qquad (3)$$

By Theorem B, which is proved already in Section 6, we have $\kappa^\vee(v^\vee) = (\Leftrightarrow 1)^{r(v^\vee)-s(v^\vee)}$ for $v^\vee \in O_1^\vee(\mathbb{F}_q)$. Hence (3) can be also written as

$$(1 \Leftrightarrow C) \sum_{v^\vee \in O_1^\vee(\mathbb{F}_q)} (\Leftrightarrow 1)^{s(v^\vee)} = 0. \qquad (4)$$

Hence by (6.2, (13)), we get $C = 1$, and we have proved Theorem A2. □

7.4. We record (7.2, (3)) (with $C = 1$) as a theorem, which refines both (3.5.3, (4)) and Theorem A2.

THEOREM A2†. *Assume the notation of* (3.1.2) *and* (5.2.3.1, (5)). *If the characteristic of* $\mathbb{F}_q$ *is sufficiently large, then we have for all* $\chi \in \mathrm{Hom}(\mathbb{F}_q^\times, \overline{\mathbb{Q}_\ell}^\times)$ *that*

$$\begin{aligned}
&\mathcal{F}_\psi(j_!^\vee i_*^\vee L(\chi(f^\vee))(m)[m]) \\
&\quad \cong (-1)^m q^{-(m+r)/2} \prod_{j\geqslant 1} G(\chi^j, \psi)^{e(j)} \\
&\qquad \otimes Rj_*\left\{L\left(\chi\left(\frac{b_0}{\prod_{j\geqslant 1}(j^j)^{e(j)}} f(\,)^{-1}\right)\right) \otimes F^* L(\kappa^\vee)\right\}[n] \quad \text{on } V. \qquad (1)
\end{aligned}$$

7.5. REMARK. The character sum in Theorem A2 can also be calculated differently, following the method used to determine the sum $S_h^\vee(\chi, v^\vee)$ in (5.2.1). For this, one has to replace in the statement of Lemma (4.3.3) and Proposition (4.3.5) the sheaf $L(\omega)$ by the constant sheaf $\mathbb{C}_{O_1}$ on O_1, and the Bernstein polynomial $b(s)$ by $b(s, F^{\vee *}\delta_\omega)$ which is defined in [Gyo3, (6.11.3) and (6.14)]. Moreover in the proof of Lemma (4.3.3), one has to replace $f^{\vee\alpha}$ by $f^{\vee\alpha+k}F^{\vee *}\delta_\omega$, with $k \in \mathbb{N}$ big enough, and use [Gyo3, (6.21) and (6.19)] instead of [Gyo1, (3.23) and (3.1)]. An argument as in (5.2.1), with $L(\omega)$ replaced by $\mathbb{C}_{O_1}$, then yields an expression for $\sum_{v^\vee \in O_1^\vee(\mathbb{F}_q)} \chi(f^\vee(v^\vee))\psi(\langle v^\vee, v\rangle)$ involving $b(s, F^*\delta_{\omega^\vee})$. Comparing with Theorem A2, we conclude that *the roots* mod $\mathbb{Z}$ *of* $b(s, F^*\delta_{\omega^\vee})$ *are the same as the roots* mod $\mathbb{Z}$ *of* $b(s)$.

7.6. REMARK. Assume the characteristic of $\mathbb{F}_q$ is sufficiently large and let $\chi \in \mathrm{Hom}(\mathbb{F}_q^\times, \mathbb{C}^\times)$. Suppose that

$$(\text{order of } \chi)^{-1} \not\equiv \alpha_j \bmod \mathbb{Z}, \qquad (1)$$

for each α_j in (1.2, (4)). *Then the character sum in Theorem A2 vanishes for* $v \in (V\backslash\Omega)(\mathbb{F}_q)$. The proof is the same as in Remark (5.2.3.3). Actually, even more is true: *If* (1) *holds, then* $\mathcal{F}_\psi(j_!^\vee i_*^\vee L(\chi(f^\vee)))$ *is zero on* $V\backslash\Omega$. This is a direct consequence of Theorem (3.5.3, (4)) and the fact that $Rj_*(f^*L_\chi \otimes F^*L(\omega^\vee)) = j_!(f^*L_\chi \otimes F^*L(\omega^\vee))$ which follows from [Gyo3, (6.21, (1) and (2))] (with L and u_0 as in loc. cit. (6.11.3)). Indeed if $\alpha \in \mathbb{Q}$ has order in $\mathbb{Q}/\mathbb{Z}$ equal to the order of χ, then (1) and Remark (7.5) imply that $\alpha \in A_+ \cap A_-$ with the notation of loc. cit. and hence $Rj_*(\mathbb{C}f^\alpha \otimes F^*\mathbb{C}\omega^\vee) = j_!(\mathbb{C}f^\alpha \otimes F^*\mathbb{C}\,\omega^\vee)$.

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8. Frobenius determinant and the Hessian

8.1. FURTHER APPLICATION OF LAUMON'S PRODUCT FORMULA

8.1.1. Let X be a scheme of finite type over $\mathbb{F}_q$, $f : X \to \mathbb{P}^1_{\mathbb{F}_q}$ a morphism, and $K \in D^b_c(X, \overline{\mathbb{Q}_\ell})$. Let $u \in |X|$ be a closed point of X. We denote by $\Phi_{f,u}(K)$ the stalk at u of the complex of vanishing cycles on $f^{-1}(f(u))$ associated to f and K, cf. [Del1]. Note that $\Phi_{f,u}(K)$ is a $G_{f(u)}$ module (in the sense of (3.1.1) with $s = f(u)$), hence we can consider the local constant $\varepsilon_{\psi,0}(T_{f(u)}, \Phi_{f,u}(K), \omega)$ introduced in (3.1.4), for any rational differential form $\omega \neq 0$ on $\mathbb{P}^1_{\mathbb{F}_q}$. The following lemma is a direct consequence of Laumon's Product Formula (3.1.5).

8.1.2. LEMMA. *Let X be a scheme, $f : X \to \mathbb{G}_{m,\mathbb{F}_q}$ a proper morphism and $K \in D^b_c(X, \overline{\mathbb{Q}_\ell})$. Suppose that $R^i f_! K$ has tame ramification at* 0 *and* ∞ *for all i, and that*

$$[(Rf_!K)_{\bar\eta_0}] = \sum_{\substack{N \in \mathbf{N} \\ (N,q)=1}} \alpha_N [V_N], \quad \textit{and} \quad [(Rf_!K)_{\bar\eta_\infty}] = \sum_{\substack{N \in \mathbf{N} \\ (N,q)=1}} \beta_N [V_N], \tag{1}$$

where the integers α_N, β_N are zero for almost all N. Moreover suppose that $\Phi_{f,u}(K)$ is zero for all $u \in |X|$ outside a finite subset Σ of $|X|$. Then

$$\begin{aligned} \varepsilon_0(X, K) \;=\; & q^{\sum_N N\alpha_N} \prod_N (\varepsilon_{\psi,0}(T_0, V_N, x^{-1}\,\mathrm{d}x)^{\alpha_N} \\ & \times \varepsilon_{\psi,0}(T_0, V_N, \Leftrightarrow x^{-1}\,\mathrm{d}x)^{\beta_N}) \\ & \times \prod_{u \in \Sigma} \varepsilon_{\psi,0}(T_{f(u)}, \Phi_{f,u}(K), x^{-1}\,\mathrm{d}x). \end{aligned} \tag{2}$$

(See (3.1.3) for the definition of ε_0, and (3.1.1) for V_N).

