

ON THE RECONSTRUCTION CONJECTURE FOR SEPARABLE GRAPHS

V. KRISHNAMOORTHY[†] and K. R. PARTHASARATHY

(Received 9 August 1979; revised 26 June 1980)

Communicated by J. N. Crossley

Abstract

Some sufficient conditions for the reconstructability of separable graphs are given proceeding along the lines suggested by Bondy, Greenwell and Hemminger. It is shown that the structure and automorphism group of a central block plays an important role in the reconstruction.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 05 C 60.

1. Introduction

1.1. Notation and Terminology

We consider only finite, unoriented, ordinary graphs and follow Harary (1969) for basic terminology and use the following additional terminology.

The maximal subgraph without pendant vertices of a graph G is called the *pruned graph* of G . The block or cutpoint corresponding to the unique central point of the block-cutpoint-tree ($bc(G)$) of G is the *autocenter* of G denoted by $C(G)$ or C . The autocenter of the pruned graph is the *pruned center* P of G . Let $v \in V(P)$. If P is a block the maximal connected subgraph of G whose intersection with P is v is the *branch* of P at v . If $P (= v)$ is a cutpoint, then any maximal connected subgraph of G which contains v as a noncutpoint is a branch at v . Let B be a block of G and X be a block or cutpoint of G . The *block distance* $D(X, B)$ is the number of blocks between X and B , including B and excluding X if X is a block. This is k if the distance between the corresponding

©Copyright Australian Mathematical Society 1981

[†] Research supported by CSIR, India, while the author V. Krishnamoorthy was at Indian Institute of Technology, Madras.

points of $bc(G)$ is $2k$ or $2k - 1$. The *block length* of a branch at X is defined as $\max D(X, B)$ where B is any block in that branch. A branch at C of maximum length, say r , is called a *radial branch* and r is called the *block radius* of G . A block of G is called a *peripheral block* if it corresponds to a peripheral point (end of a diameter) of $bc(G)$. Figure 1 may be helpful in understanding these concepts.

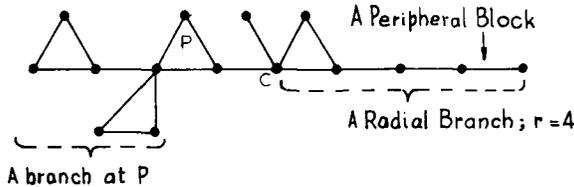


FIGURE 1

By a branch in this paper we always mean a branch at the pruned center P .

Let

$$\begin{aligned}
 V_1 &= \{u \in V \mid \deg u = 1\}, \\
 \mathcal{G} &= \{G - u \mid u \in V(G)\}, \\
 \mathcal{G}_1 &= \{G - u \mid \deg u = 1\} \quad \text{and} \\
 \mathcal{G}_{\bar{P}} &= \{G - u \mid u \notin V(P)\},
 \end{aligned}$$

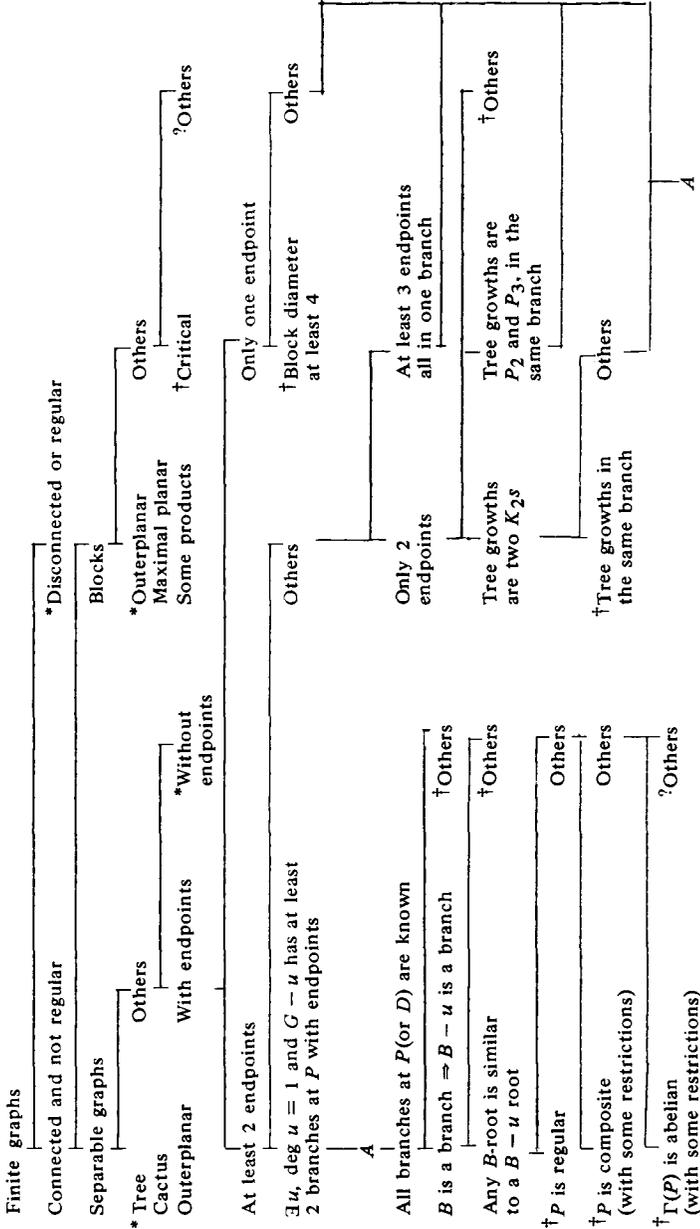
where the multiplicities of the $G - u$'s are also taken into account.

