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# CONGRUENCE LATTICES OF FINITE SEMIMODULAR LATTICES

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ABSTRACT. We prove that every finite distributive lattice can be represented as the congruence lattice of a finite (planar) *semimodular* lattice.

1**. Introduction.** A classical result of R. P. Dilworth (*circa* 1940, unpublished, see [1], pp. 455–457) states that a finite distributive lattice $D$ can be represented as the congruence lattice of a finite lattice $L$.

There are a number of papers strengthening this result by requiring that the lattice $L$ representing $D$ have special properties. The lattice $L$ constructed by Dilworth is *atomistic*. A *sectionally complemented* lattice $L$ is constructed in G. Grätzer and E. T. Schmidt [7], while a *planar* lattice is constructed in G. Grätzer and H. Lakser [4]. A "small" lattice $L$ is constructed in G. Grätzer, H. Lakser, and E. T. Schmidt [5]: if $D$ has $n$ join-irreducible elements, the lattice $L$ is of size $O(n^2)$. (This is "best possible", according to G. Grätzer, I. Rival, and N. Zaguia [6].)

In this paper, we construct a *semimodular* lattice $L$:

THEOREM. *Every finite distributive lattice $D$ can be represented as the congruence lattice of a finite* semimodular *lattice $S$. In fact, $S$ can be constructed as a* planar *lattice of size $O(n^3)$, where $n$ is the number of join-irreducible elements of $D$.*

This result, with size $O(n^4)$, was announced in [9]; the present paper contains an improved construction, due to the second author, yielding size $O(n^3)$. It would be interesting to decide whether the size $O(n^2)$ is possible for (planar) semimodular lattices.

2**. Preliminaries.** We use the basic concepts and notations as in [2]; in particular, for a finite distributive lattice $D$, $\mathrm{J}(D)$ denotes the poset of join-irreducible elements. $\mathrm{Con}\,L$ denotes the congruence lattice of the lattice $L$. For a prime interval $\mathfrak{p} = [a, b]$, $\Theta(\mathfrak{p}) = \Theta(a, b)$ is the smallest congruence collapsing $a$ and $b$. $\mathfrak{C}_2$ denotes the two-element chain.

It is convenient to describe congruences of a finite lattice using coloring:

Let $L$ be a finite lattice and let $\Gamma$ be a finite set; the elements of $\Gamma$ will be called *colors*. A *coloring* $\mu$ of $L$ over $\Gamma$ is a map

$$\mu\colon \mathfrak{P}(L) \longrightarrow \Gamma$$

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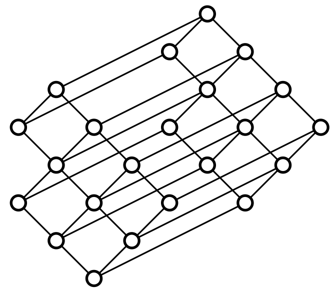

FIGURE 1: $D$

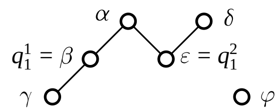


FIGURE 2: $J$

of the set of prime intervals $\mathfrak{P}(L)$ of $L$ into $\Gamma$ satisfying the condition: if two prime intervals generate the same congruence relation of $L$, then they have the same color; that is,

$$\mathfrak{p}, \mathfrak{q} \in \mathfrak{P}(L) \quad \text{and} \quad \Theta(\mathfrak{p}) = \Theta(\mathfrak{q}) \quad \text{imply that} \quad \mathfrak{p}\mu = \mathfrak{q}\mu.$$

Since the join-irreducible congruences of $L$ are exactly those that can be generated by prime intervals, equivalently, $\mu$ can be regarded as a map of the set $\mathrm{J}(\mathrm{Con}\, L)$ of join-irreducible congruences of $L$ into $\Gamma$:

$$\mu\colon \mathrm{J}(\mathrm{Con}\, L) \longrightarrow \Gamma.$$

In view of this condition, it is enough to define $\mu$ on sufficiently many prime intervals so that every prime interval is projective to one on which $\mu$ is defined.