Proof. For any $s \in |\mathbb{G}_{m,\mathbb{F}_q}|$ we have a distinguished triangle

$$(Rf_!K)_s \to (Rf_!K)_{\bar\eta_s} \to \bigoplus_{\substack{u \in \Sigma \\ f(u)=s}} \Phi_{f,u}(K) \overset{+1}{\Leftrightarrow} \tag{3}$$

of G_s-modules, cf. [Del1, (2.1.2.4)]. Hence

$$\begin{aligned} & \varepsilon_{\psi,0}(T_s, Rf_!K, \omega) \\ & \quad = \varepsilon_{\psi,0}(T_s, (Rf_!K)_s, \omega) \prod_{\substack{u \in \Sigma \\ f(u)=s}} \varepsilon_{\psi,0}(T_s, \Phi_{f,u}(K), \omega), \quad \text{i.e.,} \end{aligned}$$

$$\begin{aligned} & \varepsilon_{\psi}(T_s, Rf_!K, \omega) \\ & \quad = \varepsilon_{\psi}(T_s, (Rf_!K)_s, \omega) \prod_{\substack{u \in \Sigma \\ f(u)=s}} \varepsilon_{\psi,0}(T_s, \Phi_{f,u}(K), \omega). \end{aligned}$$

Thus if ω has no pole and no zero at s, then

$$\varepsilon_\psi(T_s, Rf_!K, \omega) = \prod_{\substack{u\in\Sigma \\ f(u)=s}} \varepsilon_{\psi,0}(T_s, \Phi_{f,u}(K), \omega), \tag{4}$$

by [Lau1, (3.1.5.6)] with $K = \overline{\mathbb{Q}_\ell}$. Here note that the action of I_s on $(Rf_!K)_s$ is trivial by definition. Apply Laumon's Product Formula (3.1.5) to the differential $x^{-1}dx$ and the complex $j_!(Rf_!K)$, with j the immersion $\mathbb{G}_{m,\mathbb{F}_q} \to \mathbb{P}^1_{\mathbb{F}_q}$. The Lemma follows now directly from (3.1.4, (4)). □

The following lemma is a special case of Lemma 1 of [Sai1], see also the last three lines on page 401 of loc. cit. (Note however that the ε-factors in loc. cit. are with respect to a Haar measure which is q times the one used in [Lau1, 3.1.5.8], and hence differ from our ε-factors $\varepsilon_{\psi,0}(T_0, K, \omega)$ by a factor $q^{\mathrm{rank}\, K}$, cf. [Del5, 5.3].)

8.1.3. LEMMA. *Assume the notation of* (3.1.4) *with q odd, and let $N \in \mathbb{N}$ be coprime with q. Then*

$$\varepsilon_{\psi,0}(T_0, V_N, x^{-1}\,\mathrm{d}x) = \Leftrightarrow q^{-N} G(\chi_{1/2}, \psi)^{N-1} \chi_{1/2}(2^{N-1}N), \tag{1}$$

$$\varepsilon_{\psi,0}(T_0, V_N, \Leftrightarrow x^{-1}\,\mathrm{d}x) = \Leftrightarrow q^{-N} G(\chi_{1/2}, \psi)^{N-1} \chi_{1/2}((\Leftrightarrow 2)^{N-1}N). \tag{2}$$

(See (1.5) for the definition of $\chi_{1/2}$ and (3.1.2) for $G(\chi_{1/2}, \psi)$.)

REMARK. We briefly sketch a different proof of Lemma (8.1.3): As in [Sai1, p. 402 line 14–19] one easily reduces to the case N odd, by induction on $\mathrm{ord}_2 N$ and [Lau1, 3.1.5.4(iv)]. Thus suppose that N is odd. It is an elementary exercise to verify that det V_N is unramified and that det $(\mathrm{Frob}_q, (V_N)_1) = (\frac{q}{N})$, where $(\frac{q}{N})$ denotes the Jacobi symbol. Hence by [Lau1, 3.1.5.5] and the quadratic reciprocity law, it suffices to prove that $\varepsilon_{\psi,0}(T_0, V_N, \mathrm{d}x) = \Leftrightarrow q^{(N-1)/2}$. Let C be an irreducible component of V_N. Then C is 'induced' by a multiplicative character χ of $\mathbb{F}_{q^r}^\times$, for some r, see e.g. [Lau1, p. 198 line 5–10]. Let m be the order of χ. Then the order of q in $(\mathbb{Z}/m\mathbb{Z})^\times$ equals r, because C is irreducible. By loc. cit. we have $\varepsilon_{\psi,0}(T_0, C, \mathrm{d}x) = \chi(\Leftrightarrow 1)G_r(\chi, \psi)$, with $G_r(\chi, \psi)$ the Gauss sum over $\mathbb{F}_{q^r}$. Note that $\chi(\Leftrightarrow 1) = 1$, since m is odd. When the irreducible component $\overline{C}$ of V_N, 'induced' by χ^{-1}, does not coincide with C, then $C \oplus \overline{C}$ yields a contribution $G_r(\chi, \psi)G_r(\bar{\chi}, \psi) = q^r$. But if $C = \overline{C}$ and $m > 1$, then r is even, $q^{r/2} \equiv \Leftrightarrow 1$ mod m, and a result of Stickelberger (see [BerEva, (10.3)] or [BauMcE, last line of p. 165]) yields $G_r(\chi, \psi) = q^{r/2}$. Multiplying all these contributions (together with $\Leftrightarrow 1$ for the component V_1 of V_N) we obtain $\varepsilon_{\psi,0}(T_0, V_N, \mathrm{d}x) = \Leftrightarrow q^{(N-1)/2}$, when N is odd. □

8.1.4. For any smooth $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}$ on $\mathbb{G}_{m,\mathbb{F}_q}$, with tame ramification at 0 and ∞, we have

$$\varepsilon_{\psi,0}(T_0,\mathcal{F},\mathrm{d}x)=\varepsilon_0(\mathbb{G}_{m,\mathbb{F}_q},\mathcal{F}\otimes L_{\overline{\psi}}).$$

(See (3.1.2) for $L_{\overline{\psi}}$, recalling that $\overline{\psi}=\psi^{-1}$.) Indeed this follows by straightforward adaptation of the proof of (3.5.3.1) in [Lau1, p. 198]: We may assume that $\mathcal{F}$ is irreducible. In the notation of loc. cit., take a finite extension k_1 ($\subset\overline{\mathbb{F}_q}=:\overline{k}$) of $\mathbb{F}_q=:k$ and a smooth $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}_1$ of rank 1 on $\mathbb{G}_{m,k_1}$ such that $f_*\mathcal{F}_1=\mathcal{F}$. Here $f\colon\mathbb{G}_{m,k_1}\to\mathbb{G}_{m,k}$ is the morphism induced by $f\colon\mathrm{Spec}(k_1)\to\mathrm{Spec}(k)$. Then we have, with obvious notation, the following isomorphisms

$$\begin{aligned}&R\Gamma_c(\mathbb{G}_{m,k}\otimes_k\overline{k},\mathcal{F}\otimes L_{\overline{\psi}})\\&\quad\cong R\Gamma_c(\mathbb{G}_{m,k_1}\otimes_k\overline{k},\mathcal{F}_1\otimes f^*L_{\overline{\psi}})\\&\quad\cong f_*R\Gamma_c(\mathbb{G}_{m,k_1}\otimes_{k_1}\overline{k},\mathcal{F}_1\otimes L_{\overline{\psi}\circ\mathrm{Tr}}),\end{aligned}\tag{1}$$

which are compatible with the Frobenius action. Cf. [Del4, (1.7.7)]. Here Tr denotes the trace $k_1\to k$. Since $\mathcal{F}_1$ is isomorphic to a Kummer torsor on $\mathbb{G}_{m,\overline{k}}$ (cf. $\ell\ell$.20–23 of [Lau1, p. 198]), we can determine the rank of (1), and we get $\varepsilon_0(\mathbb{G}_{m,k},\mathcal{F}\otimes L_{\overline{\psi}})=\varepsilon_0(\mathbb{G}_{m,k_1},\mathcal{F}_1\otimes L_{\overline{\psi}\circ\mathrm{Tr}})$. Then the remaining adaptation is easy and hence omitted.

8.2. FROBENIUS DETERMINANT FOR A NON-DEGENERATE CRITICAL POINT

8.2.1. Let X be a scheme of finite type over $\mathbb{F}_q$, with q odd, $f: X\to\mathbb{A}^1_{\mathbb{F}_q}$ a morphism and $u\in X(\mathbb{F}_q)$. Suppose that X is smooth at u (over $\mathbb{F}_q$) and of dimension m. Choose a regular system of parameters $x_1,x_2,\ldots,x_m$ for $\mathcal{O}_{X,u}$. Assume that u is a non-degenerate critical point of f, meaning that u is a critical point and the Hessian

$$\Delta_u(f):=\det\left(\frac{\partial^2 f(u)}{\partial x_i\partial x_j}\right)_{i,j=1,\ldots,m}$$

is non-zero. Note that the image of $\Delta_u(f)$ in $\mathbb{F}_q^\times/\mathbb{F}_q^{\times 2}$ does not depend on the choice of $x_1,\ldots,x_m$. Finally let $K\in D^b_c(X,\overline{\mathbb{Q}_\ell})$ be smooth at u, meaning that the cohomology sheaves of K are smooth in a neighborhood of u.