G is said to be reconstructable (or 1-reconstructable or \bar{P} -reconstructable) if G is reconstructable from the collection \mathcal{G} (or \mathcal{G}_1 or $\mathcal{G}_{\bar{P}}$, respectively). Clearly $\mathcal{G} \supseteq \mathcal{G}_{\bar{P}} \supseteq \mathcal{G}_1$. Thus, 1-reconstructability $\Rightarrow \bar{P}$ -reconstructability \Rightarrow reconstructability.

1.2. Review and Scope

The reconstruction conjecture states that every graph G is reconstructable. This has been settled in the affirmative for a few classes of graphs, in particular for separable graphs without pendant vertices (Bondy (1969)). Bondy's conjecture that G is 1-reconstructable if $|V_1|$ is sufficiently large was disproved in Bryant (1971). However, 1-reconstructability of some classes of separable graphs with pendant vertices was established in Greenwell and Hemminger (1969), making crucial use of the notion of pruned center. In this paper we first show that not all graphs are \bar{P} -reconstructable and then give some sufficient conditions for the \bar{P} -reconstructability of graphs.

This shows that to settle the reconstruction conjecture for separable graphs eventually one will have to consider cards from the deck $\mathcal{G}_P = \{G - u \mid u \in V(P)\}$. One attempt to settle the conjecture may be by establishing the reconstructability of graphs with various mutually exclusive and exhaustive assumptions on the nature of P .



A 'Tree' of Results

*Known result; † proved in the thesis; ? unsettled.

In this paper we show that G is reconstructable if P is regular and \bar{P} -reconstructable if P has an abelian automorphism group. In Krishnamoorthy (1976), one of us has established the reconstructability of a graph whose P is a composite block (that is, P is expressible as the Cartesian product of graphs). The 'tree' of results in that thesis is given here for an over all view of the Reconstruction Conjecture with reference to the present work.

2. A counter example

In the counter example constructed in Bryant (1971) to show that there are graphs which are not 1-reconstructable, P is a block and all the nontrivial branches are isomorphic to K_2 . Clearly these graphs are also non \bar{P} -reconstructable. The following steps explain the construction of some graphs which are not \bar{P} -reconstructable, and in which the branches are not restricted to be K_2 .

Step 1: Let $\{H_1, H_2, \dots, H_m\}$ be a set of distinct nonisomorphic branches (connected rooted graphs) such that if u is any noncutpoint (excluding the root) of H_i , then $H_i - u$ is some H_j , where $1 \leq i, j \leq m$.

Step 2: Consider a block H , such that $\{v_1, \dots, v_m\} \subseteq V(H)$ and the automorphism group of H when restricted to $\{v_1, \dots, v_m\}$ gives A_m , the alternating group on m letters. Existence of such a graph is proved in Bouwer (1969) and Krishnamoorthy and Parthasarathy (1978).

Step 3: Identifying the root of H_i with v_i , for $1 \leq i \leq m$, we get G . Interchange the positions of any two branches to get G' , non-isomorphic to G . H_i can be chosen such that H is the pruned center of G .

It can be easily seen that the collection $\mathcal{G}_{\bar{P}}$ is the same as the corresponding collection for G' and hence G is not \bar{P} -reconstructable.

Note 2.1: We can adjust the set $\{H_i\}$ in such a way as to include any given set of branches.

Note 2.2: The counter example can be modified so that the branches occur at different similarity classes of $V(P)$. When we talk of points of P similar to each other we consider P as a graph.