Let $A$ and $B$ be lattices, $D_A$ a dual ideal of $A$, $I_B$ an ideal of $B$, and $D_B$ a dual ideal of $B$. Let us assume that $D_A$, $I_B$, and $D_B$ are isomorphic. We now define what it means that we obtain *$C$ by gluing $B$ to $A$ $k$-times*. For $k = 1$, let $C$ be the gluing of $A$ and $B$ over $D_A$ and $I_B$ with the dual ideal $D_B$ regarded as a dual ideal $D_C$ of $C$. Now if $C_{k-1}$ with the dual ideal $D_{C_{k-1}}$ is the gluing of $B$ to $A$ $k-1$-times, then we glue $C_{k-1}$ and $B$ over $D_{C_{k-1}}$ and $I_B$ to obtain $C$ the gluing of $B$ to $A$ $k$-times with the dual ideal $D_B$ regarded as a dual ideal $D_C$ of $C$. Observe that if $A$ and $B$ are semimodular, then so is $C$. Since we construct the lattice $S$ of the Theorem from semimodular components using gluing, the semimodularity of $S$ follows.

3**. The construction.** We construct the semimodular lattice $S$ of the Theorem in several steps. The construction is easy to follow on pictures but somewhat notational in a formal presentation. So we suggest that the reader follow it on the example we present; the example is the smallest one that illustrates various aspects of the construction. This example represents the 22-element distributive lattice $D$ of Figure 1 as the congruence lattice of a semimodular lattice. The poset $J$ of join-irreducibles has six elements, and it is shown in Figure 2.

Take the eight-element, nonmodular, semimodular lattice $S_8$ of Figure 3. $S_8$ has an ideal, $I_{S_8} = (b]$, and a dual ideal, $D_{S_8} = [c)$, both isomorphic to $\mathfrak{C}_2$; we shall utilize these

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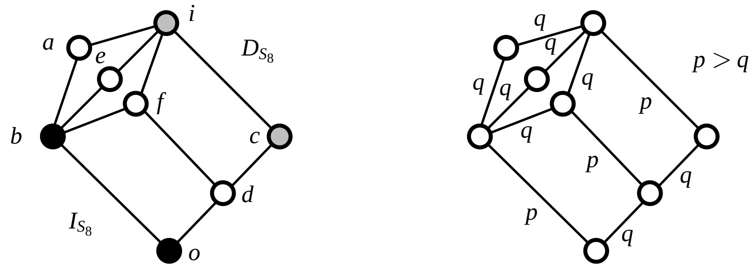


FIGURE 3: $S_8$

for repeated gluings. The elements of $I_{S_8}$ are black filled and the elements of $D_{S_8}$ are shaded in Figure 3. It is easy to see that the congruence lattice of $S_8$ is the three-element chain. Using the notation $\mathrm{J}(\mathrm{Con}\,\mathfrak{C}_3) = \{p, q\}$, with $p > q$, we also show the colored $S_8$ in Figure 3.

Let $D$ be a finite distributive lattice, and let $J = \mathrm{J}(D)$ be the poset of its join-irreducible elements, $n = |J|$. We enumerate

$$p_1, p_2, \ldots, p_m$$

the non-minimal elements of $J$. For every $p_i$, $i = 1, 2, \ldots, m$, let

$$v(p_i) = \{q_i^1, q_i^2, \ldots, q_i^{k_i}\}$$

denote the set of all lower covers of $p_i$ in $J$; since $p_i$ is non-minimal, it follows that $k_i > 0$. Let

$$r_1, r_2, \ldots, r_t$$

enumerate all elements of $J$ that are incomparable with all other elements.

In the example, $m = 3$, $t = 1$. Let

$$\begin{gathered} p_1 = \alpha, \quad v(\alpha) = \{\beta, \epsilon\}, \quad q_1^1 = \beta, \quad q_1^2 = \epsilon, \\ p_2 = \beta, \quad v(\beta) = \{\gamma\}, \\ p_3 = \delta, \quad v(\delta) = \{\epsilon\}. \end{gathered}$$

So $k_1 = 2$, $k_2 = k_3 = 1$.

*Step 1.* For every $i$, with $1 \leq i \leq m$, we construct a lattice $A_i$ with an ideal $I_i$ and a dual ideal $D_i$, where $I_i$ is a chain of length $2(k_i + \cdots + k_m)$ and $D_i$ is a chain of length $2(k_{i+1} + \cdots + k_m)$.