The following lemma is implicit in [Sai2, proof of Lemma 7], and is an easy consequence of the material in [Del2].

8.2.2. LEMMA. *Assume the notation and hypothesis of* (8.2.1). *Then*

$$\varepsilon_{\psi,0}(T_{f(u)},\Phi_{f,u}(K),\mathrm{d}x)^{(-1)^{m-1}}$$

$$= \det(\mathrm{Frob}_q, K_u)((\Leftrightarrow 1)^{m-1}\chi_{1/2}((\Leftrightarrow 2)^m \Delta_u(f))G(\chi_{1/2},\psi)^m)^{\mathrm{rank}(K_u)} \quad (1)$$

and when $f(u) \neq 0$ *we also have*

$$\varepsilon_{\psi,0}(T_{f(u)}, \Phi_{f,u}(K), x^{-1}\,\mathrm{d}x)^{(-1)^{m-1}}$$

$$= \det(\mathrm{Frob}_q, K_u)$$

$$\times((\Leftrightarrow 1)^{m-1}\chi_{1/2}((\Leftrightarrow 2)^m f(u)^m \Delta_u(f))G(\chi_{1/2},\psi)^m)^{\mathrm{rank}(K_u)}. \quad (2)$$

Proof. It is well-known that $\Phi_{f,u}(\overline{\mathbb{Q}_\ell})[m \Leftrightarrow 1]$ is concentrated in degree zero, has rank 1, and has the same geometric monodromy as $L_{\chi^m_{1/2}}$ at 0, cf. [Del2]. Hence from [Lau1, 3.1.5.5], it follows that (1) implies (2). Moreover by [Lau1, 3.1.5.6] we may suppose that $K = \overline{\mathbb{Q}_\ell}$. Clearly we can assume that $X = \mathbb{A}^m_{\mathbb{F}_q}$, $f = \sum_{i=1}^m a_i x_i^2$, and $u = 0$. It is an easy exercise to calculate the action of $G_{f(u)}$ on $\Phi_{f,u}(\overline{\mathbb{Q}_\ell})$, using the material in [Del2], and to deduce (1) from it. However we will give a different proof of (1): Although f is not proper, we have a distinguished triangle (8.1.2, (3)) with $s = 0$ and $\Sigma = \{0\}$, see [Del2, Prop. 2.2.3] and [Del1, Prop. 2.1.9]. Hence (8.1.2, (4)), (8.1.4) and [Lau1, 3.1.5.4(iii)] yield

$$\begin{aligned}\varepsilon_{\psi,0}(T_0, \Phi_{f,0}(\overline{\mathbb{Q}_\ell}), \mathrm{d}x) &= \varepsilon_\psi(T_0, Rf_!\overline{\mathbb{Q}_\ell}, \mathrm{d}x)\\ &= \varepsilon_0(\mathbb{A}^1_{\mathbb{F}_q}, (Rf_!\overline{\mathbb{Q}_\ell}) \otimes L_{\overline{\psi}})\\ &= \varepsilon_0(\mathbb{A}^m_{\mathbb{F}_q}, f^* L_{\overline{\psi}}).\end{aligned}$$

This gives (1), because $H_c^{\cdot}(\mathbb{A}^m_{\mathbb{F}_q}, L(\psi(\Leftrightarrow \sum_{i=1}^m a_i x_i^2)))$ is concentrated in degree m with dimension 1 and eigenvalue of Frobenius equal to $(\Leftrightarrow 1)^m \chi_{1/2}((\Leftrightarrow 1)^m \prod_i a_i) \times G(\chi_{1/2}, \psi)^m$. □

8.2.3. PROPOSITION. *Assume the notation and hypothesis of Lemma* (8.1.2), *with* q *odd and* $\Sigma = \{u\}, u \in X(\mathbb{F}_q)$. *Suppose that* X *and* K *are smooth at* u, *that* X *has dimension* m *at* u, *and that* u *is a non-degenerate critical point of* f. *Put* $\rho = \mathrm{rank}(K_u)$. *Then*

$$m\rho + \sum_N (\alpha_N \Leftrightarrow \beta_N) \equiv 0 \bmod 2, \textit{ and} \quad (1)$$

$$\varepsilon_0(X, K) = (\Leftrightarrow 1)^\rho q^{w/2} \chi_{1/2}(c) \det(\mathrm{Frob}_q, K_u)^{(-1)^{m-1}}, \quad (2)$$

where

$$w = \Leftrightarrow m\rho(\Leftrightarrow 1)^m \Leftrightarrow \sum_N (\alpha_N + \beta_N), \quad (3)$$

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$$c = (\Leftrightarrow 1)^{(m\rho+\sum_N(\alpha_N-\beta_N))/2}(\prod_N N^{\alpha_N-\beta_N})(f(u)^m\Delta_u(f))^\rho. \tag{4}$$

Proof. Consider the determinant of $L_{\chi_{1/2}} \otimes Rf_!K$. This is a $\overline{\mathbb{Q}_\ell}$-sheaf on $\mathbb{G}_{m,\mathbb{F}_q}$ which is smooth outside $f(u)$. Its geometric monodromy at respectively $0, f(u), \infty$ equals the geometric monodromy of respectively

$$(L_{\chi_{1/2}})^{\otimes \sum_N \alpha_N}, \quad (L_{\chi_{1/2}})^{\otimes \rho m}, \quad (L_{\chi_{1/2}})^{\otimes \sum_N \beta_N}$$

at 0, because the determinant of V_N has the same geometric monodromy at 0 as $L_{\chi_{1/2}^{N-1}}$, and because of (8.1.2,(3)), and the first sentence in the proof of (8.2.2). This directly implies assertion (1). Using (8.1.2), (8.1.3), (8.2.2) and the formula $G(\chi_{1/2},\psi)^2 = \chi_{1/2}(\Leftrightarrow 1)q$, we obtain an expression for $\varepsilon_0(X,K)$, which simplifies drastically by using the congruence (1) and the relation $\text{rank}(Rf_!K)_{\bar\eta_0} = \sum_N N\alpha_N = \sum_N N\beta_N$. This yields the assertion (2). □

9. Proof of Theorem C

9.1. CALCULATION OF THE HESSIAN OF $f|_{H(v_0^\vee)\cap O_1}$ AT ITS CRITICAL POINT

The main purpose of (9.1) is to provide (9.1.7, (1)) as a preliminary for the proof of Theorem C.

9.1.0. *Notation and conventions*

We continue to assume the notation of Section 1 and of (3.5.1).

(1) $H(v_0^\vee) := \{v \in V \mid \langle v_0^\vee, v\rangle = 1\}$ for $0 \neq v_0^\vee \in V^\vee$.

(2) $\text{sing}\,(f|_{H(v_0^\vee)\cap O_1}) := \{\text{critical points of } f|_{H(v_0^\vee)\cap O_1}\}$. (We shall show in (9.1.1) that $H(v_0^\vee)\cap O_1$ is always a non-singular variety.)

(3) $B_{v_0^\vee}(y,y') := \langle y, (F_*^\vee)_{v_0^\vee}(y')\rangle$ for $v_0^\vee \in \Omega^\vee$ and $y, y' \in V^\vee$. (Recall that $(F_*^\vee)_{v_0^\vee} : T_{v_0^\vee}\Omega^\vee (= V^\vee) \to T_{F^\vee(v_0^\vee)}V (= V)$ is the linear mapping induced by $F^\vee : \Omega^\vee \to V$.)

(4) For a symmetric matrix $A \in M_n(\mathbb{F}_q)$, if ${}^tXAX = \text{diag}(a_1,\ldots,a_m,0,\ldots,0)$ with $X \in \text{GL}_n(\mathbb{F}_q)$ and $a_i \in \mathbb{F}_q^\times$, then we put $\Delta(A) := \prod_{i=1}^m a_i$ $(\in \mathbb{F}_q^\times/\mathbb{F}_q^{\times 2})$, and call it the discriminant of A. This definition of 'discriminant' is equivalent to the one given in (1.5).