3. \bar{P} -reconstructability

The above counter example leads to the natural question: When is G \bar{P} -reconstructable?

We first assume that there exists a pendant vertex v such that $G - v$ has at least two branches with pendant vertices (Assumption I). This gives us all the branches

of G (at P) (see Greenwell and Hemminger (1969), p. 105), and hence if P is a cutpoint G is obviously \bar{P} -reconstructable. Henceforth let P be a block. Once we know all the branches and P , we can find whether C is a cutpoint or not and whether C intersects P . This is found by noting that in whatever way the branches are attached to different points of P , $D(C, P)$ is going to be the same.

THEOREM 3.1. *If $C \cap P \neq C$, then G is 1-reconstructable.*

PROOF. If u is a pendant vertex of G , it is obvious that $C(G - u)$ will be the same as $C(G)$ or $C(G - u)$ will be a cutpoint on $C(G)$ if $C(G)$ is a block and it will be a block containing $C(G)$ if $C(G)$ is a cutpoint. Since $C \cap P \neq C$ and P is a block, in any $G - u \in \mathcal{G}_1$, $C(G - u) \neq P$. Hence in any $G - u \in \mathcal{G}_1$, we can find $P(G - u) (= P)$ and fix the branch containing $C(G - u)$ and $C(G)$. Choosing a $G - u \in \mathcal{G}_1$ such that the branch at P containing $C(G - u)$ is not a branch of G , we can reconstruct G , by replacing this branch by the suitable branch of G .

Henceforth we assume that $P \cap C = C$.

THEOREM 3.2. *If there is a branch B containing a noncutpoint u such that $B - u$ is not a branch of G , then G is \bar{P} -reconstructable.*

PROOF. *Case 1.* $C(G) = P$. In the collection of branches of G , replacing one B by $B - u$, we can find $r(G - u)$ and determine whether $C(G - u)$ is a block or a cutpoint.

Case 1a. Suppose $r(G - u) = r(G)$ and $C(G - u)$ is a block. Then $C(G - u)$ is the same as P . Find a $G' \in \mathcal{G}$ with $r(G') = r(G)$ and $C(G')$ as a block isomorphic to P . Moreover choose G' such that a branch $B - u$ appears instead of a B . G is reconstructed from this G' by replacing the branch $B - u$ by B .

Case 1b. $r(G - u) = r(G)$ and $C(G - u)$ is a cutpoint. This implies that u is the only pendant vertex in the only peripheral block in B and there exists only one more branch with peripheral blocks and $P(G - u) = P(G)$ and $C(G - u)$ is a cutpoint on $P(G)$. Considering such a graph $G' \in \mathcal{G}$, since we can find out $P(G - u) = P(G)$ in that G' , we can reconstruct G by replacing the branch $B - u$ by B .

Case 1c. $r(G - u) > r(G)$. As before we know whether $C(G - u)$ is a cutpoint or a block. Choose a $G' \in \mathcal{G}$ which has a block isomorphic to P , having all the branches of G except one B replaced by $B - u$. If the choice of such a block isomorphic to P in G' is unique, then G is reconstructed in the usual way. Suppose there are at least two choices for P in G' . It can be easily checked (Krishnamoorthy (1976)) that the other choices for P are only among the blocks

in between P and B_v (where $G' = G - v$ and B_v is the block containing the point v) or the blocks in $B_v - v$. Hence the existence of the second choice for P in G' implies that the number of points in B is more than the total number of points in all the other branches and P . Hence if B contains a pendant vertex w , then clearly $B - w$ is also not a branch and in this case we will be in one of the previous cases. So, let there be no pendant vertex in B .

Since there are at least two pendant vertices in the other branches and as these branches appear in $B - v$ (by the second choice for P), and since B does not have pendant vertices, these pendant vertices of $B - v$ have come from the points of degree 2 in B_v and they are adjacent to v in G . Now we can choose a point w of degree 2 (adjacent to at most one point of degree 2) in B_v which gives only one end vertex in $B - w$. In this case, there will be only one choice for P in $G - w$ and hence G is reconstructed.

Case 2. $C(G)$ is a cutpoint on P . The proof is not much different from that of Case 1 and is omitted.

This completes the proof.

Henceforth we assume that if E is a branch of G , then so also is any $E - w$, $w \in V(E)$ (Assumption II). This leads to the following observations.

COROLLARY 3.3. P is the block with maximum number of points in G .

Note 3.4. The deck $\mathcal{G}_{\bar{P}}$ can be recognized in \mathcal{G} .

Note 3.5. In any $G' \in \mathcal{G}_{\bar{P}}$, P can be identified.

