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LETTER TO THE EDITOR

Dear Editor,

On the covariances of outdegrees in random plane recursive trees

In 2005 Janson [3], extending the earlier work of Mahmoud *et al.* [4], established the joint asymptotic normality of the outdegrees of a random plane recursive tree (we refer to [3] for references, discussion, and statements, and to [2] for a much wider context). In particular, he gave the following formula for the entries of the limiting covariance matrix [3, Theorem 1.3]:

$$\tilde{\sigma}_{ij} = 2\sum_{k=0}^{i}\sum_{l=0}^{j}\frac{(-1)^{k+l}}{k+l+4}\binom{i}{k}\binom{j}{l}\left(\frac{2(k+l+4)!}{(k+3)!(l+3)!} - 1 - \frac{(k+1)(l+1)}{(k+3)(l+3)}\right). \tag{1}$$

Since this formula is not very convenient to work with (in particular the behavior of $\tilde{\sigma}_{ij}$ as i and/or j grow to ∞ is not immediately clear), we found it worthwhile to point out that it may be simplified considerably. Throughout, $(x)_m = x(x-1)\dots(x-(m-1))$ denotes the falling factorial.

Proposition 1. *For all integers $i \ge 0$, $j \ge 0$, we have*

$$\tilde{\sigma}_{ij} = \frac{16}{(i+3)_3(j+3)_3} - \frac{24}{(i+j+4)_4} \quad \text{if } i \ne j,$$
$$\tilde{\sigma}_{jj} = \frac{4}{(j+3)_3} + \frac{16}{(j+3)_3^2} - \frac{24}{(2j+4)_4}.$$

For the proof we will need two identities involving binomial coefficients that we present in the following two lemmas.

Lemma 1. *For all integers $k \ge 0$, $a \ge 0$, and $j \ge k$,*

$$\sum_{l=0}^{j}(-1)^l\binom{j}{l}\binom{k+l+a}{l+a} = \begin{cases} 0 & \text{if } j > k, \\ (-1)^j & \text{if } j = k. \end{cases}$$

Proof. This is a special case of equation (5.24) in [1] as we have found thanks to the encouragement by one of the referees to search for a source in the literature. It corresponds to $m = 0$ and $s = n + a$ in the notation used in [1]. However, to keep this letter self-contained we supply a short proof. We proceed by induction over k for all a and $j \ge k$. If $k = 0$ the equality holds for all $a \ge 0$ since its left-hand side is $(1-1)^j$ if $j > 0$ and 1 if $j = 0$. Assume that it holds for nonnegative integers up to k and all values of a and $j \ge k$. Let $a \ge 0$ be any integer. For $j \ge k+1$,

$$\sum_{l=0}^{j}(-1)^l\binom{j}{l}\binom{k+1+l+a}{l+a} = \frac{k+1+a}{k+1}\sum_{l=0}^{j}(-1)^l\binom{j}{l}\binom{k+l+a}{l+a}$$
$$+ \sum_{l=0}^{j}(-1)^l\left(\frac{j!}{l!(j-l)!}\right)\left(\frac{l(k+l+a)!}{(k+1)!\,(l+a)!}\right).$$

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The first sum is 0 by the inductive hypothesis. We cancel the ls in the second sum and write it as

$$\sum_{l=1}^{j}(-1)^l\left(\frac{j!}{(l-1)!\,(j-l)!}\right)\left(\frac{(k+l-1+a+1)!}{(k+1)k!\,(l-1+a+1)!}\right)$$
$$= -\frac{j}{k+1}\sum_{l=0}^{j-1}(-1)^l\binom{j-1}{l}\binom{k+l+a+1}{l+a+1}.$$

By the inductive hypothesis (applied to k, $a+1$, and $j-1$) this sum is 0 if $j-1>k$ and is $(-1)^{j-1}$ if $j-1=k$. This proves that the original expression is 0 if $j>k+1$ and is $(-1)^j$ if $j=k+1$ thus completing the induction.