(5) If q is odd, for two symmetric matrices $A_i \in M_{n_i}(\mathbb{F}_q)$ $(i = 1,2)$, we define the equivalence relation $A_1 \sim A_2$ as follows. Put $\ker A_i := \{x \in \mathbb{F}_q^{n_i} \mid A_i x = 0\}$, and let Q_i be the non-degenerate quadratic form on $\mathbb{F}_q^{n_i}/\ker A_i$ induced by A_i. If Q_1 and Q_2 are equivalent as quadratic forms, then we define $A_1 \sim A_2$. In particular, if $n_1 = n_2 = 1$, (i.e., A_1 and A_2 are scalars), then $A_1 \sim A_2 \Leftrightarrow A_1 = A_2 \times (\text{square in } \mathbb{F}_q^\times)$. As is well known, $A_1 \sim A_2 \Leftrightarrow$ '$\text{rank}\,A_1 = \text{rank}A_2$ and $\Delta(A_1) \sim \Delta(A_2)$' for two symmetric matrices A_1 and A_2.

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(6) For a function $\varphi(x_1, \dots, x_n)$ on a non-singular variety with local coordinates $x_1, \dots,\ x_n$, and for a critical point u of φ, put $\mathrm{Hess}_u(\varphi) := \left(\frac{\partial^2\varphi}{\partial x_i \partial x_j}(u)\right)_{1\leqslant i,j\leqslant n}$ and $\Delta_u(\varphi) := \Delta\,(\mathrm{Hess}_u(\varphi))$. Here φ is a regular function, or more generally any function such that the $\partial^2\varphi/\partial x_i\partial x_j$ are defined.

(7) Throughout (9.1), every variety, say X, and every morphism is assumed to be defined over $\mathbb{F}_q$, and X is identified with $X(k)$, where k is a (fixed) algebraic closure of $\mathbb{F}_q$. We always assume that char $\mathbb{F}_q \gg 0$.

9.1.1. LEMMA. *For any* $0 \neq v_0^\vee \in V^\vee$, $H(v_0^\vee)$ *intersects* O_1 *transversally. In particular* $H(v_0^\vee) \cap O_1$ *is a non-singular variety.*

Proof. We may and do assume that $m := \dim O_1$ is strictly smaller than $n := \dim V$. Since $k^\times O_1 = O_1$ [Gyo1, (1.4, (2))], any affine hyperplane tangent to O_1 contains the origin of V. Hence $H(v_0^\vee)$ is not tangent to O_1. □

9.1.2. LEMMA. *For* $v_0^\vee \in \Omega^\vee$, $\mathrm{sing}\,(f|_{H(v_0^\vee)\cap O_1}) = \{u\}$, *where* $u := d^{-1}\cdot F^\vee(v_0^\vee)$ *with* $d := \deg f$.

Proof. For $v \in O_1$, consider the following conditions

(1) $v \in\ \mathrm{sing}(f|_{H(v_0^\vee)\cap O_1})$,
(2) $F(v) \in kv_0^\vee + (T_vO_1)^\perp$,
(3) $v \in H(v_0^\vee) \cap O_1$,
(4) $F(v) \in d\cdot v_0^\vee + (T_vO_1)^\perp$,
(5) $d\cdot v_0^\vee \in F^{\vee\,-1}(v)$,
(6) $d^{-1}\cdot F^\vee(v_0^\vee) = v$.

Since $kv_0^\vee + (T_vO_1)^\perp = (T_v(H(v_0^\vee) \cap O_1))^\perp$ by (9.1.1), we get (1) ⇔ [(2) and (3)]. Assume (2) and (3). Then $F(v) \in cv_0^\vee + (T_vO_1)^\perp$ for some $c \in k$. Since $k^\times \cdot v \subset O_1$,

$$v \in T_vO_1. \tag{7}$$

Hence $d = \langle F(v), v\rangle = \langle cv_0^\vee, v\rangle = c$, and especially [(2) and (3)] ⇔ [(3) and (4)]. Define an isomorphism $\Phi^\vee : (TO_1)^\perp \to \Omega^\vee$ as in (2.2), and let $\pi : (TO_1)^\perp \to O_1$ be the projection. Then

$$F(v) + (T_vO_1)^\perp = \Phi^\vee(\pi^{-1}(v)) = F^{\vee\,-1}(v) \tag{8}$$

by (2.2). Hence (4) ⇔ (5) ⇔ (6). If (6) is satisfied, then $\langle v_0^\vee,\ v\rangle = d^{-1} \times\langle\, v_0^\vee, F^\vee(v_0^\vee)\rangle = 1$. Hence (1) ⇔ [(3) and (6)] ⇔ (6). □

9.1.3. LEMMA. *Let* $v_1^\vee \in O_1^\vee$ *and* $v^\vee \in F^{\vee\,-1}F^\vee(v_1^\vee)$ $(= v_1^\vee + (T_{F^\vee(v_1^\vee)}O_1)^\perp$ *by* (9.1.2, (8)). *Then*

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(1) $\ker B_{v^\vee} = \ker B_{v_1^\vee} = \ker\,(F_*^\vee)_{v^\vee} = \ker\,(F_*^\vee)_{v_1^\vee} = (T_{F^\vee(v_1^\vee)}O_1)^\perp$, and
(2) $B_{v^\vee}$ *is non-degenerate on* $T_{v_1^\vee}O_1^\vee$.

Proof. (1) By (2.2, (1)), we get $\ker(F_*^\vee)_{v^\vee} = \ker(F_*^\vee)_{v_1^\vee} = (T_{F^\vee(v_1^\vee)}O_1)^\perp$. By (9.1.0, (3)), we get $\ker B_{v^\vee} = \ker(F_*^\vee)_{v^\vee}$ for any $v^\vee \in \Omega^\vee$. Since

$$V^\vee = (T_{v_1^\vee}O_1^\vee) \oplus (T_{F^\vee(v_1^\vee)}O_1)^\perp \tag{3}$$

(cf. (6.1.4, (1))), (2) follows from (1). □

9.1.4. LEMMA. *Let* $v_1^\vee \in O_1^\vee(\mathbb{F}_q)$, $u_i^\vee, w_i^\vee \in V^\vee(\mathbb{F}_q)$ $(1 \leqslant i \leqslant m)$, *and* $\varphi(v^\vee) := \det\,(B_{v^\vee}(u_i^\vee, w_j^\vee))_{1\leqslant i,j\leqslant m}$ *for* $v^\vee \in \Omega^\vee$. *If* $\varphi(v_1^\vee) \neq 0$, *then* $\varphi(v^\vee) = \varphi(v_1^\vee)$ *for* $v^\vee \in F^{\vee -1}F^\vee(v_1^\vee)$. *In particular,* $(B_{v^\vee}(u_i^\vee, u_j^\vee))_{1\leqslant i,j\leqslant m} \sim (B_{v_1^\vee}(u_i^\vee, u_j^\vee))_{1\leqslant i,j\leqslant m}$, *if* $v^\vee$ *is* $\mathbb{F}_q$*-rational.*

Proof. By (9.1.3, (1) and (3)), we may assume from the beginning that

$$u_i^\vee, w_i^\vee \in T_{v_1^\vee}O_1^\vee. \tag{1}$$

Then $\{u_1^\vee, \ldots, u_m^\vee\}$ and $\{w_1^\vee, \ldots, w_m^\vee\}$ are linear bases of $T_{v_1^\vee}O_1^\vee$. For $g \in G_{v_1^\vee} = G_{F^\vee(v_1^\vee)}$, let $g^{-1}u_i^\vee = \sum_{j=1}^m a_{ij}(g)u_j^\vee$, $g^{-1}w_i^\vee = \sum_{j=1}^m b_{ij}(g)w_j^\vee$, $A(g) := (a_{ij}(g))_{1\leqslant i,j\leqslant m}$, and $B(g) := (b_{ij}(g))_{1\leqslant i,j\leqslant m}$. Then

$$\begin{aligned}\varphi(gv^\vee) &= \det(B_{gv^\vee}(u_i^\vee, w_j^\vee)) = \det(B_{v^\vee}(g^{-1}u_i^\vee, g^{-1}w_j^\vee))\\ &= \det A(g)\cdot\varphi(v^\vee)\cdot\det B(g).\end{aligned} \tag{$2_{v^\vee}$}$$

Dividing $(2_{v^\vee})$ by $(2_{v_1^\vee})$, we get

$$\varphi(gv^\vee)/\varphi(v_1^\vee) = \varphi(v^\vee)/\varphi(v_1^\vee) \quad \text{for } v^\vee \in F^{\vee -1}F^\vee(v_1^\vee). \tag{3}$$