Note 3.6. In any $G - u$, where $u \in V(P)$ and $P - u$ is a block, $P - u$ can be identified.

Theorem 3.2 is sharpened as follows.

THEOREM 3.7. *Suppose there is a branch B of G , rooted at $v \in V(P)$ and a noncutpoint u of B such that no branch isomorphic to $B - u$ is rooted at a point similar to v in P . Then G is \bar{P} -reconstructable.*

PROOF. Let S_1, S_2, \dots, S_k be the similarity classes in P where P is considered as a graph. These are known, as P is known. Note that there will be nontrivial ($\neq K_1$) branches attached to at least two S_i 's, because of Theorem 3.2 and the assumption following it. First we find the branches at each S_i separately.

Consider a $G' \in \mathcal{G}_{\bar{P}}$ which contains maximum number of points at the branches in an S_i . Clearly this gives all the branches at S_i , as the deleted point has come from a branch attached to some S_j , where $i \neq j$.

Let B be a branch attached to a point in S_1 , say, such that $B - u$ is not a branch in any point in S_1 . Now considering a $G' \in \mathcal{G}_{\bar{P}}$ where at S_1 , one $B - u$

appears instead of a B , we can reconstruct G . (Note that by Note 3.4, P is identified in any $G' \in \mathcal{G}_{\bar{P}}$.) This completes the proof.

Henceforth we assume: *If B is a branch with a noncutpoint u , at a point v of P , then there is a branch isomorphic to $B - u$ at a point similar to v in P (Assumption III).*

4. Reconstructability

Theorem 3.7 highly restricts the separable graphs that are left out to be considered for reconstruction. It is of interest to know under what conditions on P , G will be \bar{P} -reconstructable or reconstructable.

The following example shows that P is regular does not imply the \bar{P} -reconstructability of G .

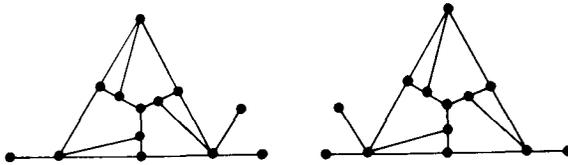


FIGURE 2

However we have

THEOREM 4.1. *If P is regular then G is reconstructable.*

PROOF. *Case 1.* P is regular of degree > 3 . Then P is a noncritical block (Behzad and Chartrand (1971), Theorem 2.10). So, there is a point u in P such that $P - u$ is a block. Now consider a $G' \in \mathcal{G}$ such that there is no block isomorphic to P and there is a block isomorphic to $P - u$. Since P is regular, it can be reconstructed from the $P - u$ in G' and the missing branch if any, can be attached to the new vertex added to $P - u$, to get G .

Case 2. P is a cycle. In this case P is the largest cycle in G and proceeding exactly as in the case of unicyclic graphs (Manvel (1969)) G is reconstructed.

The proof is complete.

5. Abelian groups

We first show that under certain conditions, the only counter examples one can have to \bar{P} -reconstructability are the ones described in Section 2. We know

that the group A_n plays a vital role in these examples and in an attempt to avoid this group, we consider abelian groups in this section.

Let Γ be the restriction of $\Gamma(P)$ to S where $S = A \cup B$ and $A = \{x \in V(P) | x \text{ is a nontrivial root in } P\}$ and $B = \{y \in V(P) | y \text{ is similar (in } P) \text{ to some } x \in A\}$. Let G satisfy assumption I. Hence we know P and all the branches of G . Suppose G is not \bar{P} -reconstructable. This implies that Assumptions II and III are also satisfied and if H is any graph with $\mathcal{G}_{\bar{P}} = \mathcal{H}_{\overline{P(H)}}$, then clearly $P(H) = P$, the branches of H are the same as those of G and the only difference between G and H is in the attachment of the branches to P .

THEOREM 5.1. *Let G be non \bar{P} -reconstructable and $\mathcal{G}_{\bar{P}} = \mathcal{H}_{\bar{P}}$. Let S defined above be a similarity class. If every branch (including the trivial one) occurs only once at the points of S then Γ is the alternating group A_n , where $|S| = n$.*

PROOF. By assumption II, if B is a branch of G , so also is any $B - u$, u a noncutpoint of B . Let B_i denote the branch at the point i of P . Let

$$S = \{1, \dots, n\} \quad \text{where } |V(B_i)| < |V(B_j)| \quad \text{if } i < j.$$

Hence $B_1 = K_1$, $B_2 = K_2$ and $|V(B_3)| = 3$. Consider the $G' \in \mathcal{G}_{\bar{P}}$ where one K_2 -branch (that is, a branch isomorphic to K_2) is missing. In this G' , there are two points in S with trivial branches and since, by assumption, G is not \bar{P} -reconstructable, by attaching a K_2 to the point 1, we must get H of the theorem. This further implies that $(1\ 2) (= (1\ 2)(3)(4) \cdots (n)) \notin \Gamma$. Now, consider a G' in which B_3 is missing and instead a K_2 appears at 3. Since the corresponding H' is isomorphic to this G' , we get $(1\ 2\ 3) \in \Gamma$. Hence Γ gives A_3 on $\{1, 2, 3\}$ when restricted as identity on $\{4, 5, \dots, n\}$ pointwise.