Now we shall twice use the construction, *gluing k-times*, described in Section 2. To form $A_i$, glue $S_8$ to itself $(k_i - 1)$-times with the ideal $I_{S_8}$ and the dual ideal $D_{S_8}$, to obtain the lattice $A_i^1$ with a dual ideal $D_{A_i^1}$. Now take

$$\mathfrak{C}_2^2 = \{\langle 0, 0\rangle, \langle 0, 1\rangle, \langle 1, 0\rangle, \langle 1, 1\rangle\}$$

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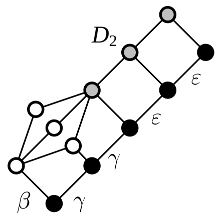


FIGURE 4: $A_2$

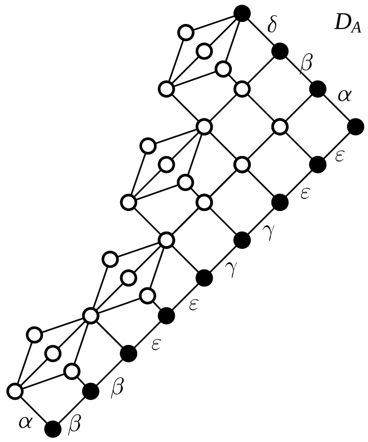


FIGURE 5: $A$

with the ideal

$$I_{\mathfrak{C}_2^2} = \{\langle 0,0\rangle, \langle 1,0\rangle\}$$

and the dual ideal

$$D_{\mathfrak{C}_2^2} = \{\langle 0,1\rangle, \langle 1,1\rangle\},$$

and glue $2(k_{i+1} + \cdots + k_m)$-times $\mathfrak{C}_2^2$ to $A_i^1$. The ideal $I_i$ is generated by the element $\langle 0,1\rangle$ of the top $\mathfrak{C}_2^2$, while $D_i$ is generated by the unit element of $A_i^1$.

We define a coloring $\mu_i$ of $A_i$ as follows. On any copy of $S_8$, $[o,b]\mu_i = p_i$ and on the $j$-th copy of $S_8$,

$$[o,d]\mu_i = [d,c]\mu_i = q_i^j;$$

on the first two copies of $\mathfrak{C}_2^2$,

$$\big[\langle 0,1\rangle, \langle 1,1\rangle\big]\mu_i = q_{i+1}^1,$$

on the next two copies,

$$\big[\langle 0,1\rangle, \langle 1,1\rangle\big]\mu_i = q_{i+1}^2,$$

after $k_{i+1}$ pairs, the next two satisfy

$$\big[\langle 0,1\rangle, \langle 1,1\rangle\big]\mu_i = q_{i+2}^1,$$

and so on.

Figure 4 shows $A_2$ for the example. The elements forming $I_2$ are black filled; the elements forming $D_2$ are shaded. Note that $I_2$ is of length $2(k_2 + k_3) = 4$, while $D_2$ is of length $2k_3 = 2$.

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LEMMA 1. *$\mu_i$ is a coloring of $A_i$. The join-irreducible congruences of $A_i$ are generated by prime intervals of $I_i$ and by $[o, b]$ of the bottom $S_8$ in $A_i$. If $\mathfrak{p}$ and $\mathfrak{q}$ are $[o, b]$ or a prime interval $[o, d]$ or $[d, c]$ of a copy of $S_8$ in $A_i$, then $\Theta(\mathfrak{p}) \geq \Theta(\mathfrak{q})$ iff $\mathfrak{p}\mu_i \geq \mathfrak{q}\mu_i$. In particular, $\Theta(o, b) \succ \Theta(o, d)$ in $\mathrm{J}(\mathrm{Con}\, A_i)$, where o, b, d are in a copy of $S_8$ in $A_i$. If $\mathfrak{p}$ is a prime interval $[\langle 0, 1\rangle, \langle 1, 1\rangle]$ in a copy of $\mathfrak{C}_2^2$, then $\Theta(\mathfrak{p})$ is incomparable to any $\Theta(\mathfrak{q})$, where $\mathfrak{q}$ is $[o, b]$ or a prime interval of $I_i$ different from $\mathfrak{p}$.*

PROOF. This is trivial since every prime interval of $S_8$ is projective to one of $[o, b]$, $[o, d]$, $[d, c]$. ∎

*Step 2.* We define the lattice $A$ by gluing together the (colored) lattices $A_i$, $1 \leq i \leq m$.