(Note that $gv_1^\vee = v_1^\vee$.) Since $F^\vee(O_0^\vee) = O_1$ [Gyo1, (1.18, (2))], $G_{v_1^\vee}$ acts homogeneously on the open dense subset $O_0^\vee \cap F^{\vee -1}F^\vee(v_1^\vee)$ of $F^{\vee -1}F^\vee(v_1^\vee)$, and hence (3) implies that $\varphi(v^\vee)/\varphi(v_1^\vee) = 1$ for all $v^\vee \in F^{\vee -1}F^\vee(v_1^\vee)$. □

9.1.5. REMARK. Let $\{u_i^\vee\}$ and $\{w_i^\vee\}$ be linear bases of $T_{v_1^\vee}O_1^\vee$. Then $\varphi(v_1^\vee) \neq 0$ by (9.1.3, (2)), $\varphi \equiv \varphi(v_1^\vee)$ on $F^{\vee -1}F^\vee(v_1^\vee)$ by (9.1.4), and $\det A(g) = \det B(g) = \det\,(g^{-1}|T_{v_1^\vee}O_1^\vee)$ for $g \in G_{v_1^\vee}$. Hence (9.1.4, (2)) yields

$$\det(g|T_{v_1^\vee}O_1^\vee)^2 = 1 \quad (g \in G_{v_1^\vee}). \tag{1}$$

Since $\langle F^\vee(v_1^\vee), v_1^\vee\rangle = d \neq 0$, it follows that

$$T_{v_1^\vee}O_1^\vee = kv_1^\vee \oplus (F^\vee(v_1^\vee)^\perp \cap T_{v_1^\vee}O_1^\vee) \tag{2}$$

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$$\det(g|T_{v_1^\vee}O_1^\vee) = \det(g|F^\vee(v_1^\vee)^\perp \cap T_{v_1^\vee}O_1^\vee) \quad (g \in G_{v_1^\vee}). \tag{3}$$

9.1.6. LEMMA. *Take* $u \in O_1(\mathbb{F}_q)$, $v_0^\vee \in \Omega^\vee(\mathbb{F}_q)$ *and* $v_1^\vee \in O_1^\vee(\mathbb{F}_q)$ *so that* $F^\vee(v_0^\vee) = F^\vee(v_1^\vee) = d \cdot u$ *with* $d = \deg f$. *Then*

$$\mathrm{Hess}_u(f|_{H(v_1^\vee)\cap O_1}) \sim \mathrm{Hess}_u(f|_{H(v_0^\vee)\cap O_1}). \tag{1}$$

Proof. First we take as $v_1^\vee$ an arbitrary point of $O_1^\vee$. Let $\{v_2^\vee, \ldots, v_m^\vee\}$ be a linear basis of $F^\vee(v_1^\vee)^\perp \cap T_{v_1^\vee}O_1^\vee$, and $x_i := \langle v_i^\vee, \ \rangle \Leftrightarrow \langle v_i^\vee, u\rangle$ ($1 \leqslant i \leqslant m$). Then $\{v_1^\vee, v_2^\vee, \ldots, v_m^\vee\}$ is a linear basis of $T_{v_1^\vee}O_1^\vee$ by (9.1.5, (2)), and $\{x_1, \ldots, x_m\}$ gives a local coordinate system of O_1 at u, since $v_i^\vee = dx_i \equiv 0$ on $(T_{v_1^\vee}O_1^\vee)^\perp$ and $(T_uO_1) \oplus (T_{v_1^\vee}O_1^\vee)^\perp = V$. (Cf. (9.1.3, (3)).) For $v^\vee \in F^{\vee -1}F^\vee(v_1^\vee)$, put $z_1^{v^\vee} := \langle v^\vee, \ \rangle \Leftrightarrow 1$ and $z_i^{v^\vee} := x_i$ ($2 \leqslant i \leqslant m$). Then $\{z_i^{v^\vee}\}_{1\leqslant i\leqslant m}$ (resp. $\{z_i^{v^\vee}\}_{2\leqslant i\leqslant m}$) gives a local coordinate system of O_1 (resp. $H(v^\vee) \cap O_1$) at u. (Note that $v^\vee \in F^{\vee -1}F^\vee(v_1^\vee) = (T_{F^\vee(v_1^\vee)}O_1)^\perp + F(F^\vee(v_1^\vee)) = (T_uO_1)^\perp + v_1^\vee$, by (9.1.2, (8)) and hence $v_1^\vee \Leftrightarrow v^\vee$ is perpendicular to T_uO_1, i.e., $(dx_1)_u = (dz_1^{v^\vee})_u$. Note also that

$$\begin{aligned} z_1^{v^\vee}(u) &= \langle v^\vee, u\rangle \Leftrightarrow 1 = \langle v_1^\vee, u\rangle \Leftrightarrow 1 \quad \text{since}\, u \in T_uO_1 \perp (v_1^\vee \Leftrightarrow v^\vee) \\ &= \langle v_1^\vee, d^{-1} \cdot F^\vee(v_1^\vee)\rangle \Leftrightarrow 1 = 0 \end{aligned}$$

by Euler's identity.) Fix $g \in G_{v_1^\vee}$ and put $z_i'(v) := z_i^{gv^\vee}(gv)$. Then $\{z_i'\}_{1\leqslant i\leqslant m}$ (resp. $\{z_i'\}_{2\leqslant i\leqslant m}$) is a local coordinate system of O_1 (resp. $H(v^\vee) \cap O_1$) at $g^{-1}u = u$. Moreover we have $H(v^\vee)\cap O_1 = \{z_1^{v^\vee} = 0\} = \{z_1' = 0\}$. For $v^\vee \in F^{\vee -1}F^\vee(v_1)$, put

$$\xi(v^\vee) := \det\left(\frac{\partial^2(f|_{H(v^\vee)\cap O_1})}{\partial z_i^{v^\vee}\partial z_j^{v^\vee}}(u)\right)_{2\leqslant i,j\leqslant m}. \tag{2}$$

Since $f^\vee(v_1^\vee) = f^\vee(gv_1^\vee) = \phi(g)^{-1}f^\vee(v_1^\vee) \neq 0$, we have $\phi(g) = 1$ and $f(gv) = f(v)$. In other words, the function f (resp. $z_i^{gv^\vee}$) on $H(gv^\vee)\cap O_1$ is identified with f (resp. z_i') on $H(v^\vee) \cap O_1$, via $g\colon H(v^\vee) \cap O_1 \overset{\cong}{\Leftrightarrow\!\!\to} H(gv^\vee) \cap O_1$. Therefore, noting that $gu = u$ is a critical point of $f|_{H(v^\vee)\cap O_1}$ by (9.1.2), we get

$$\begin{aligned} \xi(gv^\vee) &= \det\left(\frac{\partial^2(f|_{H(v^\vee)\cap O_1})}{\partial z_{i'}'\partial z_{j'}'}(u)\right)_{i',j'} \\ &= \det\left(\frac{\partial z_i^{v^\vee}}{\partial z_{i'}'}(u)\right)_{i',i} \cdot \det\left(\frac{\partial^2(f|_{H(v^\vee)\cap O_1})}{\partial z_i^{v^\vee}\partial z_j^{v^\vee}}(u)\right)_{i,j} \\ &\quad \times \det\left(\frac{\partial z_j^{v^\vee}}{\partial z_{j'}'}(u)\right)_{j,j'}, \end{aligned} \tag{3}$$

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where i, j, i' and j' run over $\{2, \ldots, m\}$. Since $(dz_i^{v^\vee})_u = v_i^\vee$ and $(dz_i')_u = g^{-1}v_i^\vee$ for $2 \leqslant i \leqslant m$,

$$\det\left(\frac{\partial z_i^{v^\vee}}{\partial z_{i'}'}(u)\right)_{2\leqslant i,i'\leqslant m} = \det(g|F^\vee(v_1^\vee)^\perp \cap T_{v_1^\vee}O_1^\vee). \tag{4}$$

By (3), (4), and (9.1.5, (1) and (3)),

$$\xi(gv^\vee) = \xi(v^\vee) \quad (v^\vee \in F^{\vee-1}F^\vee(v_1^\vee), g \in G_{v_1^\vee}). \tag{5}$$

Since $G_{v_1^\vee}$ acts prehomogeneously on $F^{\vee-1}F^\vee(v_1^\vee)$ (cf. the argument at the end of the proof of (9.1.4)), (5) implies that $\xi(v^\vee)$ is independent of $v^\vee$. Thus we get the result, if we assume

$$\xi(v_1^\vee) \neq 0, \tag{6}$$

which will be proved in the course of proving the next lemma. □

9.1.7. LEMMA. *Let* $v_0^\vee \in \Omega^\vee(\mathbb{F}_q), u := d^{-1}\cdot F^\vee(v_0^\vee), H(v_0^\vee) := \{v \in V | \langle v_0^\vee, v\rangle = 1\}$, *and* $\{y_1, \ldots, y_n\}$ *be a linear coordinate system of* $V^\vee$ *defined over* $\mathbb{F}_q$. *Then* u *is a non-degenerate critical point of* $f|_{H(v_0^\vee)\cap O_1}$, *and*

$$\Leftrightarrow d \cdot f(u)^{m-1}\Delta_u(f|_{H(v_0^\vee)\cap O_1}) \sim \Delta\left(\frac{\partial^2 \log f^\vee}{\partial y_i \partial y_j}(v_0^\vee)\right)_{1\leqslant i,j\leqslant n}. \tag{1}$$

(*See* (9.1.0, (5) *and* (6)) *for* Δ *and* Δ_u.)