Let the following statement be the basis for induction. “ Γ gives A_r on $\{1, 2, \dots, r\}$, when it is restricted as identity on $\{r + 1, \dots, n\}$ pointwise, where $3 < r < n$.”

We show that Γ when restricted as identity on $\{r + 2, \dots, n\}$ pointwise, gives A_{r+1} on $\{1, 2, \dots, r + 1\}$. Let u be a noncutpoint of B_{r+1} . Let $B_{r+1} - u \cong B_k$ where $2 \leq k < r$. Then $G - u \cong H - u$ implies that $(1\ 2)(k\ r + 1) \in \Gamma$. Using this, it can be easily shown that Γ contains all the even permutations on $\{1, 2, \dots, r + 1\}$ when restricted as identity on $\{r + 2, \dots, n\}$ pointwise. Since this group cannot be the symmetric group S_{r+1} (since it does not contain $(1\ 2)$) it is A_{r+1} . Thus, the proof is complete by induction.

Now, we assume that Γ is abelian. (This is slightly more general than assuming that $\Gamma(P)$ is abelian).

If Γ is transitive (on S), that is when S is a similarity class of P , we have $|\Gamma| = |S|$ and there is only one element in Γ taking a point $u \in S$ to a point

$v \in S$ (Wielandt (1964), Proposition 4.4). Hence labeling any arbitrary point of S , say v , as e , the identity element of Γ , we can uniquely label the other points of S with the other elements of Γ as follows. If $\alpha(v) = w$, then label w as α .

If Γ is intransitive, that is when S is a union of (more than one) similarity classes, restricting Γ to each of the similarity classes, we can proceed just as above.

THEOREM 5.2. *Let Γ be transitive on S . Let there be at least 3 branches in G with at least one of them different from K_2 . Then G is \bar{P} -reconstructable.*

PROOF. Let B be a branch such that $|V(B)| > 3$.

Case 1. Suppose there is only one B -branch in G . We can assume that B has maximum number of points among all such branches appearing only once. From a G' where one K_2 -branch is missing, we can find the positions (in terms of labeling) of the other branches with respect to the B -root (that is, root of a B -branch) at e . If there is a branch D , apart from the missing K_2 , such that $D - u \not\cong B$, then from the corresponding $G - u$ where one D -branch is missing, we can find the position of the missing K_2 in G' and hence G is reconstructed.

Suppose there is only one K_2 -branch and for every other branch $D \not\cong B$, $D - u \cong B$. This implies that $|V(B)| = 3$ and $|V(D)| = 4$ and the number of D -branches is at least two. Now, consider all the graphs (in $\mathcal{G}_{\bar{P}}$) in which two B -branches appear. In each of these graphs label the unique K_2 -root as e and the B -roots accordingly. Only the labels of the B -roots are common in all these graphs. This indicates the position of the missing D -branch in any such graph and hence G is reconstructed.

Case 2. Every non K_2 -branch appears at least twice. Let B be such a branch with maximum number of points. Consider a G' where one K_2 -branch is missing. Let $\{e, \alpha_1, \dots, \alpha_k\}$ be the labels of the B -roots starting with some B -root as e .

Subcase 2.a. Suppose starting with some other B -root, say α_i , we get the same labels for the B -roots. This implies that

$$\{e, \alpha_1, \dots, \alpha_k\} = \{\alpha_i^{-1}, \alpha_i^{-1} \alpha_1, \dots, \alpha_i^{-1} \alpha_k\}.$$

That is, in any labelling of S if there is a B -branch at β , then there are B -branches at $\alpha_i \beta$ and $\alpha_i^{-1} \beta$ also. Hence in any G' where one B -branch is missing we can easily find the position of the missing B -branch and hence G is reconstructed.

Subcase 2.b. Suppose starting with each B -root as e , we let different sets of labels for the B -roots. Let $\{e, \alpha_1, \dots, \alpha_k\}$ be the labels of B -roots starting from a certain B -root, say v . If all the B -branches are present in a G' , then we can distinguish the point v in G' .