For $1 \leq i \leq m$, we define, by induction, the lattice $\overline{A}_i$, which contains $A_i$, and, therefore, $D_i$, as a dual ideal. Let $\overline{A}_1 = A_1$. Assume that $\overline{A}_i$ with $D_i$ as a dual ideal has been defined. Observe that both $D_i$ and $I_{i+1}$ are chains of length $2(k_{i+1} + \cdots + k_m)$, and so they are isomorphic; in fact, this isomorphism preserves colors. We glue $\overline{A}_i$ to $A_{i+1}$ over $D_i$ and $I_{i+1}$ to obtain $\overline{A}_{i+1}$. Define $A = \overline{A}_m$ and $I_A = I_1$.

Observe that $\mu_i$ on $D_i$ agrees with $\mu_{i+1}$ on $I_{i+1}$; therefore, the $\mu_i$, $1 \leq i \leq m$, define a coloring $\mu_A$ of $A$.

Let $D_A$ be the dual ideal of $A$ generated by the element $\langle 0, 1\rangle$ of the top $\mathfrak{C}_2^2$ in $A_1$. $D_A$ is a chain of length $m$. The prime interval $[o, b]$ in the bottom $S_8$ in $A_i$ ($1 \leq i \leq m$) is projective to a unique prime interval $\mathfrak{p}$ of $D_A$; define $\mathfrak{p}\mu_A = [o, b]\mu_A$.

Figure 5 show this lattice for the example. The elements of $I_A$ and $D_A$ are black filled.

LEMMA 2. *$\mu_A$ is a coloring of $A$. The join-irreducible congruences of $A$ are generated by prime intervals of $I_A$ and $D_A$. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime intervals in $I_A$ and $D_A$.*

*(i) If $\mathfrak{p}$ and $\mathfrak{q}$ are prime intervals of $D_A$, then $\Theta(\mathfrak{p})$ and $\Theta(\mathfrak{q})$ are incomparable.*

*(ii) If $\mathfrak{p}$ is a prime interval of $D_A$ and $\mathfrak{q}$ is a prime interval of $I_A$, then $\Theta(\mathfrak{p})$ and $\Theta(\mathfrak{q})$ are comparable iff $\mathfrak{p} \subseteq A_i$, for some $1 \leq i \leq m$, $\mathfrak{q}$ is perspective to some $[o, d]$ or $[d, c]$ in some $S_8$ in $A_i$; in which case, $\Theta(\mathfrak{p}) \succ \Theta(\mathfrak{q})$ in $\mathrm{J}(\mathrm{Con}\, A)$.*

*(iii) If $\mathfrak{p}$ and $\mathfrak{q}$ are prime intervals of $I_A$, then $\Theta(\mathfrak{p}) \geq \Theta(\mathfrak{q})$ iff $\mathfrak{p}$ and $\mathfrak{q}$ are perspective to prime intervals $\mathfrak{p}'$ and $\mathfrak{q}'$ in some $A_i$, respectively, for some $1 \leq i \leq m$, and $\mathfrak{p}'$ and $\mathfrak{q}'$ are adjacent edges of some $S_8$ in $A_i$; in which case, $\Theta(\mathfrak{p}) = \Theta(\mathfrak{q})$.*

PROOF. This is obvious from the statement that if $A$ and $B$ are glued together over the dual ideal $D$ of $A$ and the ideal $I$ of $B$, then a congruence of the glued lattice is obtained from a congruence $\Theta$ of $A$ and a congruence $\Phi$ of $B$ with the property that the restriction of $\Theta$ to $D$ agrees with the restriction of $\Phi$ to $I$. ∎

Observe that the congruence lattice of $A$ is still quite different from $D$ in two ways: the congruences that correspond to the $r_i$ are still missing; prime intervals in $I_A \cup D_A$ of the same color generate incomparable congruences with one exception: they are adjacent intervals in $I_A$, perspective to the two prime intervals of some $S_8$ in some $A_i$. For instance, in the example, see Figure 5, the prime interval of $D_A$ of color $\beta$ generates a congruence incomparable to the congruence generated by a prime interval of $I_A$ of color $\beta$; also, a prime interval of color $\epsilon$ in the top part of $I_A$ generates a congruence incomparable to the congruence generated by a prime interval of color $\epsilon$ in the lower part of $I_A$.

