A CLASS OF LOOPS WITH THE ISOTOPY-ISOMORPHY PROPERTY

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1. Introduction. A loop L has the isotopy–isomorphy property provided each loop isotopic to L is isomorphic to L. A familiar problem is that of characterizing those loops having this property.

It is well known (**1**, p. 56) that the loop isotopes of $(L, \cdot)$ are those loops $L(a, b, *)$ defined by $x * y = x/b \cdot a\backslash y$ for some a, b in L. In this paper we first show (Corollary to Theorem 1) that a loop L with identity element 1 has the isotopy–isomorphy property if L is isomorphic to $L(1, x)$ and to $L(x, 1)$ for each x in L. We then determine necessary and sufficient conditions (Theorems 2 and 3) for L to be isomorphic to these isotopes under translations (i.e. permutations of the form $x\nu = cx$ or $x\nu = xc$ for c fixed). Finally these results, together with some from Osborn (**4**), are used to show that loops satisfying the identity $c(cx)^\rho = (cy)[c(xy)]^\rho$ are isomorphic to all their isotopes (for notation see **4**).

2. Results. Bryant and Schneider (**3**) have shown that if L is a loop, and if $L(a, b, \circ)$ is isomorphic to $L(c, d, *)$ under θ, then $L(e, f, \triangle)$ is isomorphic to $L((eb)\theta/d, c\backslash(af)\theta, \square)$ under θ. Our first theorem is a consequence of this result.

THEOREM 1. *Suppose there exist a, b in L such that $L(a, 1)$ is isomorphic to $L(a, y)$ and $L(1, b)$ is isomorphic to $L(x, b)$ for all x, y in L. Then L is isomorphic to all its loop isotopes.*

Proof. Consider an arbitrary isotope $L(r, s)$ of L. We shall show that $L(r, s)$ is isomorphic to $L(a, b)$. Note that $L(a, b)$ is isomorphic to $L(a, y)$ and $L(x, b)$ for all x, y in L. In particular, $L(a, b)$ is isomorphic to $L(a, s)$ under some isomorphism θ, and $L(a, b)$ is isomorphic to $L((rs)\theta^{-1}/b, b)$. Also note that $(ab)\theta = (as)$, since (ab) and (as) are the identities of $L(a, b)$ and $L(a, s)$. Finally, by Bryant and Schneider's theorem, $L((rs)\theta^{-1}/b, b)$ is isomorphic under θ to $L(\{[(rs)\theta^{-1}]/b \cdot b\}\theta/s, a\backslash(ab)\theta) = L(r, s)$. Thus $L(a, b)$ is isomorphic to $L(r, s)$ for all r, s in L.

If we let $a = b = 1$ in Theorem 1 we have:

COROLLARY. *If a loop L is isomorphic to $L(1, x)$ and to $L(x, 1)$ for each x in L, then L is isomorphic to all its loop isotopes.*

Received January 25, 1965.

https://doi.org/10.4153/CJM-1966-057-0 Published online by Cambridge University Press

As a result of this Corollary we can concentrate our efforts on determining conditions under which a loop L is isomorphic to these special isotopes. In particular, the following two theorems exhibit necessary and sufficient conditions for the existence of such isomorphisms which are translations.

THEOREM 2. (i) *L is isomorphic to $L(1, c, *)$ under $x\nu = xc$ if and only if c is in the right nucleus of L.* (ii) *L is isomorphic to $L(c, 1, *)$ under $x\nu = cx$ if and only if c is in the left nucleus of L.*

Proof. (i) If c is in the right nucleus we have $(xy)\nu = (xy)c = x(yc) = (xc/c)(yc) = (xc)*(yc) = x\nu * y\nu$. Conversely if ν is an isomorphism, then $(xy)c = (xy)\nu = x\nu * y\nu = (xc)*(yc) = (xc/c)(yc) = x(yc)$.

The proof of (ii) is similar.

We note that Theorem 2 provides a proof of the familiar fact that a group is isomorphic to all its isotopes.

THEOREM 3. (i) *L is isomorphic to $L(1, c, *)$ under $x\nu = cx$ if and only if*

(1) $$(cx)/c\cdot cy = c(xy) \qquad \text{for all } x, y \text{ in } L.$$

(ii) *L is isomorphic to $L(c, 1, *)$ under $x\nu = xc$ if and only if*

(2) $$xc\cdot c\backslash(yc) = (xy)c \qquad \text{for all } x, y \text{ in } L.$$

Proof. (i) If (1) holds, then

$$(xy)\nu = c(xy) = (cx)/c\cdot cy = (cx)*(cy) = x\nu * y\nu.$$

Conversely, if ν is an isomorphism, then

$$c(xy) = (xy)\nu = x\nu * y\nu = (cx)*(cy) = (cx)/c\cdot cy.$$

The proof of (ii) is similar.

The following theorem is our main result.

THEOREM 4. *If a loop L has the property that, for all c, x, y in L,*

(3) $$c(cx)^\rho = (cy)[c(xy)]^\rho,$$

then L is isomorphic to all its isotopes.