Proof. By Euler's identity, we can show that

$$\sum_{i,j=1}^{n} y_i \cdot \frac{\partial^2 \log f^\vee}{\partial y_i \partial y_j} \cdot y_j = \Leftrightarrow d. \tag{2}$$

In particular, $B_{v_1^\vee}(v_1^\vee, v_1^\vee) = \Leftrightarrow d \neq 0$, where $v_1^\vee$ is the unique element of $O_1^\vee \cap F^{\vee-1}F^\vee(v_0^\vee)$. Note that $v_1^\vee$ is $\mathbb{F}_q$-rational. By (9.1.2, (7)), $v_1^\vee \in T_{v_1^\vee}O_1^\vee$. By (9.1.3), $B_{v_1^\vee}|_{T_{v_1^\vee}}O_1^\vee$ is non-degenerate. Hence we can take an $\mathbb{F}_q$-rational linear basis $\{v_1^\vee, \ldots, v_m^\vee\}$ of $T_{v_1^\vee}O_1^\vee$ which contains $v_1^\vee$, and such that

$$(B_{v_1^\vee}(v_i^\vee, v_j^\vee))_{1\leqslant i,j\leqslant m} = \mathrm{diag}(a_1, \ldots, a_m) \quad (a_i \neq 0, a_1 = \Leftrightarrow d). \tag{3}$$

Let $\{v_{m+1}^\vee, \ldots, v_n^\vee\}$ be an $\mathbb{F}_q$-rational linear basis of $(T_{F^\vee(v_1^\vee)}O_1)^\perp$. By (9.1.3, (3)), $\{v_1^\vee, \ldots, v_n^\vee\}$ is a linear basis of $V^\vee$. Let $\{v_1, \ldots, v_n\}$ be its dual basis,

$$x_i := \langle v_i^\vee, \ \rangle \quad \text{and} \quad y_i := \langle v_i, \ \rangle. \tag{4}$$

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Then

$$(B_{v_1^\vee}(v_i^\vee, v_j^\vee))_{1\leqslant i,j\leqslant n} = \mathrm{diag}(a_1,\dots,a_m,0,\dots,0) \quad \text{by } (9.1.3,\ (1)), \tag{5}$$

$$(F_*^\vee)_{v_1^\vee}(v_j^\vee) = a_j v_j, \text{where}\, a_j := 0\ (j > m) \tag{6}$$

(cf. (5) and (9.1.0, (3))), and

$$\begin{aligned}
&\left(\frac{\partial^2 \log f^\vee}{\partial y_i \partial y_j}(v_0^\vee)\right)_{1\leqslant i,j\leqslant n}\\
&\quad = (B_{v_0^\vee}(v_i^\vee, v_j^\vee))_{1\leqslant i,j\leqslant n} \quad \text{by } (6.1.4,\ (2))\\
&\quad\quad \sim (B_{v_0^\vee}(v_i^\vee, v_j^\vee))_{1\leqslant i,j\leqslant m} \quad \text{by } (9.1.3,\ (1))\\
&\quad\quad \sim (B_{v_1^\vee}(v_i^\vee, v_j^\vee))_{1\leqslant i,j\leqslant m} \quad \text{by } (9.1.3,\ (2)) \text{ and } (9.1.4).
\end{aligned} \tag{7}$$

On the other hand, since $FF^\vee|_{O_1^\vee}$ is the identity,

$$v_j^\vee = (F_*)_{F^\vee(v_1^\vee)}(F_*^\vee)_{v_1^\vee}(v_j^\vee) = (F_*)_{F^\vee(v_1^\vee)}(a_j v_j) \quad (1 \leqslant j \leqslant m), \tag{8}$$

by (6). Hence

$$\begin{aligned}
&a_2 \dots a_m \sim (a_2 \dots a_m)^{-1}\\
&\quad = \det(\langle v_i, (F_*)_{F^\vee(v_1^\vee)}(v_j)\rangle)_{2\leqslant i,j\leqslant m}\\
&\quad = \det\left(\frac{\partial^2 \log f}{\partial x_i \partial x_j}(F^\vee(v_1^\vee))\right)_{2\leqslant i,j\leqslant m}.
\end{aligned} \tag{9}$$

Since every $\partial^2 \log f/\partial x_i \partial x_j$ is homogeneous of degree $\Leftrightarrow$2,

$$a_2 \dots a_m \sim \det\left(\frac{\partial^2 \log f}{\partial x_i \partial x_j}(u)\right)_{2\leqslant i,j\leqslant m} \neq 0. \tag{10}$$

(Note that $u = d^{-1} \cdot F^\vee(v_0^\vee) = d^{-1} \cdot F^\vee(v_1^\vee)$.) Take a local coordinate system $\{z_1, \dots, z_n\}$ of V at u so that

$$z_1 := \langle v_1^\vee,\ \rangle \Leftrightarrow 1 (= x_1 \Leftrightarrow 1), \tag{11}$$

$$z_i = \langle v_i^\vee,\ \rangle \Leftrightarrow \langle v_i^\vee, u\rangle \quad (= x_i \Leftrightarrow \langle v_i^\vee, u\rangle) \quad (2 \leqslant i \leqslant m), \tag{12}$$

$$(\mathrm{d}z_i)_u = v_i^\vee (= \mathrm{d}x_i) \quad (m+1 \leqslant i \leqslant n), \quad \text{and} \tag{13}$$

$$O_1 = \{z_{m+1} = \dots = z_n = 0\}. \tag{14}$$

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Then $H(v_1^\vee) \cap O_1 = \{z_1 = z_{m+1} = \cdots = z_n = 0\}$. Now, let us calculate

$$\begin{aligned}\frac{\partial^2 \log f}{\partial z_i \partial z_j} &= \sum_{1 \leqslant i', j' \leqslant n} \frac{\partial x_{i'}}{\partial z_i} \cdot \frac{\partial^2 \log f}{\partial x_{i'} \partial x_{j'}} \cdot \frac{\partial x_{j'}}{\partial z_j} \\ &\quad + \sum_{1 \leqslant i' \leqslant n} \frac{\partial \log f}{\partial x_{i'}} \cdot \frac{\partial^2 x_{i'}}{\partial z_i \partial z_j} \end{aligned} \tag{15}$$

at u for $2 \leqslant i, j \leqslant m$. By (11)–(13),

$$\frac{\partial x_{i'}}{\partial z_i}(u) = \delta_{i,i'} \quad (1 \leqslant i, i' \leqslant n). \tag{16}$$

By (11) and (12)

$$\frac{\partial^2 x_{i'}}{\partial z_i \partial z_j} = 0 \quad (1 \leqslant i' \leqslant m). \tag{17}$$

Since $f^\vee(F(v)) = b_0 f(v)^{-1}$ by (2.1), f is constant on $F^{-1}F(u) = u + (T_{F(u)} O_1^\vee)^\perp = u + (T_{v_1^\vee} O_1^\vee)^\perp$. (Indeed, $F(u) = F(d^{-1} \cdot F^\vee(v_0^\vee)) = d \cdot F F^\vee(v_0^\vee) = d \cdot v_1^\vee$, and $O_1^\vee = d \cdot O_1^\vee$.) Since $u + (T_{v_1^\vee} O_1^\vee)^\perp$ can be expressed as $\{x_j = c_j$ $(1 \leqslant j \leqslant m)\}$ with some constants $\{c_j\}$,

$$f_{x_{i'}}(u) = 0 \quad (m+1 \leqslant i' \leqslant n). \tag{18}$$

By (15)–(18)

$$\frac{\partial^2 \log f}{\partial z_i \partial z_j}(u) = \frac{\partial^2 \log f}{\partial x_i \partial x_j}(u) \quad (2 \leqslant i, j \leqslant m). \tag{19}$$