*Step 3.* We extend $A$ to a lattice $B$ with an ideal $I_B$ which is a chain and which has the property that every prime interval of $B$ is projective to a prime interval of $I_B$.

This step is easy. We form the lattice $D_A^2$ with the ideal

$$I_{D_A^2} = \{\langle x, 0_{D_A}\rangle \mid x \in D_A\},$$

where $0_{D_A}$ is the zero of $D_A$. Let $1_{D_A}$ denote the unit element of $D_A$ and, for $x \in D_A$, $x < 1_{D_A}$, let $x^*$ denote the cover of $x$ in $D_A$. For every $x \in D_A$, $x < 1_{D_A}$, we add an element $m_x$ to $D_A^2$ so that the elements

$$\langle x, x\rangle, \langle x, x^*\rangle, \langle x^*, x\rangle, x_m, \langle x^*, x^*\rangle$$

form a sublattice isomorphic to $\mathfrak{M}_3$ with $\langle x, x\rangle$ as zero and $\langle x^*, x^*\rangle$ as unit. Let $M$ be the resulting lattice. Obviously, $M$ is a finite planar modular lattice whose congruence lattice is isomorphic to the congruence lattice of $D_A$. $I_{D_A^2}$ is also an ideal of $M$; we shall denote it by $I_M$.

Figure 6 shows $M$ for the example. The elements of $I_M$ are black filled.

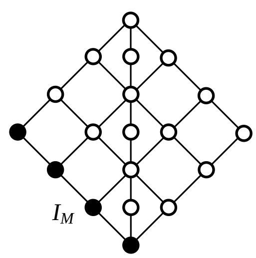


FIGURE 6: $M$

We glue $A$ to $M$ over $D_A$ and $I_M$ to obtain $B$. Let $I_B$ be defined as the ideal generated by $\langle 0, 1_{D_A}\rangle$. We define $\mu_B$ as an extension of $\mu_A$; every prime interval $\mathfrak{p}$ of $M$ is projective to exactly one prime interval $\overline{\mathfrak{p}}$ of $I_M$, we define $\mathfrak{p}\mu_B = \overline{\mathfrak{p}}\mu_A$.

LEMMA 3. *$\mu_B$ is a coloring of $B$. The join-irreducible congruences of $B$ are generated by prime intervals of $I_B$. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime intervals in $I_B$.*

*(i) If $\mathfrak{p}$ and $\mathfrak{q}$ are prime intervals of $M$, then $\Theta(\mathfrak{p})$ and $\Theta(\mathfrak{q})$ are incomparable.*

*(ii) If $\mathfrak{p}$ is a prime interval of $M$ and $\mathfrak{q}$ is a prime interval of $I_A$, then $\Theta(\mathfrak{p})$ and $\Theta(\mathfrak{q})$ are related exactly as $\Theta_A(\overline{\mathfrak{p}})$ and $\Theta_A(\mathfrak{q})$ are related in $A$.*

*(iii) If $\mathfrak{p}$ and $\mathfrak{q}$ are prime intervals of $I_A$, then $\Theta(\mathfrak{p})$ and $\Theta(\mathfrak{q})$ are related exactly as $\Theta_A(\mathfrak{p})$ and $\Theta_A(\mathfrak{q})$ are related in $A$.*

PROOF. This is obvious from the congruence structure of $M$. ∎

*Step 4.* We extend $B$ to the lattice $S$ of the Theorem.