Proof. If we let $c = 1$, we see that (3) implies the weak inverse property. We next show that (3) implies its dual,

(4) $$(xc)^\lambda c = [(yx)c]^\lambda(yc).$$

Suppose L satisfies (3). If we replace x by $(yz)^\lambda$ and use the weak inverse property to obtain $xy = z^\lambda$, we have that (3) implies $c[c(yz)^\lambda]^\rho = (cy)(cz^\lambda)^\rho$. This is equivalent to the statement that

(5) $$(L(c),\ \lambda L(c)\rho,\ \lambda L(c)\rho L(c))$$

is an autotopism of L. For weak-inverse-property loops, Osborn (**4**, p. 296) has shown that

$$L^{-1}(c) = \lambda R(c)\rho \tag{6}$$

and that $(\lambda^2, \lambda^2, \lambda^2)$ and (ρ^2, ρ^2, ρ^2) are autotopisms. Using these results together with the fact that the autotopisms of a loop form a group, we show that each of the following is an autotopism of L:

$$(L^{-1}(c), \lambda L^{-1}(c)\rho, L^{-1}(c)\lambda L^{-1}(c)\rho), \tag{7}$$

$$(\lambda R(c)\rho, \lambda^2 R(c)\rho^2, \lambda R(c)\lambda R(c)\rho^2), \tag{8}$$

$$(\lambda R(c)\lambda, \lambda^2 R(c), \lambda R(c)\lambda R(c)), \tag{9}$$

$$(\rho R(c)\lambda, R(c), \rho R(c)\lambda R(c)). \tag{10}$$

(7) is the inverse of (5). (8) is obtained by using (6) in (7). The multiplication of (8) on the right by $(\lambda^2, \lambda^2, \lambda^2)$ yields (9), and the multiplication of (9) on the left by (ρ^2, ρ^2, ρ^2) yields (10). (10) is equivalent to $(z^\rho c)^\lambda (yc) = [(zy)^\rho c]^\lambda c$. If we let $z^\rho = yx$ and use the weak inverse property to obtain $x = (zy)^\rho$, we have the desired dual. By a similar argument it could be shown that (4) implies (3).

If (U, V, W) is an autotopism of a weak-inverse-property loop L, then $(\rho W\lambda, U, \rho V\lambda)$ and $(V, \lambda W\rho, \lambda U\rho)$ are also autotopisms of L (**4**, p. 296). Thus if a loop has property (3), then $(L(c), \lambda L(c)\rho, \lambda L(c)\rho L(c))$ and therefore $(L(c)\rho L(c)\lambda, L(c), L(c))$ are autotopisms of L. The latter autotopism can be written $(L(c)R^{-1}(c), L(c), L(c))$, since $R^{-1}(c) = \rho L(c)\lambda$ (**4**, p. 296). But this autotopism is equivalent to (1). Similarly, if L has property (3), then it has property (4). Thus $(\rho R(c)\lambda, R(c), \rho R(c)\lambda R(c))$ and therefore

$$(R(c), R(c)\lambda R(c)\rho, R(c))$$

are autotopisms of L. The latter can be written $(R(c), R(c)L^{-1}(c), R(c))$. This autotopism is equivalent to (2).

The desired result now follows from Theorem 3 and the Corollary to Theorem 1.

Osborn (**4**, pp. 300–302) has defined a loop H and has shown that the homomorphs of H are the weak-inverse-property loops with one generator which are isomorphic to all their isotopes. It is not hard to show that H has property (3). *Thus* (3) *is both a necessary and a sufficient condition that a one-generator weak-inverse-property loop have the isotopy–isomorphy property.*

We finally prove the following theorem.

THEOREM 5. *Let L be a Moufang loop. Then L has property* (3) *if and only if for each a in L, a^2 is in the nucleus of L.*

https://doi.org/10.4153/CJM-1966-057-0 Published online by Cambridge University Press

Proof. We first show that $c\cdot xc^{-1} = cx\cdot c^{-1}$ for all x, c in L. Since L is Moufang, L has the inverse property. Therefore

$$(c\cdot xc^{-1})c = (cx)(c^{-1}c) = cx = (cx\cdot c^{-1})c;$$

thus $c\cdot xc^{-1} = cx\cdot c^{-1}$.

In a Moufang loop L, (3) can be written

$$(cy)((y^{-1}x^{-1})c^{-1}) = c(x^{-1}c^{-1}).$$

If we let $x^{-1} = yz$, we obtain

$$(cy)(zc^{-1}) = c(yz\cdot c^{-1}) = (c\cdot yz)c^{-1}. \tag{11}$$

(11) is equivalent to the requirement that $(L(c), R(c^{-1}), L(c)R(c^{-1}))$ be an autotopism of L. Since L is Moufang, $(L(c), R(c), R(c)L(c))$ is an autotopism. The product $(L(c)L(c), I, L(c)L(c))$ is an autotopism of L. Thus

$$c(cx)\cdot y = c\cdot c(xy),$$

which is equivalent to the requirement that $(cc)x\cdot y = (cc)(xy)$, since $c(cz) = (cc)z$ in a Moufang loop. Thus c^2 is in the left nucleus of L, and, because the nuclei of a Moufang loop coincide, all squares are in the nucleus of L.

Conversely, if L is a Moufang loop all of whose squares are in the nucleus, then $(L(c), R(c), R(c)L(c))$ and $(L(c)L(c), I, L(c)L(c))$ are autotopisms of L. Multiplying the second by the inverse of the first, we obtain the autotopism $(L(c), R(c^{-1}), L(c)R(c^{-1}))$, which is (11).

It follows immediately from Theorems 4 and 5 that a Moufang loop all of whose squares lie in the nucleus is isomorphic to all its isotopes. This result has been mentioned by Bruck (**2**, p. 60), who attributes it to a correspondence from J. Tits.

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