Thus we get

$$\begin{aligned} &\left(\frac{\partial^2 \log f^\vee}{\partial y_i \partial y_j}(v_0^\vee)\right)_{1 \leqslant i,j \leqslant n} \\ &\quad \sim \operatorname{diag}(\Leftrightarrow d, a_2, \ldots, a_m) \quad \text{by (3) and (7)} \\ &\quad \sim (\Leftrightarrow d) \oplus \left(\frac{\partial^2 \log f}{\partial x_i \partial x_j}(u)\right)_{2 \leqslant i,j \leqslant m} \quad \text{by (10)} \\ &\quad = (\Leftrightarrow d) \oplus \left(\frac{\partial^2 \log f}{\partial z_i \partial z_j}(u)\right)_{2 \leqslant i,j \leqslant m} \quad \text{by (19)} \\ &\quad \sim (\Leftrightarrow d) \oplus \operatorname{Hess}_u(\log f|_{H(v_1^\vee) \cap O_1}). \end{aligned} \tag{20}$$

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Comparing the second and the last members of (20), we can see that the rank of $\mathrm{Hess}_u(\log f|_{H(v_1^\vee)\cap O_1})$ is $m \Leftrightarrow 1$, and hence we get (9.1.6, (6)). In particular, we get (9.1.6, (1)), which together with (20) yields

$$\begin{aligned}
&\Delta\left(\frac{\partial \log f^\vee}{\partial y_i \partial y_j}(v_0^\vee)\right) \sim (\Leftrightarrow d)\Delta(\mathrm{Hess}_u(\log f|_{H(v_0^\vee)\cap O_1}))\\
&\quad = (\Leftrightarrow d)\Delta(f(u)^{-1}\cdot \mathrm{Hess}_u(f|_{H(v_0^\vee)\cap O_1}))\\
&\quad \sim (\Leftrightarrow d) f(u)^{m-1}\Delta_u(f|_{H(v_0^\vee)\cap O_1}).
\end{aligned}$$

(The equality in the second line can be proved by a direct calculation using the fact that u is a critical point of $f|_{H(v_0^\vee)\cap O_1}$). □

9.2. EXISTENCE OF A GOOD COMPACTIFICATION

9.2.1. LEMMA. *Assume the notation of* (5.1.1). *Let* $K \in D_c^b(V, \overline{\mathbb{Q}_\ell})$ *be* α*-homogeneous and* $f\colon V \to \mathbb{A}_{\mathbb{C}}^1$ *a homogeneous polynomial over* $\mathbb{C}$ *of degree* d. *Let* $v^\vee \in V^\vee(\mathbb{C})$ *be general enough. Then* $f|_{H(v^\vee)\setminus f^{-1}(0)}$ *has a compactification*

$$\begin{array}{ccc}
H(v^\vee)\setminus f^{-1}(0) & \xrightarrow{\ \iota\ } & Z\\
{\scriptstyle f}\downarrow & \swarrow{\scriptstyle \pi} & \\
\mathbb{G}_{m,\mathbb{C}}, & &
\end{array}$$

with Z *a scheme over* $\mathbb{C}$, π *proper,* ι *an open immersion making the above diagram commutative, and* π *locally acyclic relative to* $\iota_!(K|_{H(v^\vee)\setminus f^{-1}(0)})$ *at each point* u *outside the image of* ι *in* Z. *(This means that the stalk* $\Phi_{\pi,u}(K)$ *at* u*, of the complex of vanishing cycles on* $\pi^{-1}(\pi(u))$ *associated to* K*, is zero* (*cf.* [Del1]).)

Proof. Put

$$Y = \{(x_0\colon x_1 : \cdots : x_n, t) \in \mathbb{P}_{\mathbb{C}}^n \times \mathbb{G}_{m,\mathbb{C}} \mid f(x_1, \ldots, x_n) = x_0^d t\}.$$

Thus Y is the part 'above' $\mathbb{G}_{m,\mathbb{C}}$ of the closure in $\mathbb{P}_{\mathbb{C}}^n \times \mathbb{A}_{\mathbb{C}}^1$ of the graph in $V \times \mathbb{A}_{\mathbb{C}}^1$ of f. Hence the natural projection $p\colon Y \to \mathbb{G}_{m,\mathbb{C}}$ is proper and yields a compactification of $f|_{V\setminus f^{-1}(0)}$ via the diagram

$$\begin{array}{ccc}
V\setminus f^{-1}(0) & \xrightarrow{\ i\ } & Y\\
& {\scriptstyle f}\searrow & \downarrow{\scriptstyle p}\\
& & \mathbb{G}_{m,\mathbb{C}},
\end{array}$$

where i is the open immersion defined by $(x_1, \ldots, x_n) \mapsto (1{:}\, x_1 : \cdots : x_n, f(x_1, \ldots, x_n))$. Put

$$Z := \{(x_0{:}\, x_1 : \cdots : x_n, t) \in Y \mid \langle v^\vee, (x_1, \ldots, x_n)\rangle = x_0\} \subset Y.$$

Then $\pi := p|_Z$ is a compactification of $f|_{H(v^\vee)\setminus f^{-1}(0)}$. It suffices to prove that π is locally acyclic relative to $i_!(K|_{V\setminus f^{-1}(0)})$ on $Z_\infty := Z \cap$ (locus of $x_0 = 0$). Put

$$\begin{aligned}
&X := \{(x_0{:}\, x_1 : \cdots : x_n) \in \mathbb{P}^n_{\mathbb{C}} \mid f(x_1, \ldots, x_n) = x_0^d\},\\
&X_\infty := X \cap (\text{locus of } x_0 = 0), \qquad Y' := X \times \mathbb{G}_{m,\mathbb{C}},\\
&Z' := \{(x_0{:}\, x_1 : \cdots : x_n, t) \in Y' \mid \langle v^\vee, (x_1, \ldots, x_n)\rangle = x_0 t^{-1}\} \subset Y',\\
&Y'_\infty := Y' \cap (\text{locus of } x_0 = 0) = X_\infty \times \mathbb{G}_{m,\mathbb{C}},\\
&Z'_\infty := Z' \cap (\text{locus of } x_0 = 0).
\end{aligned}$$

Let

$$p'{:}\, Y' \to \mathbb{G}_{m,\mathbb{C}}, \quad \text{and} \quad \pi'{:}\, Z' \to \mathbb{G}_{m,\mathbb{C}}$$

be the natural projections. We have a commutative diagram

$$\begin{array}{ccc}
Y' & \xrightarrow{\ \theta\ } & Y\\
{\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p}\\
\mathbb{G}_{m,\mathbb{C}} & \xrightarrow{\ \pi_d\ } & \mathbb{G}_{m,\mathbb{C}},
\end{array}$$

where θ is given by $(x_0{:}\, x_1 : \cdots : x_n, t) \mapsto (x_0 : tx_1 : tx_2 : \cdots : tx_n, t^d)$, and π_d by $t \mapsto t^d$. Note that θ is locally bianalytic, inducing a locally bianalytic map from Z' onto Z. Moreover, since K is α-homogeneous,

$$\theta^* i_!(K|_{V\setminus f^{-1}(0)}) = i'_!(K|_{f^{-1}(1)} \boxtimes L_\alpha),$$

where i' is the open immersion

$$i'{:}\, f^{-1}(1) \times \mathbb{G}_{m,\mathbb{C}} \to Y'{:}\, (x_1, \ldots, x_n, t) \mapsto (1{:}\, x_1 : \cdots : x_n, t).$$

Hence it suffices to prove that π' is locally acyclic on Z'_∞ relative to (the restriction to Z' of) $i'_!(K|_{f^{-1}(1)} \boxtimes L_\alpha)$.

By stratification it suffices to prove, for any $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}$ on X which is smooth on a locally closed smooth subscheme $W \subset X\setminus X_\infty = f^{-1}(1)$ but zero outside W, that π' is locally acyclic on Z'_∞ relative to (the restriction from Y' to Z' of) $\mathcal{F} \boxtimes \overline{\mathbb{Q}_\ell}$.