Suppose there is a branch D different from B and K_2 . Since $|V(B)|$ is maximum, any $D - u \cong B$. Now from a G' where one K_2 -branch is missing, find the labels of the D -roots with the point v as e . Now from a G'' where one D -branch is missing, G is easily reconstructed.

Suppose the branches different from B are all isomorphic to K_2 . Hence $|V(B)| = 3$ (by Assumption II). If there are at least two K_2 -branches, then comparing two graphs where one K_2 -branch is missing in each, we can reconstruct G (as we can fix v in both these graphs). So, let there be only one K_2 -branch in G . Suppose $\{\beta_1, \dots, \beta_k\}$ are the labels of the B -roots where the K_2 -root is e (we do not know these labels now). Consider a G' in which there are two K_2 -branches. So there are two choices for fixing the K_2 -branch. One choice gives the labels $\{\beta_1, \dots, \beta_k\}$ for the B -roots and the other choice gives $\{\beta_i^{-1}\beta_1, \dots, \beta_i^{-1}\beta_k, \beta_i^{-1}\} - \{e\}$ where $1 \leq i \leq k$. Considering some other similar G'' , we can get

$$\{\beta_1, \dots, \beta_k\} \quad \text{and} \quad \{\beta_j^{-1}\beta_1, \dots, \beta_j^{-1}\beta_k, \beta_j^{-1}\} - \{e\},$$

where $j \neq i$. If

$$\{\beta_i^{-1}\beta_1, \dots, \beta_i^{-1}\beta_k, \beta_i^{-1}\} \neq \{\beta_j^{-1}\beta_1, \dots, \beta_j^{-1}\beta_k, \beta_j^{-1}\}$$

then we can distinguish $\{\beta_1, \dots, \beta_k\}$ and using this in the G' considered, G is reconstructed. Otherwise,

$$\{\beta_i^{-1}, \beta_i^{-1}\beta_1, \dots, \beta_i^{-1}\beta_k\} = \{\beta_j^{-1}, \beta_j^{-1}\beta_1, \dots, \beta_j^{-1}\beta_k\},$$

where $1 \leq i, j \leq k$.

This implies

$$\{\beta_i^{-1}, \beta_i^{-1}\beta_1, \dots, \beta_i^{-1}\beta_k\} = \{e, \beta_1^{-1}, \beta_2^{-1}, \dots, \beta_k^{-1}\}.$$

Multiplying all the elements by β_i , we get

$$\{e, \beta_1, \dots, \beta_k\} = \{\beta_i, \beta_i\beta_1^{-1}, \beta_i\beta_2^{-1}, \dots, \beta_i\beta_k^{-1}\}.$$

Hence every $\beta_i\beta_j^{-1} \in \{e, \beta_1, \dots, \beta_k\}$ (where $1 \leq i, j \leq k$) and

$$\{\beta_i^{-1}, \beta_i^{-1}\beta_1, \dots, \beta_i^{-1}\beta_k\} = \{e, \beta_1, \dots, \beta_k\}.$$

This shows $\{e, \beta_1, \dots, \beta_k\}$ is a subgroup of Γ .

Hence starting from any B -root as e , we will get the labels of the B -roots as $\{e, \beta_1, \dots, \beta_k\} - \{\beta_i\}$ for some i , and the K_2 -branch will be at β_i . Now G is easily reconstructed from the G' where the K_2 -branch is missing.

THEOREM 5.3. *Let Γ be intransitive. Let there be at least 3 branches with at least one of them different from K_2 . Then G is \bar{P} -reconstructable.*

PROOF. As we can deal with each of the similarity classes of S separately, by the previous theorem it is enough to consider the case where each of the

similarity classes having a non K_2 -branch has just two nontrivial branches. This implies that any (non-trivial) non K_2 -branch has just 3 points. Let S_1 be a similarity class of S with the non K_2 -branch B . From a G' where one K_2 -branch is missing from some other similarity class, we can find the label of the K_2 -root at S_1 , with the B -root at S_1 as e . Now from the graph in which the K_2 -branch at S_1 is missing, G is reconstructed. This completes the proof.

The remaining case is when all the branches are isomorphic to K_2 . In this case we prove a theorem under the assumption that Γ is cyclic and transitive.