Lemma 2. *For all integers $j\geq 0$, $i\geq 0$, and $a\geq 1$, we have*

$$\sum_{l=0}^{j}\frac{(-1)^l}{(l+a)\binom{l+a+i}{i}}\binom{j}{l} = \frac{1}{a\binom{i+j+a}{a}} = \frac{(a-1)!}{(i+j+a)_a}.$$

Proof. We use induction over $j\geq 0$ for all $a\geq 1$ and $i\geq 0$. (Alternatively i can stay fixed throughout). When $j=0$ both sides are $1/(a\binom{a+i}{i})$. Assume that the statement holds for all integers up to j and all $a\geq 1$. We will prove that it holds for $j+1$ and all integers $a\geq 1$. We have

$$\begin{aligned}
\sum_{l=0}^{j+1}\frac{(-1)^l}{(l+a)\binom{l+a+i}{i}}\binom{j+1}{l} &= \sum_{l=0}^{j+1}\frac{(-1)^l}{(l+a)\binom{l+a+i}{i}}\left\{\binom{j}{l}+\binom{j}{l-1}\right\}\\
&= \sum_{l=0}^{j}\frac{(-1)^l}{(l+a)\binom{l+a+i}{i}}\binom{j}{l} + \sum_{l=1}^{j+1}\frac{(-1)^l}{(l+a)\binom{l+a+i}{i}}\binom{j}{l-1}\\
&= \frac{1}{a\binom{i+j+a}{a}} + \sum_{l=1}^{j+1}\frac{(-1)^{l-1+1}}{(l-1+a+1)\binom{l-1+a+1+i}{i}}\binom{j}{l-1}\\
&= \frac{1}{a\binom{i+j+a}{a}} - \sum_{l=0}^{j}\frac{(-1)^l}{(l+a+1)\binom{l+a+1+i}{i}}\binom{j}{l}\\
&= \frac{1}{a\binom{i+j+a}{a}} - \frac{1}{(a+1)\binom{i+j+a+1}{a+1}}\\
&= \frac{(a-1)!\,(i+j)!}{(i+j+a)!}\left\{1-\frac{a}{i+j+a+1}\right\}\\
&= \frac{(a-1)!\,(i+j+1)!}{(i+j+a+1)!}\\
&= \frac{1}{a\binom{i+j+1+a}{a}},
\end{aligned}$$

where we have used the inductive hypothesis, first with j and a and then with j and $a+1$. This proves Lemma 2.

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Proof of Proposition 1. Assume, without loss of generality, that $0 \le i \le j$. We split the right–hand side of (1) as

$$4\sum_{k=0}^{i}\sum_{l=0}^{j}\frac{(-1)^{k+l}}{k+l+4}\binom{i}{k}\binom{j}{l}\frac{(k+l+4)!}{(k+3)!\,(l+3)!} \tag{2}$$

$$-2\sum_{k=0}^{i}\sum_{l=0}^{j}\frac{(-1)^{k+l}}{k+l+4}\binom{i}{k}\binom{j}{l}\left(1+\frac{(k+1)(l+1)}{(k+3)(l+3)}\right). \tag{3}$$

We claim that (2) is 0 unless $i = j$ in which case it is $4/(j+3)_3$. To see this note that

$$\frac{(k+l+4)!}{(k+l+4)(k+3)!(l+3)!} = \frac{1}{(k+3)_3}\binom{k+l+3}{l+3},$$

so that

$$\sum_{k=0}^{i}\sum_{l=0}^{j}\frac{(-1)^{k+l}}{k+l+4}\binom{i}{k}\binom{j}{l}\frac{(k+l+4)!}{(k+3)!(l+3)!} = \sum_{k=0}^{i}\frac{(-1)^k}{(k+3)_3}\binom{i}{k}\sum_{l=0}^{j}(-1)^l\binom{j}{l}\binom{k+l+3}{l+3}.$$