This is also an easy step. We take a chain $C$ of length $n$ and we color $C$ over $J$ so that the coloring is a bijection. We form the lattice $C \times I_B$. For every pair of prime intervals,

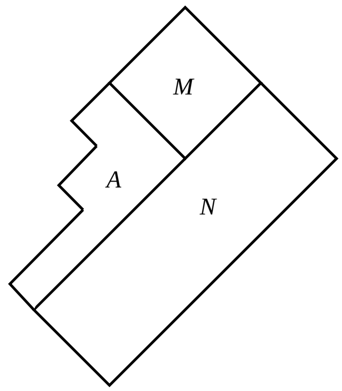


FIGURE 7: $S$

$\mathfrak{p} = [a, b]$ of $C$ and $\mathfrak{q} = [c, d]$ of $I_B$, if $\mathfrak{p}$ and $\mathfrak{q}$ have the same color, then we add an element $m(\mathfrak{p}, \mathfrak{q})$ to $C$ over $J$ so that the elements

$$\langle a, c\rangle, \langle b, c\rangle, \langle a, d\rangle, m(\mathfrak{p}, \mathfrak{q}), \langle b, d\rangle$$

form a sublattice isomorphic to $\mathfrak{M}_3$. Let $N$ denote the resulting lattice. $N$ is obviously modular and planar. Set

$$I_N = \{\langle x, 0_{I_B}\rangle \mid x \in C\},$$
$$D_N = \{\langle 1_C, x\rangle \mid x \in I_B\},$$

where $0_{I_B}$ is the zero of $I_B$ and $1_C$ is the unit of $C$. Then $I_N$ is the ideal of $N$ (isomorphic to $C$) and $D_N$ is a dual ideal of $N$ (isomorphic to $I_B$). Every prime interval of $N$ is projective to a prime interval of $I_N$, so we have a natural coloring $\mu_N$ on $N$. Note that this coloring agrees with the coloring $\mu_B$ on $D_N$ under the isomorphism with $I_B$.

We glue $N$ to $B$ over $D_N$ and $I_B$ to obtain $S$ with the coloring $\mu_S$. Set $I_S = I_N$. Figure 7 is a sketch of $S$.

It is clear from the construction and from the lemmas that every prime interval of $S$ is projective to a prime interval of $I_S$ and that distinct prime intervals of $I_S$ generate distinct join-irreducible congruences of $S$.

It remains to see that if $\mathfrak{p}$ and $\mathfrak{q}$ are distinct prime intervals, then $\Theta(\mathfrak{p}) \geq \Theta(\mathfrak{q})$ iff $\mathfrak{p}\mu_S \geq \mathfrak{q}\mu_S$. Since $J$ is finite, it is sufficient to prove that $\Theta(\mathfrak{p}) \succ \Theta(\mathfrak{q})$ in $\mathrm{J}(\mathrm{Con}\, S)$ iff $\mathfrak{p}\mu_S \succ \mathfrak{q}\mu_S$ in $\mathrm{J}(D)$. But this is clear since if $\mathfrak{p}\mu_S \succ \mathfrak{q}\mu_S$ in $\mathrm{J}(D)$, then $p\mu_S = p_i$, for some $1 \leq i \leq m$, and $q\mu_S = q_i^j$, for some $1 \leq j \leq k_i$, so $\Theta(\mathfrak{p}) \succ \Theta(\mathfrak{q})$ was guaranteed in $A_i$.

To establish that the size of $S$ is $O(n^3)$, we give a very crude upper bound for $|S|$. $2n^2 + 1$ is an upper bound for $|I_i|$, $1 \leq i \leq m$, so $3(2n^2 + 1)$ is an upper bound for $|A_i|$ and $3(2n^2 + 1)n$ is an upper bound for $|A|$. Since $|D_A| \leq n + 1$, we get the upper bound $(n+1)^2+n+1$ for $|M|$. Finally, $|I_B| \leq 2n^2+1+n+1 = 2n^2+n+2$, so $|N| \leq 2(2n^2+n+2)(n+1)$. Therefore,

$$3(2n^2 + 1)n + (n + 1)^2 + n + 1 + 2(2n^2 + n + 2)(n + 1)$$

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is an upper bound for $S$ and it is a cubic polynomial in $n$. This completes the proof of the Theorem.

It is not difficult to find better upper bounds for $|S|$; for instance,

$$|S| \leq 3n^3 + 2n^2 - 7n + 4.$$

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