THEOREM 5.4. *Let Γ be cyclic and transitive. Let all the branches be isomorphic to K_2 . If the number of branches is at least 4, then G is \bar{P} -reconstructable.*

PROOF. Since Γ is cyclic and transitive on S , any generator of Γ contains a single cycle of length $|S|$. Let $(1\ 2\ \cdots\ k)$ be such a generator. In any $P(G')$, $G' \in \mathcal{G}_{\bar{P}}$, we can fix the orientation given by this cycle. We can *imagine* a directed cycle corresponding to $(1\ 2\ \cdots\ k)$ and for our purpose, think that P is replaced by this directed cycle.

(Note that here we cannot proceed as in the case of unicyclic graphs, by deleting the points of the cycle, since the cycle is an imaginary one.)

Two nontrivial roots are said to be consecutive if there is a path joining them in the (imagined) directed cycle such that the other points of the path are trivial roots. A new distance between two consecutive roots is defined as the length of such a path.

Since all the branches are isomorphic to K_2 , it is enough to find the distances between the consecutive roots in the cyclic order. First, we determine these distances without any order. For this, choose a $G' \in \mathcal{G}_{\bar{P}}$ which has a maximum distance, say p , between two consecutive roots. Since this p is a sum of two adjacent distances of G , the other distances of this G' are distances in G also. Let r, s be two adjacent distances of G given by G' . Consider a $G'' \in \mathcal{G}_{\bar{P}}$ with maximum number of distances equal to $r + s$ and minimum number of distances equal to r . Clearly by splitting a distance $r + s$ in G'' as r and s , we get all the distances of G .

In any G' , which has a distance p , we can find out which combination of distances (without any order), say m and n , has given rise to the distance p , by comparing the distances of G and G' . If $m = n$, G is easily reconstructed. Hence for definiteness let $m > n$. Let c_{mn} be the number of combinations of adjacent distances m and n in G giving rise to p where the distance m occurs first (in the cyclic order). Let c_{nm} be defined similarly. Let $c = c_{mn} + c_{nm}$. Clearly c is the number of graphs in $\mathcal{G}_{\bar{P}}$ with a distance p appearing, instead of an m and an n .

Of course, we don't know the numbers c_{mn} and c_{nm} now, though we know c . Let c'_{mn} , c'_{nm} , c' denote the corresponding numbers for any $G' \in \mathcal{G}_{\bar{p}}$ with a distance p . These numbers are, of course, known.

Case 1. Suppose $c > 1$. Once we know the numbers c_{mn} and c_{nm} we can reconstruct G as follows. Five different cases arise.

- (a) $c' = c - 1$; $c'_{mn} = c_{mn} - 1$; $c'_{nm} = c_{nm}$.
- (b) $c' = c - 1$; $c'_{mn} = c_{mn}$; $c'_{nm} = c_{nm} - 1$.
- (c) $c' = c - 3$; $c'_{mn} = c_{mn} - 1$; $c'_{nm} = c_{nm} - 2$.
- (d) $c' = c - 3$; $c'_{mn} = c_{mn} - 2$; $c'_{nm} = c_{nm} - 1$.
- (e) $c' = c - 2$; $c'_{mn} = c_{mn} - 1$; $c'_{nm} = c_{nm} - 1$.

In the first case the distance p in G' is replaced by m and then n to get G . The second case is similar. In case c , the distances in G' will be of the form $\dots n, p, m \dots$ and they are replaced by n, m, n, m, \dots to get G . In case d , the distances $\dots m, p, n, \dots$ of G' are replaced by $\dots m, n, m, n, \dots$. Suppose every G' considered is in case e . The four possibilities for the consecutive distances with p in the middle are

- (1) $\dots r, m + n, m, \dots, r \neq m, n$
- (2) $\dots m, n + m, r, \dots; r \neq m, n$
- (3) $\dots r, n + m, n, \dots; r \neq m$ and
- (4) $\dots n, m + n, r, \dots; r \neq m$.

The splitting of $m + n$ is easily found in the first two cases and also in the other cases if $r \neq n$. Suppose $r = n$. Now, consider a G'' with a distance p and minimum number of n 's occurring consecutively after p . In this G'' clearly m appears first in the splitting of p . Hence G is reconstructed.

Now our aim is to find c_{mn} and c_{nm} . If there are at least two different combinations of adjacent distances of G , such that $m + n = p = x + y$ then from a G' where x and y give rise to p , we can find c_{mn} and c_{nm} . Suppose only the combination m and n gives p . If there is a G' where $c' = c - 2$, then $c_{mn} = c'_{mn} + 1$ and $c_{nm} = c'_{nm} + 1$. If there is G' where $c' = c - 3$, then c_{mn} and c_{nm} are easily found out from our discussion of the reconstruction of cases c and d . So, in every G' considered, let $c' = c - 1$. We can easily find from all these graphs whether c_{mn} or c_{nm} is zero or not. If both are different from zero, a G' with maximum c'_{mn} gives c_{mn} and hence c_{nm} also. Suppose $c_{mn} = 0$. Then in any G' , $c'_{nm} = c_{nm} - 1$. Hence c_{nm} is found.