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Let $\overline{W}$ be the closure of W in X. By embedded resolution of singularities [Hir], there exists a smooth scheme $\widetilde{X}$ over $\mathbb{C}$ and a proper morphism $h\colon \widetilde{X} \to \overline{W}$ which is an isomorphism above W, such that $h^{-1}(\overline{W}\backslash W)$ has normal crossings in $\widetilde{X}$ (meaning that its irreducible components are smooth and intersect transversally). Consider

$$\begin{aligned}
&\widetilde{Y} := \widetilde{X} \times \mathbb{G}_{m,\mathbb{C}}, \qquad g : \widetilde{Y} \to Y' : (x,t) \mapsto (h(x),t),\\
&\widetilde{Z} := g^{-1}(Z') \subset \widetilde{X} \times \mathbb{G}_{m,\mathbb{C}}, \qquad \widetilde{Z}_\infty := g^{-1}(Z'_\infty), \qquad \widetilde{Y}_\infty := g^{-1}(Y'_\infty),\\
&\widetilde{\pi} := \pi' \circ (g|_{\widetilde{Z}}) : \widetilde{Z} \to \mathbb{G}_{m,\mathbb{C}} : (x,t) \mapsto t.
\end{aligned}$$

Note that $g^*(\mathcal{F} \boxtimes \overline{\mathbb{Q}_\ell})$ is zero on the divisor $D := h^{-1}(\overline{W}\backslash W)_{\mathrm{red}} \times \mathbb{G}_{m,\mathbb{C}}$ of $\widetilde{Y}$, and smooth on $\widetilde{Y}\backslash D$. Moreover g is a closed immersion outside D. Because the morphism $g_0\colon \widetilde{Z} \to Z'$ induced by g is proper, the functor Rg_{0*} commutes with the functor $R\Psi$ of nearby cycles, see [Del1, 2.1.7.1]. Hence it suffices to prove that $\widetilde{\pi}$ is locally acyclic on $\widetilde{Z}_\infty$ relative to (the restriction from $\widetilde{Y}$ to $\widetilde{Z}$ of) $g^*(\mathcal{F} \boxtimes \overline{\mathbb{Q}_\ell})$.

Note that Z' and $\widetilde{Z}$ depend on $v^\vee$. The linear system on Y' generated by all the divisors Z' for $v^\vee$ running through $V^\vee(\mathbb{C})$, has no base points. Hence the same is true for the linear system on $\widetilde{Y}$ generated by all the $\widetilde{Z}$. Moreover the divisor D has normal crossings in $\widetilde{Y}$. Thus by Bertini's Theorem (cf. [GriHar, p. 137]), for $v^\vee$ general enough, $\widetilde{Z}$ is smooth over $\mathbb{C}$ and intersects each irreducible component of D transversally, and $D \cap \widetilde{Z}$ is a divisor on $\widetilde{Z}$ with normal crossings. Clearly $g^*(\mathcal{F}\boxtimes\overline{\mathbb{Q}_\ell})$ is zero on $D\cap\widetilde{Z}$ and smooth on $\widetilde{Z}\backslash(D\cap\widetilde{Z})$. Moreover for any $z \in \widetilde{Z}_\infty$, we claim that the restriction of $\widetilde{\pi}$ to the intersection E of all irreducible components of $D \cap \widetilde{Z}$ containing z, is smooth at z. Since the smoothness is characterized by the surjectivity of $d\widetilde{\pi}$, this implies that $\widetilde{Z}$ (resp. $D \cap \widetilde{Z}$) is smooth (resp. relatively normal crossing) at z over $\mathbb{G}_m$, and hence implies by [Del1, Lemme 2.1.11] that $\widetilde{\pi}$ is locally acyclic on $\widetilde{Z}_\infty$ relative to (the restriction to $\widetilde{Z}$ of) $g^*(\mathcal{F} \boxtimes \overline{\mathbb{Q}_\ell})$. Thus it remains to prove the claim. Locally at z, the scheme E equals the intersection E' of $\widetilde{Z}$ and the irreducible components of D containing z. But at least one of these components of D is contained in $\widetilde{Y}_\infty$, because $\widetilde{Y}_\infty \subset D$ since $W \subset X\backslash X_\infty$. Hence the scheme E' equals the intersection of $\widetilde{Z}\cap\widetilde{Y}_\infty$ and the irreducible components of D containing z. One verifies from the definitions that $\widetilde{Z}\cap\widetilde{Y}_\infty = \widetilde{Z}_\infty$ is a cartesian product with second factor $\mathbb{G}_{m,\mathbb{C}}$. The same is obviously true for D and hence also for E'. Thus $\widetilde{\pi}|_E$, being the projection onto $\mathbb{G}_{m,\mathbb{C}}$, is smooth at z, which proves the claim. □

9.3. Proof of Theorem C

We now return to the notation and convention of (9.1.0), in order to prove

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Theorem C. In particular, we assume that char $\mathbb{F}_q \gg 0$. From (5.2.3.1, (3) and (4)) it follows (replacing the triple (G, ρ, V) by its dual $(G, \rho^\vee, V^\vee)$) that it suffices to prove

$$\operatorname{sign}(\varepsilon_0(H(v^\vee) \cap O_1, L(\omega))) = -\chi_{1/2}\left((-1)^{(m+r)/2} \prod_{j \geqslant 1} j^{e(j)}\right), \tag{1}$$

for at least one $v^\vee \in \Omega^\vee(\mathbb{F}_q)$. We have seen in (9.1.2) and (9.1.7) that $f|_{H(v^\vee)\cap O_1}$ has only one critical point $u = d^{-1}F^\vee(v^\vee)$ and that this is a nondegenerate critical point. Hence, for $v^\vee$ the reduction mod p of a general enough point, say $v_0^\vee$, in $\Omega^\vee(\mathbb{Q})$, we can apply Proposition (8.2.3) (with m replaced by $m-1$) to a suitable compactification, say $\tilde{f}: X \to \mathbb{G}_{m,\mathbb{F}_q}$, of $f|_{H(v^\vee)\cap O_1}$. (More precisely, let $\pi: Z \to \mathbb{G}_{m,\mathbb{Q}}$ be the compactification of $f: H(v_0^\vee)\backslash f^{-1}(0) \to \mathbb{G}_{m,\mathbb{Q}}$ obtained by Lemma (9.2.1) with $K = j_! i_* L(\omega)$. Consider its 'reduction modulo p' for $p \gg 0$. Then restrict to the closure of $H(v^\vee) \cap O_1$ in Z. Note that the acyclicity of $\tilde{f}$ on $X\backslash(H(v^\vee) \cap O_1)$ follows from the acyclicity of π by [Del1, 2.1.7.1].) This yields

$$\begin{aligned}&\operatorname{sign}(\varepsilon_0(H(v^\vee) \cap O_1, L(\omega)))\\ &\quad = -\chi_{1/2}(c' f(u)^{m-1} \Delta_u(f|_{H(v^\vee)\cap O_1})) \det(\mathrm{Frob}_q, L(\omega)_u)^{(-1)^m},\end{aligned} \tag{2}$$

with

$$c' = (-1)^{(m-1+\Sigma_N(\alpha_N - \beta_N))/2} \prod_N N^{\alpha_N - \beta_N}. \tag{3}$$

The lemmas (9.1.7) and (3.5.4) imply that

$$\begin{aligned}&\chi_{1/2}(f(u)^{m-1} \Delta_u(f|_{H(v^\vee)\cap O_1})) \det(\mathrm{Frob}_q, L(\omega)_u)^{(-1)^m}\\ &\quad = \chi_{1/2}(-dh^\vee(v^\vee) h(d^{-1}F^\vee(v^\vee)))\\ &\quad = \chi_{1/2}(-d),\end{aligned} \tag{4}$$

where the last equality follows from Lemma (6.1.10), and the equality $h(cv) = h(v)$ in $\mathbb{F}_q^\times/\mathbb{F}_q^{\times 2}$ for all $c \in \mathbb{F}_q^\times$ and $v \in O_1(\mathbb{F}_q)$, which can be proved by a direct calculation or by (6.1.9, (1)). Moreover from (3) and Proposition (4.3.5) one gets

$$\chi_{1/2}(c') = \chi_{1/2}\left(-(-1)^{(m+r)/2} d \prod_{j \geqslant 1} j^{e(j)}\right). \tag{5}$$

The formula (1) follows now from (2), (4) and (5). This terminates the proof of Theorem C. □

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