Since $k \le i$ and we assumed that $i \le j$, by Lemma 1, the inner sum is 0 unless $i = j$ and if that is the case only the term $k = i = j$ in the outer sum is nonzero and it is

$$\frac{(-1)^j}{(j+3)_3}\binom{j}{j}\sum_{l=0}^{j}(-1)^l\binom{j}{l}\binom{j+l+3}{l+3} = \frac{(-1)^{2j}}{(j+3)_3} = \frac{1}{(j+3)_3}$$

by Lemma 1. To handle (3), we write

$$1+\frac{(k+1)(l+1)}{(k+3)(l+3)} = 2\frac{(k+1)(l+1)+(k+l+4)}{(k+3)(l+3)},$$

so that (3) is

$$-4\sum_{k=0}^{i}(-1)^k\frac{k+1}{k+3}\binom{i}{k}\sum_{l=0}^{j}(-1)^l\frac{l+1}{(l+3)(k+l+4)}\binom{j}{l} \tag{4}$$

$$-4\sum_{k=0}^{i}(-1)^k\frac{1}{k+3}\binom{i}{k}\sum_{l=0}^{j}(-1)^l\frac{1}{l+3}\binom{j}{l}. \tag{5}$$

By Lemma 2 (used with $a = 3$ and $i = 0$), (5) can be written as

$$-4\left(\frac{2}{(i+3)_3}\right)\left(\frac{2}{(j+3)_3}\right) = -\frac{16}{(i+3)_3(j+3)_3}.$$

To handle (4), we first note that

$$\sum_{l=0}^{j}(-1)^l\frac{l+1}{(l+3)(k+l+4)}\binom{j}{l} = \frac{k+3}{(k+1)(k+4)\binom{k+j+4}{j}} - \frac{2}{3(k+1)\binom{j+3}{j}}.$$

This follows from partial fraction decomposition

$$\frac{l+1}{(l+3)(k+l+4)} = \left(\frac{k+3}{k+1}\right)\left(\frac{1}{k+l+4}\right) - \frac{2}{(k+1)(l+3)}$$

and

$$\sum_{l=0}^{j} \frac{(-1)^l}{k+l+4}\binom{j}{l} = \frac{1}{(k+4)\binom{k+j+4}{j}}, \qquad \sum_{l=0}^{j} \frac{(-1)^l}{l+3}\binom{j}{l} = \frac{1}{3\binom{j+3}{j}},$$

which is Lemma 2 used twice with $a = k + 4$ and $i = 0$ for the first equality and with $a = 3$ and $i = 0$ for the second equality. Therefore, (4) can be written as

$$-4\sum_{k=0}^{i}(-1)^k \frac{1}{(k+4)\binom{k+j+4}{j}}\binom{i}{k} + \frac{16}{(j+3)_3}\sum_{k=0}^{i}(-1)^k \frac{1}{k+3}\binom{i}{k}.$$

Applying Lemma 2 (with $a = 4$ and general i) to the first term and with $a = 3$ and $i = 0$ to the second term, we find that (4) is

$$-\frac{24}{(i+j+4)_4} + \frac{32}{(i+3)_3(j+3)_3}.$$

Hence, the combined contribution of (4) and (5) is

$$-\frac{16}{(i+3)_3(j+3)_3} + \frac{32}{(i+3)_3(j+3)_3} - \frac{24}{(i+j+4)_4} = \frac{16}{(i+3)_3(j+3)_3} - \frac{24}{(i+j+4)_4},$$

which completes the proof.

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Yours sincerely,

Hüseyin Acan* and Paweł Hitczenko**

*School of Mathematical Sciences,
Monash University,
Melbourne, VIC 3800,
Australia.

Email address: huseyin.acan@monash.edu.au

**Department of Mathematics,
Drexel University,
Philadelphia, PA 19104,
USA.

Email address: phitczenko@math.drexel.edu

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