Case 2. $c = 1$. Consider the unique G_1 with the distance p . Let the distances in this be $\dots r, p, s, t, \dots$. Suppose $r \neq s$. Then consider a G' which does not have the consecutive distances m and n and which has minimum number of distances equal to m . In this if the new distance formed is $m + r$ then m occurs first and then n , and if the new distance formed is $m + s$, then m occurs after n . Suppose $r = s$. Clearly $r \neq m, n$ (since $c = 1$). Two cases arise depending on t . If $t \neq n$, consider a G' where there are minimum number of distances equal to m , maximum number of distances equal to $m + s$ and the combination (m, n)

does not occur. If in this G' the distance n occurs after a distance $m + s$, then m occurs first and then n in p . Otherwise n occurs first and then m in p .

Suppose $t = n$. Then n will occur after $m + s$, whether m occurs first or not in p . But now, consider a G' in which the combination (m, n) does not occur and in which the distances n and s combine to give $n + s$. If there is an m after an $n + s$, in the G' , clearly n occurs first in the (m, n) combination. Otherwise n occurs first.

This completes the proof.

Note 5.5. Theorem 5.4 is not true when there are only three K_2 -branches. For example, consider P where $S = \{1, 2, \dots, 6\}$ and Γ is the cyclic group on 6 elements whose generator is $(1\ 2\ 3\ 4\ 5\ 6)$. By choosing the distances as 1, 2 and 3 (in that order) we get G . If we chose the distances 1, 3 and 2 (in that order) we get $H \cong G$. It is easy to see that G is not \bar{P} -reconstructable.

This counterexample can be modified such that the pruned center is a cycle. The G and H for this case are shown in Figure 3.

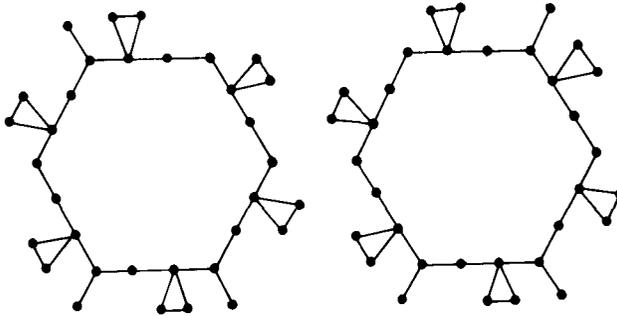


FIGURE 3

This shows that the corresponding subcases of Theorem 4.10 and Corollaries 4.13 and 4.14 in Greenwell and Hemminger (1969) are not true.

References

- M. Behzad and G. Chartrand (1971), *Introduction to the theory of graphs* (Allyn and Bacon, Boston).
- J. A. Bondy (1969), 'On Ulam's conjecture for separable graphs', *Pacific J. Math.* **31**, 281–288.
- I. Z. Bouwer (1969), 'Section graphs for finite permutation groups', *The many facets of graph theory*, edited by G. Chartrand and S. F. Kapoor, pp. 55–61, *Lecture Notes in Mathematics* 110, Springer Verlag, Berlin.
- R. M. Bryant (1971), 'On a conjecture concerning the reconstruction of graphs', *J. Combinatorial Theory Ser. B* **11**, 139–141.

- D. L. Greenwell and R. L. Hemminger (1969), 'Reconstructing graphs', *The many facets of graph theory*, edited by G. Chartrand and S. F. Kapoor, pp. 91–114, Lecture Notes in Mathematics 110, Springer Verlag, Berlin.
- F. Harary (1969), *Graph theory* (Addison-Wesley, Reading, Mass.).
- V. Krishnamoorthy (1976), 'The reconstruction conjecture and some related problems' (Ph. D. thesis, I. I. I. Madras).
- V. Krishnamoorthy and K. R. Parthasarathy (1978), '*F*-sets in graphs', *J. Combinatorial Theory Ser. B* **24**, 53–60.
- B. Manvel (1969), 'Reconstruction of unicyclic graphs', *Proof techniques in graph theory*, edited by F. Harary, pp. 103–107 (Academic Press, New York).
- H. Wielandt (1964), *Finite permutation groups* (Academic Press, New York).

Faculty of Science
Madras Institute of Technology
Campus
Perarignar Anna University
of Technology
Madras-44
India

Department of Mathematics
Indian Institute of Technology
Madras-36
India